Can. J. Math., Vol. XXVII, No. 4, 1975, pp. 797–804

HOMEOMORPHISMS ON THE SOLID DOUBLE TORUS

DONALD MYERS

1. Introduction. A finite set of generators for the isotopy classes of self-homeomorphisms of closed surfaces was given by Lickorish in three papers [**2**; **3**; **4**]. In [**5**] the group of isotopy classes for a particular, well-known cube with holes was presented. There the structure was "tight" enough to allow the computation of the relators as well as the generators. In this paper we give a finite set of generators for the group of isotopy classes of self-homeomorphisms on the solid double torus, the cube with two handles. Let us remark that the group of isotopy classes for the solid torus is well-known.

Most of the notation that we will use is as in [**1**] and [**5**]. A non-trivial disk is a properly embedded disk whose boundary is not null-homotopic in the manifold's boundary.

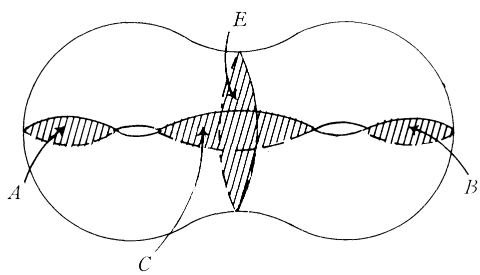

FIGURE 1

2. Homeomorphisms on the solid double torus. Let T be the solid double torus as shown in Figure 1. Let A, B, C, and E denote the properly embedded disks shown in Figure 1 and let G and H be the properly embedded annuli of Figure 2. For a properly embedded disk (or annulus) S we cut the manifold at S and twist one of the components of S in the cut manifold 360°, then glue the manifold back together again at S. This disk (or annulus)-homeomorphism induces a C-homeomorphism (two C-homeomorphisms, respectively) on ∂T [**2**]. Let b, e, g, h denote the disk and annulus-homeomorphisms at B, E, G, H respectively. Let R be the rotation of T in S^3 which takes each of A, B, C onto themselves but interchanges the components of

Received August 10, 1973 and in revised form, January 23, 1974.

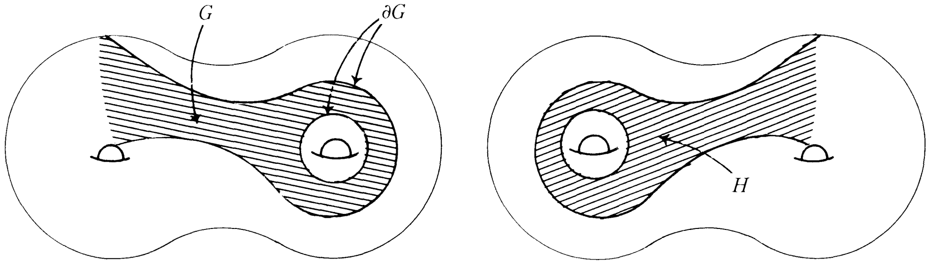

FIGURE 2a FIGURE 2b

$T - (A \cup B \cup C)$. If we let $\pi_1(T)$, the first homotopy group, have the presentation $(\bar{a}, \bar{b}: -)$ where $\bar{a}$ and $\bar{b}$ are loops hitting A and B respectively in a single point, then R_*, the induced map, takes each of $\bar{a}$ and $\bar{b}$ onto its inverse.

Let V be the homeomorphism of T onto itself which is the identity on the component of $(T - E)$ containing B and which takes $\bar{a}$ to $\bar{a}^{-1}$ in $\pi_1(T)$. Geometrically this is accomplished by holding E fixed and twisting the handle A 180° in S^3. Note that $V^2 = e$.

Let N be the group of all homeomorphisms of T onto itself generated by g, h, b, R, V, and all homeomorphisms isotopic to the identity. Two properly embedded disks D_1 and D_2 are said to be N-equivalent if there exists an element f of N such that $f(D_1) = D_2$. The homeomorphism g (or g^{-1}, depending upon the direction of twist used to define g) along with an isotopy shows that B and C are N-equivalent. A and C are also N-equivalent using the homeomorphism h. Also it is easy to see that if D is a disk which is N-equivalent to B, then a disk-homeomorphism at D is in N. It will be shown that B and E represent the only two distinct classes of N-equivalent properly embedded non-trivial disks in T. Then we will show that N is precisely the group of all orientation-preserving homeomorphisms of T onto itself.

The proof of Lemma 1 is well known and that of Lemma 2 is a trivial consequence of Lemma 1.

LEMMA 1. *Let X be a disk with three holes and let S be a set of scc's in* (int X) *with the following properties:*

(1) *for any component of ∂X there is an element of S parallel to it in X,*

(2) *for any two components of ∂X there is an element of S which separates these two components from the other two components of ∂X,*

(3) *S contains a scc which bounds a disk in X.*

Let M be the group of homeomorphisms of X onto itself generated by the C-homeomorphisms about scc's in S; each of these C-homeomorphisms is assumed to be the identity on ∂X. Also include in M those homeomorphisms of X onto itself which are isotopic to the identity via isotopies which are the identity on ∂X. Then any scc in (int X) *can be taken to an element of S by an element of M, and any C-homeomorphism about a scc in* (int X) *is in M.*

https://doi.org/10.4153/CJM-1975-087-9 Published online by Cambridge University Press

Lemma 2. *Let X be ∂T cut at $\partial A \cup \partial B$. Then the group M of Lemma* 1 *may be considered to be a subgroup of N.*

Lemma 3. *Let D be a properly embedded non-trivial disk in T which misses two of A, B, C. Then D is N-equivalent to E or B depending upon whether or not it separates T.*

Proof. Since $g^{-1}C = B$ and $gA = A$ we may assume that D misses B. Similarly we may assume D misses A because of h. Cut ∂T at $\partial A \cup \partial B$ to give X. Taking *scc*'s parallel to the four components of ∂X arising from cuts at ∂A and ∂B along with ∂C, ∂E, $g^{-1}(\partial B)$, and a *scc* bounding a disk in X we get a set S for Lemma 1. See Figure 3 for $g^{-1}(\partial B)$. The result follows from Lemmas 1 and 2.

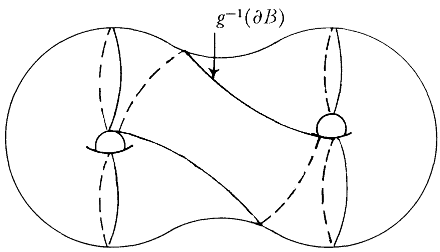

Figure 3

Theorem 1. *Any properly embedded non-trivial disk in T is N-equivalent to E or B depending upon whether or not it separates T.*

Proof. Suppose not; then there is a properly embedded disk in T that hits at least one of A and B by Lemma 3. Apply an element of N, if necessary, without increasing the number of components of $D \cap (A \cup B)$ so that we may assume that B is hit. Of all such disks pick one that hits $A \cup B$ in as few components as possible and is in general position with respect to $A \cup B$. Let X denote ∂T cut at $\partial A \cup \partial B$ where A_1 and A_2 denote the components of ∂X coming from ∂A and B_1 and B_2 denote those from ∂B. No component of $\partial D \cap X$ cuts a disk off X since otherwise we could construct an isotopy on ∂T (that is, in T) taking this arc to $\partial A \cup \partial B$ and then slightly to the other side. This either converts a component of $(A \cup B) \cap D$ into a *scc* or reduces the number of components of $(A \cup B) \cap D$. Since isotopies which are the identity on ∂T can eliminate *scc*'s in $(A \cup B) \cap D$ by starting with an innermost one, we have a contradiction in either case.

Suppose no arc of $\partial D \cap X$ has both endpoints in the same component of ∂X. Let $\pi_1(T) = (\bar{a}, \bar{b}: -)$ as before. Follow ∂D starting at a point in $\partial D \cap B_1$ and enter X. If the arc goes to B_2 then a word $\bar{b}^u$ $(u = \pm 1)$ is induced for ∂D

in $\pi_1(T)$. It leaves X and re-enters at B_1 and it may go to B_2 to give $\bar{b}^u$ again (same u). If it continues to repeat this process a nonzero power of $\bar{b}$ is induced. In going from B_1 to say A_1, it then re-enters at A_2 and $\bar{a}^w$ ($w = \pm 1$) is induced. A similar argument gives a nonzero power of $\bar{a}$ induced by ∂D before hitting one of the B_i where another nonzero power of $\bar{b}$ will be induced (the first power of $\bar{b}$ may have been zero). Thus nonzero powers of $\bar{a}$ and $\bar{b}$ are alternately induced until the *scc* is transversed. But $\pi_1(T)$ is a free group in $\bar{a}$ and $\bar{b}$; thus $\partial D \neq 1$ in $\pi_1(T)$, a contradiction.

For Z, Y in $\{A_1, A_2, B_1, B_2\}$, let $N(Z, Y)$ be the number of components of $\partial D \cap X$ with one endpoint in Z and the other in Y, and let $N(Z)$ be the number of points in $Z \cap \partial D$. From the above we assume without loss of generality that $N(B_1, B_1)$ is positive (from this it will also follow that $N(B_2, B_2)$ is also positive). Now counting the endpoints of $\partial D \cap X$ in each A_i and B_i we have the equations:

$$(1)\ N(A_1) = 2N(A_1, A_1) + N(A_1, A_2) + N(A_1, B_1) + N(A_1, B_2)$$
$$(2)\ N(A_2) = 2N(A_2, A_2) + N(A_2, A_1) + N(A_2, B_1) + N(A_2, B_2)$$
$$(3)\ N(B_1) = 2N(B_1, B_1) + N(B_1, A_1) + N(B_1, A_2) + N(B_1, B_2)$$
$$(4)\ N(B_2) = 2N(B_2, B_2) + N(B_2, A_1) + N(B_2, A_2) + N(B_2, B_1)$$
$$(5)\ N(A_1) = N(A_2)$$
$$(6)\ N(B_1) = N(B_2).$$

Equations (5) and (6) come from the fact that ∂D pierces $\partial A \cup \partial B$ at points of intersection. Combining equations (1), (2), (5) and (3), (4), (6) we have upon simplification (note that $N(Z, Y) = N(Y, Z)$):

$$(7)\ 2N(A_1, A_1) + N(A_1, B_1) + N(A_1, B_2) = 2N(A_2, A_2) + N(A_2, B_1) + N(A_2, B_2)$$
$$(8)\ 2N(B_1, B_1) + N(B_1, A_1) + N(B_1, A_2) = 2N(B_2, B_2) + N(B_2, A_1) + N(B_2, A_2).$$

Now since $N(B_1, B_1)$ is positive there is an arc u in $\partial D \cap X$ with both endpoints in B_1. Since u does not cut a disk off X it separates X into an annulus and a disk with two holes P. Suppose first that B_2 is contained in the annulus. Then $N(A_1, B_2) = N(A_2, B_2) = 0$. Also $N(B_2, B_2) = 0$ since an arc with both endpoints in B_2 lying in this annulus must cut a disk off X, a contradiction. Thus equation (8) becomes: $2N(B_1, B_1) + N(B_1, A_1) + N(B_1, A_2) = 0$ which implies that $N(B_1, B_1) = 0$ since all terms in the sum are nonnegative, a contradiction. Thus B_2 is contained in P. We may suppose A_2 is also in P since we can apply the homeomorphism V which causes a renaming of A_1 and A_2. Thus A_1 lies in the annulus and we have $N(A_1, A_2) = N(A_1, B_2) = N(A_1, A_1) = 0$. This allows us to solve for $N(A_1, B_1)$ in equation (7). Substituting this into equation (8) we have upon simplification: $N(B_2, B_2) = N(B_1, B_1) + N(A_2, A_2) + N(A_2, B_1)$. Thus $N(B_2, B_2)$ is positive and there is a component v of $\partial D \cap X$ with both endpoints in B_2, and since v lies in P and can not cut a disk off X it must separate P into two annuli. Thus v separates

A_2 and B_1 and we have $N(A_2, B_1) = 0$. Also $N(A_2, A_2) = 0$ because of the annulus. Therefore $N(B_1, B_1) = N(B_2, B_2)$. We may schematically draw X as in Figure 4 which indicates all possible remaining choices for components of $\partial D \cap X$.

By Lemma 2 we have that M is a subgroup of N. Thus by those isotopies which rotate components of ∂X, and the homeomorphisms in M, we may assume that $\partial D \cap X$ lies on ∂T as shown in Figures 4 and 5. All parallel arcs

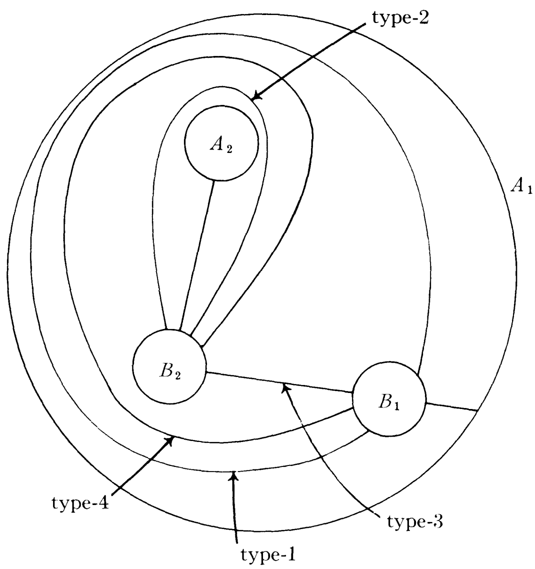

FIGURE 4

are considered to be "close" together. A type-1 arc is a component of $\partial D \cap X$ with both endpoints in B_1. A type-2 arc is one with both endpoints in B_2. The type-3 and type-4 arcs are the two types of non-isotopic components of $\partial D \cap X$ with one endpoint in B_1 and the other in B_2. The other two types of arcs are unnamed. We know that both type-1 and type-2 arcs exist.

Using the type-3 and type-4 arcs as a guide we can draw ∂G (G, the annulus for the annulus-homeomorphism g) so that $\partial G \cap (\partial D \cap X) = \emptyset$. See Figure 6. By drawing ∂G we mean to apply the appropriate isotopies.

Let B^* denote the annulus in ∂T bounded by B_1 and B_2. Now via isotopies and the homeomorphism $b|B^*$, both of which are to be the identity on (∂T − int B^*), we can insist that all arcs of $\partial D \cap B^*$ hit each g_i at most twice and if twice then with algebraic intersection zero [**2**, p. 533]. The g_i are the components of ∂G as indicated in Figure 6. Since B^* is an annulus, if g_i is hit twice

https://doi.org/10.4153/CJM-1975-087-9 Published online by Cambridge University Press

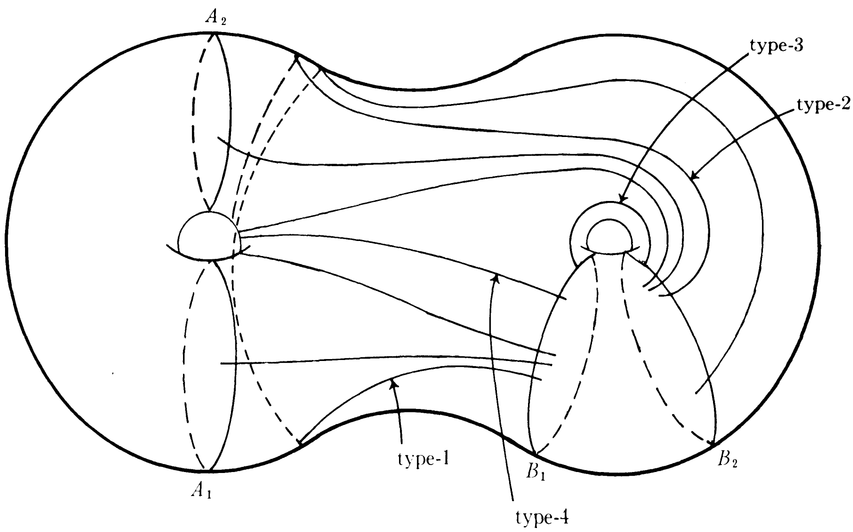

FIGURE 5

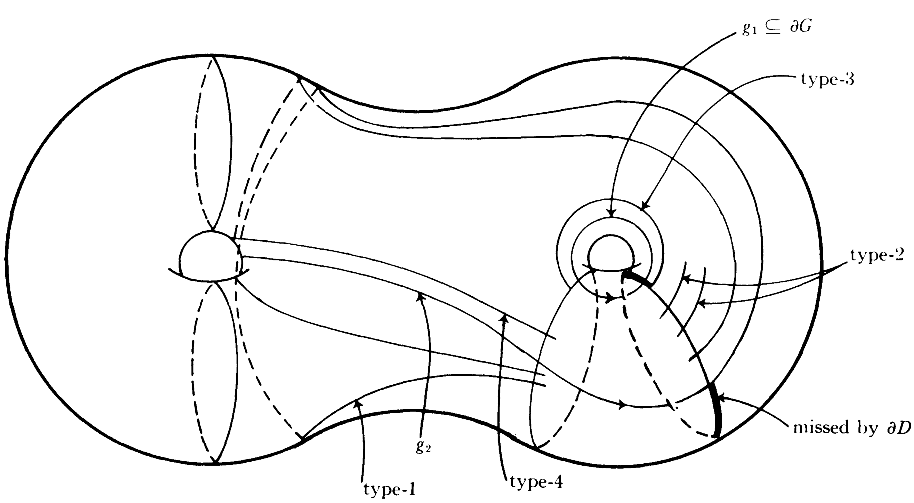

FIGURE 6

with algebraic intersection zero there is an arc in ∂D which, along with an arc in g_i, bounds a disk in the disk $(B^* - g_i)$. An isotopy then eliminates their intersection. Thus each component of $\partial D \cap B^*$ hits each g_i at most once.

Case 1. Some component of $\partial D \cap B^*$ hits both g_i. Let R, S_1, S_2, and W denote the number of arcs in $\partial D \cap B^*$ hitting both g_1 and g_2, only g_1, only g_2, and neither g_1 nor g_2, respectively. Since R is positive and one component of $(B_2 - \partial G)$ misses ∂D, either $S_1 = 0$ or $S_2 = 0$. See Figure 7

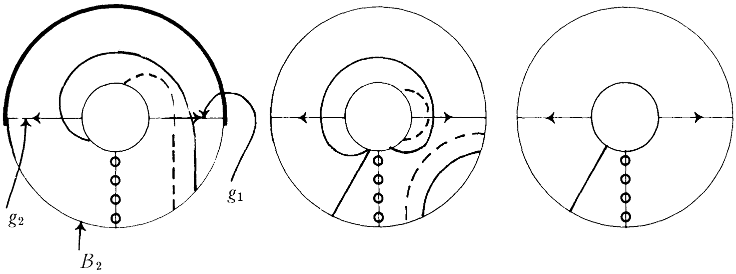

FIGURE 7a FIGURE 7b FIGURE 7c

where the heavy arc in B_2 indicates the component of $(B_2 - \partial G)$ missed by ∂D. Now apply the homeomorphism g if $S_2 = 0$ and g^{-1} if $S_1 = 0$. Now in T (that is, in ∂T) pull those arcs of $g(\partial D) \cap B^*$ off B^* which cut disks off B^*. For $Y = R, S_1, S_2, W$ let Y' be the number of arcs of $g(D) \cap B^*$ which arose from the arcs counted by Y (count Y' after the isotopies have been applied). See Figure 7b, 7c. Thus $R' = R$, $S_1' = S_2' = 0$, and $W' = W$. Thus $g(D)$ will hit $B^* \cup A$ and hence $B \cup A$ in fewer components than did D unless $R' + S_1' + S_2' + W'$ is greater than or equal to $R + S_1 + S_2 + W$. This must be the case because of our original choice of D. This implies that $S_1 = S_2 = 0$.

Case 2. No arc of $\partial D \cap B^*$ hits both g_i. Then all arcs are as in Figure 8a. Define S_1, S_2, W, S_1', S_2', and W' as in Case 1. Again we get a disk which hits $A \cup B$ in fewer components than does D unless $S_1' + S_2' + W'$ is greater than or equal to $S_1 + S_2 + W$. But applying g (Figure 8b) and an isotopy gives that $S_1' = S_2' = 0$ and $W' = W$. Again we have $S_1 = S_2 = 0$.

From Cases 1 and 2 we see that of the two components of $(B^* - \partial G)$ there is only one which contains any endpoints of the arcs $\partial D \cap B^*$, that is, of

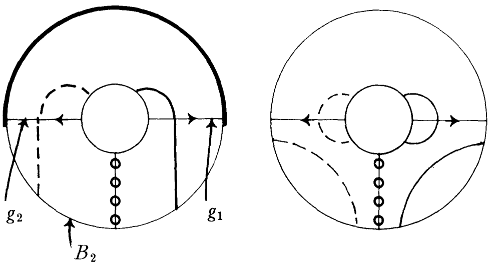

FIGURE 8a FIGURE 8b

https://doi.org/10.4153/CJM-1975-087-9 Published online by Cambridge University Press

$\partial D \cap X$. But this is impossible since we know type-1 and type-2 arcs exist and their endpoints are in different components of $(B^* - \partial G)$. This contradiction means that we can take D off B by an element of N. This is contrary to the choice of D and the theorem is established.

THEOREM 2. *N is the group of all orientation-preserving homeomorphisms of T into itself.*

Proof. Let f be an orientation-preserving homeomorphism of T onto itself. By Theorem 1, since fA is non-separating there is an element f_1 of N such that $f_1f(A) = B$. Also by Theorem 1 there is an f_2 in N such that $f_2(B) = A$. Thus $f_2f_1f(A) = A$. Now $f_2f_1f(B)$ is a nonseparating disk which misses A. Thus there is an f_3 in N such that $f_3f_2f_1f(B) = B$ and such that $f_3(A) = A$; this follows from an examination of the proof of Theorem 1 which shows that all homeomorphisms and isotopies used there could have been taken to be the identity on A if D, the disk in question, missed A.

If $f_3f_2f_1f|\partial B$ is not orientation-preserving let f_4 be R composed with a homeomorphism isotopic to the identity so that $f_4f_3f_2f_1f|\partial B = 1$. If it is orientation-preserving then f_4 is just the second mentioned homeomorphism. Let f_5 be a homeomorphism isotopic to the identity composed with V, if necessary, so that $f_5f_4f_3f_2f_1f|\partial A = 1$. Let $f_6 = f_5f_4f_3f_2f_1f$. We now have $f_6|(\partial A \cup \partial B) = 1$, $f_6(A) = A$, and $f_6(B) = B$. It suffices to show that f_6 is in N. The appropriate isotopy allows us to assume that $f_6|(A \cup B) = 1$. Since f_6 is orientation-preserving we have that $f_6|\partial T$ is also orientation-preserving. This and the fact that $f_6|(\partial A \cup \partial B) = 1$ implies f_6 does not interchange the sides $\partial A \cup \partial B$ in ∂T. An isotopy then allows us to assume that f_6 is the identity in a regular neighborhood of $A \cup B$. Thus f_6 is the identity on ∂X where X is ∂T cut at $(\partial A \cup \partial B)$. Apply f_7 in M so that $f_7f_6|T = 1$ [**2**, p. 537, statement $\psi(u)$]. Thus $f_7f_6|\partial T = 1$ and it is well-known that such a homeomorphism is isotopic to the identity. This proves that f_7f_6 (and hence f_6) is in N which is what we sought to prove.

REFERENCES

1. J. F. P. Hudson, *Piecewise linear topology* (W. A. Benjamin, Inc., New York, 1969).

2. W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. *76* (1962), 531–540.

3. ——— *Homeomorphisms of non-orientable two-manifolds*, Proc. Cambridge Philos. Soc. *59* (1963), 307–317.

4. ——— *A finite set of generators for the homeotopy group of a 2-manifold*, Proc. Cambridge Philos. Soc. *60* (1964), 769–778.

5. Donald Myers, *Homeomorphisms on a certain cube with holes*, Trans. Amer. Math. Soc. *191* (1974), 289–299.

11837 Diane Drive,
Wauwatosa, Wisconsin