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ON ANALYTIC FUNCTIONS WITH REFERENCE TO AN INTEGRAL OPERATOR

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Let $E = \{z : |z| < 1\}$ and let

$$H = \{w : \text{regular in } E,\ w(0) = 0,\ |w(z)| < 1,\ z \in E\} .$$

Let $P(A, B)$ denote the class of functions in E which can be put in the form $(1+Aw(z))/(1+Bw(z))$, $-1 \le A < B \le 1$, $w(z) \in H$. Let $S^*(A, B)$ denote the class of functions $f(z)$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ such that $zf'(z)/f(z) \in P(A, B)$. If $f(z) \in S^*(A, B)$ and $g(z) \in S^*(C, D)$ then, in this paper the radius of starlikeness of order β $(\beta \in [0, 1))$ of the following integral operator

$$F(z) = \frac{m+1}{(g(z))^m} \int_0^z t^{m-1} f(t)dt , \quad m > 1 ,$$

is determined. Conversely, a sharp estimate is obtained for the radius of starlikeness of the class of functions

$$f(z) = 2^{-1}(g(z)F(z))'$$

where $g(z)$ and $F(z)$ belong to the class $S^*(A, B)$.

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1. Introduction

Let S denote the family of functions $f(z)$ which is regular and univalent in the unit disc E and which satisfy the conditions $f(0) = 0 = f'(0) - 1$. Let $S^* \subset S$ denote the class of starlike functions, namely those members of S which map E onto a domain that is starlike with respect to the origin. Libera [6] showed that if $f(z) \in S^*$ then

$$F(z) = \frac{2}{z} \int_0^z f(t)dt$$

also belongs to S^* . The converse problem was treated by Livingston [7]. Bernadi [2] proved that, if $f(z) \in S^*$,

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt$$

also belongs to S^* .

We denote by $S^*(\alpha)$ the class of functions $f(z)$ defined in E , regular in E with normalization $f(0) = 0 = f'(0) - 1$ and $\text{Re}\{zf'(z)/f(z)\} > \alpha$, $\alpha \in [0, 1)$. Karunakaran [4] proved that if $f(z) \in S^*(\alpha)$ and $g(z) \in S^*(\gamma)$ for $\alpha, \gamma \in [0, 1)$, then

$$F(z) = \frac{2}{g(z)} \int_0^z f(t)dt$$

is β starlike for $|z| < \sigma$ where σ is a function of α, β, γ .

The following class was defined and its properties were studied by Janowski [3].

DEFINITION 1. Let

$$H = \{w : \text{regular in } E : w(0) = 0, |w(z)| < 1, z \in E\} .$$

Let $P(A, B)$ denote the class of functions in E which can be put in the form $(1+Aw(z))/(1+Bw(z))$, $-1 \le A < B \le 1$, $w(z) \in H$. Let $S^*(A, B)$ denote functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{such that} \quad zf'(z)/f(z) \in P(A, B) .$$

Equivalently $S^*(A, B)$ denotes the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

regular in the unit disc E and satisfying the conditions

$$\frac{zf'(z)}{f(z)} \propto \frac{1+Az}{1+Bz} , \quad z \in E , \quad -1 \le A < B \le 1 .$$

In this paper we determine the radius of β starlikeness of

$$F(z) = \frac{m+1}{(g(z))^m} \int_0^z t^{m-1} f(t)dt , \quad m > 1 ,$$

where $f(z) \in S^*(A, B)$ and $g(z) \in S^*(C, D)$. In the last section we examine the converse problem and obtain a sharp result.

2. Lemmas

In this section we state some lemmas which will be used to establish our theorems.

LEMMA 1. *Let* $p(z) \in P(A, B)$. *Then, for* $|z| \le r < 1$,

$$\frac{1-Ar}{1-Br} \le \text{Re } p(z) \le \frac{1+Ar}{1+Br} .$$

Proof. This follows from the fact that the function $\tau(z) = (1+Az)/(1+Bz)$ maps the disc $|z| \le r$ onto the interior of the circle with the line segment $[(1-Ar)/(1-Br), (1+Ar)/(1+Br)]$ as diameter and $p(z)$ is subordinate to $(1+Az)/(1+Bz)$.

LEMMA 2 [4]. *Suppose* $p(z) = [1+Aw(z)][1+Bw(z)]^{-1}$ *where* $-1 \le A < B \le 1$ *and* $w(z) \in H$. *Then, for* $C \ge B$,

$$\text{Re}\{Cp(z)+(A/p(z))\} - \frac{r^2|Bp(z)-A|^2-|1-p(z)|^2}{(1-r^2)|p(z)|} \ge \begin{cases} P_1(r) & \text{for } R_0 \le R_1 , \\ P_2(r) & \text{for } R_0 \ge R_1 , \end{cases}$$

where

$$P_1(r) = C\left(\frac{1+Ar}{1+Br}\right) + A\left(\frac{1+Br}{1+Ar}\right) ,$$

$$P_2(r) = \frac{2}{1-r^2} \left\{(1+A)^{\frac{1}{2}}\left[1+C-\left(D(1+C)+B^2+C\right)r^2+A\left(B^2+C\right)r^4\right]^{\frac{1}{2}}-1-BAr^2\right\} ,$$

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$$R_0^2 = \frac{(1+A)(1-Ar^2)}{(1+C)-r^2(C+B^2)}$$

and

$$R_1 = \frac{1+Ar}{1+Br} .$$

LEMMA 3 [4]. *If* $w(z) \in H$ *and* $p(z) = [1+Aw(z)][1+Bw(z)]^{-1}$ *then*

$$\text{Re}\left\{\frac{-zw'(z)}{[1+Aw(z)][1+Bw(z)]}\right\} > G ,$$

where

$$G = \frac{1}{B-A}\left\{\text{Re}\left(\frac{A}{p(z)} + Bp(z)\right) - \frac{r^2|Bp(z)-A|^2-|1-p(z)|^2}{(1-r^2)|p(z)|} - (B+A)\right\} .$$

LEMMA 4 [5]. *Let* N *and* D *be regular in* E , D *maps* E *onto a many sheeted starlike region.* $N(0) = 0 = D(0)$, $N'(0) = D'(0) = p$ *and*

$$\frac{1}{p}\left(\frac{N'(z)}{D'(z)}\right) \in P(A, B) .$$

Then

$$\frac{1}{p}\left(\frac{N(z)}{D(z)}\right) \in P(A, B) , \quad p \geq 1 .$$

LEMMA 5. *Let* $p_1(z)$ *and* $p_2(z)$ *belong to* $P(A, B)$; *then*

$$\tfrac{1}{2}(p_1(z)+p_2(z)) \in P(A, B) .$$

Proof. This follows easily from Bernadi's result [1].

3. Main theorems

THEOREM 1. *Let* $f(z) \in S^*(A, B)$ *and* $g(z) \in S^*(C, D)$ *where*

$$-1 \leq A < B \leq 1 \quad \text{and} \quad -1 \leq C < D \leq 1 .$$

Then the function $F(z)$ *defined by*

$$F(z) = \frac{m+1}{(g(z))^m}\int_0^z t^{m-1}f(t)dt$$

is starlike of order β , *for* $|z| < \sigma$ *where* σ *is given by*

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$$\sigma = \frac{L+\sqrt{L^2-(1-\beta)K}}{-K} \quad when \quad K < 0 \ , \quad L > 0 \ ,$$

where

$$L = \tfrac{1}{2}\{(m-\beta)(D-B)+(D-A)+m(B-C)\} \ ,$$

$$K = (m-\beta)BD - AD + mBC \ .$$

Proof. Let $J(z) = \int_0^z t^{m-1}f(t)dt$. Then $F(z)\left(g(z)\right)^m = (m+1)J(z)$ taking the logarithmic derivatives

$$\frac{zF'(z)}{F'(z)} = \frac{zJ'(z)}{J(z)} - m\frac{zg'(z)}{g(z)}$$

$$= m + \frac{zJ'(z)-mJ(z)}{J(z)} - m\frac{zg'(z)}{g(z)} \ .$$

Setting $N(z) = zJ'(z) - mJ(z)$ and $D(z) = J(z)$ we have $N(0) = 0 = D(0)$. By a lemma due to Bernardi [2], $D(z)$ is a $m+1$ valent starlike for $m = 1, 2, \ldots$,

$$\frac{N'(z)}{D'(z)} = \frac{zf'(z)}{f(z)} \in P(A, B) \ .$$

Therefore, by Lemma 4,

$$\frac{N(z)}{D(z)} \in P(A, B) \ .$$

Also, since $g \in S^*(C, D)$,

$$\frac{zg'(z)}{g(z)} \in P(C, D) \ .$$

Hence

$$\frac{zF'(z)}{F(z)} = m + p_1(z) - mp_2(z)$$

where

$$p_1(z) = \frac{zJ'(z)-mJ(z)}{J(z)} \in P(A, B) \ ,$$

$$p_2(z) = \frac{zg'(z)}{g(z)} \in P(C, D) \ .$$

Using Lemma 1,

$$\mathrm{Re}\left\{\frac{zF'(z)}{F(z)}\right\} \geq m + \frac{1-Ar}{1-Br} - m\frac{1+Cr}{1+Dr} \ .$$

Now $\text{Re}\{zF'(z)/F(z)\} \geq \beta$ whenever

$$(m-\beta) + \frac{1-Ar}{1-Br} + \frac{1+Cr}{1+Dr} \geq 0 ,$$

that is,

$$K^2 + 2Lr + (1-\beta) > 0 \tag{1}$$

where

$$K = (m-\beta)BD - AD + mBC ,$$

$$L = \tfrac{1}{2}\{(m-\beta)(D-B)+(D-A)+m(B-C)\} .$$

In other words $\text{Re}\{zF'(z)/F(z)\} \geq \beta$ for $|z| = r < \sigma = \dfrac{L+\sqrt{L^2-(1-\beta)K}}{-K}$, where $K < 0$ and $L > 0$.

COROLLARY 1. *When* $g(z) = z$ *in Theorem* 1 *we get*

$$F(z) = \frac{m+1}{z^m}\int_0^z t^{m-1}f(t)dt$$

and

$$\frac{zF'(z)}{F(z)} = \frac{zJ'(z)-mJ(z)}{J(z)} \in P(A, B)$$

where $J(z)$ *is as defined in Theorem* 1. *Therefore* $F \in S^*(A, B)$.

This result is a particular case of Theorem 1 in [5].

COROLLARY 2. *When* $m = 1$ *and* $A = 1 - 2\alpha$, $B = -1$, $C = 1 - 2\gamma$, $D = -1$ *in the above theorem, we get Theorem* 1 *of* [4].

4. Converse problem

THEOREM 2. *Let* $g(z) \in S^*(A, B)$ *and* $F(z) \in S^*(A, B)$. *Let us define* $f(z)$ *by* $F(z)g(z) = 2\int_0^z f(t)dt$ *or equivalently,* $f(z) = 2^{-1}(F(z)g(z))'$; *then for* $|z| = r$,

$$\text{Re}\,\frac{zf'(z)}{f(z)} \geq \begin{cases} P_1(r) & \text{for } R_0 \leq R_1 , \\ P_2(r) & \text{for } R_0 \geq R_1 , \end{cases}$$

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where

$$P_1(r) = \frac{1}{B-A}\left\{(3B-2A)\left(\frac{1+Ar}{1+Br}\right)+A\left(\frac{1+Br}{1+Ar}\right)-2B\right\},$$

$$P_2(r) = \frac{2}{B-A}\left\{\left[\frac{(1+A)(1+3B-2A)\left(1-Ar^2\right)}{1-r^2} - \frac{\left(B^2-1\right)\left(1-Ar^2\right)r^2}{1-r^2}\right]^{\frac{1}{2}} - \left[\frac{1-ABr^2}{1-r^2}\right]-B\right\},$$

$$R_0^2 = \frac{(1+A)\left(1-Ar^2\right)}{(1+3B-2A)-r^2\left(1+3B-2A+B^2\right)}$$

and

$$R_1 = \frac{1+Ar}{1+Br}.$$

These bounds are sharp for $R_0 \le R_1$.

Proof. By the definition of f we have

$$2f(z) = \left(F(z)g(z)\right)' = F'(z)g(z) + F(z)g'(z),$$

$$\text{(2)} \qquad \frac{zf(z)}{F(z)g(z)} = \tfrac{1}{2}\left\{\frac{zF'(z)}{F(z)} + \frac{zg'(z)}{g(z)}\right\}.$$

Using Lemma 5, we get

$$\text{(3)} \qquad \frac{zf(z)}{F(z)g(z)} = p(z) \in P(A, B).$$

Hence $zf(z) = F(z)g(z)p(z)$; taking logarithmic derivatives we get

$$\text{(4)} \qquad \frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zF'(z)}{F(z)} + \frac{zp'(z)}{p(z)} - 1.$$

Using (2) and (3) in (4),

$$\text{(5)} \qquad \frac{zf'(z)}{f(z)} = 2p(z) + \frac{zp'(z)}{p(z)} - 1.$$

Now

$$p(z) = \frac{1+Aw(z)}{1+Bw(z)}$$

for some $w(z)$ regular such that $w(0) = 0$ and $|w(z)| < 1$, $z \in E$. Therefore

$$\frac{zp'(z)}{p(z)} = -\frac{(B-A)zw'(z)}{[1+Aw(z)][1+Bw(z)]};$$

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using Lemma 3 we get

$$\mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geq \frac{1}{B-A}\left\{\mathrm{Re}\left(\frac{A}{p(z)} + Bp(z)\right) - \frac{r^2|Bp(z)-A|^2-|1-p(z)|^2}{(1-r^2)\,|p(z)|} - (B+A)\right\} .$$

Now

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}$$

$$= 2\ \mathrm{Re}\left(p(z)\right) + \mathrm{Re}\left(\frac{zp'(z)}{p(z)}\right) - 1 ,$$

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}$$

$$\geq 2\ \mathrm{Re}\left(p(z)\right) + \frac{1}{B-A}\left\{\mathrm{Re}\left(\frac{A}{p(z)} + Bp(z)\right) - \frac{r^2|Bp(z)-A|^2-|1-p(z)|^2}{(1-r^2)\,|p(z)|} - (B+A)-1\right\}$$

$$\geq \frac{1}{B-A}\left\{\mathrm{Re}\left((3B-2A)p(z) + \frac{A}{p(z)}\right) - \frac{r^2|Bp(z)-A|^2-|1-p(z)|^2}{(1-r^2)\,|p(z)|} - 2B\right\} ;$$

applying Lemma 2 with $C = 3B - 2A$ we get the required result. Taking the function $g_1(z)$, $F_1(z)$ defined as

$$\frac{zg_1'(z)}{g_1(z)} = \frac{1+Aw_1(z)}{1+Bw_1(z)} = \frac{zF_1'(z)}{F_1(z)}$$

we see that the corresponding functions $zf_1'(z)/f_1(z)$ attains the bound $P_1(r)$ for $w_1(z) = z$ at $z = r$ whenever $R_0 \leq R_1$.

References

[1] S.D. Bernardi, "Special classes of subordinate functions", *Duke Math. J.* 33 (1966), 55-67.

[2] S.D. Bernardi, "Convex and starlike univalent functions", *Trans. Amer. Math. Soc.* 135 (1969), 429-446.

[3] W. Janowski, "Some extremal problems for certain families of analytic functions I", *Ann. Polon. Math.* 28 (1973), 297-326.

[4] V. Karunakaran, "Certain classes of regular univalent functions", *Pacific J. Math.* 61 (1975), 173-182.

[5] G. Lakshma Reddy and K.S. Padmanabhan, "On analytic functions with reference to the Bernardi integral operator", *Bull. Austral. Math. Soc.* **25** (1982), 387-396.

[6] R.J. Libera, "Some classes of regular univalent functions", *Proc. Amer. Math. Soc.* **16** (1965), 755-758.

[7] A.E. Livingston, "On the radius of univalence of certain analytic functions", *Proc. Amer. Math. Soc.* **17** (1966), 352-357.

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