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EXISTENCE OF SOLUTIONS FOR A VECTOR SADDLE POINT PROBLEM

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We establish an existence theorem for weak saddle points of a vector valued function by making use of a vector variational inequality and convex functions.

1. INTRODUCTION

(R^m, R^m_+) is an ordered Hilbert Space with an ordering $\leqslant$ on R^m defined by the convex cone R^m_+,

$$\forall x, y \in R^m,\ y \leqslant x \Leftrightarrow x - y \in R^m_+.$$

If $\operatorname{int} R^m_+$ denotes the topological interior of the cone R^m_+, then the weak ordering $\nless R^m$ is defined by

$$\forall y, x \in R^m,\ y \nless x \Leftrightarrow x - y \notin \operatorname{int}\ R^m_+.$$

Let K and C be nonempty subsets of R^n and R^p respectively. Given a vector valued function $L : K \times C \longrightarrow R^m$ then the Vector Saddle Point Problem (in short, VSPP) is to find $x^* \in K$, $y^* \in C$ such that

$$L(x^*, y^*) - L(x, y^*) \notin \operatorname{int}\ R^m_+ \tag{1}$$

$$L(x^*, y) - L(x^*, y^*) \notin \operatorname{int}\ R^m_+, \tag{2}$$

for all $x \in K$ and $y \in C$.

The solution (x^*, y^*) of VSPP is called a weak R^m_+-saddle point of the function L.

DEFINITION 1.1: A function $f : K \longrightarrow R^m$, where K is convex set, is called R^m_+-convex if for each $x, y \in K$ and $\lambda \in [0, 1]$,

$$\lambda f(x) + (1 - \lambda) f(y) - f\big(y + \lambda(x - y)\big) \in R^m_+. \tag{3}$$

DEFINITION 1.2: A function f is said to be R^m_+-concave, if $-f$ is a R^m_+-convex.

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DEFINITION 1.3: (Tanaka [6].) A vector valued function $f : K \longrightarrow R^m$, where $K \subset R^n$ is a convex set, is called natural quasi R^m_+-convex on K if

$$f\big(\lambda x + (1-\lambda)y\big) \in \mathrm{Co}\big\{f(x), f(y)\big\} - R^m_+,$$

for every $x, y \in K$ and $\lambda \in [0,1]$, where $\mathrm{Co}\,A$ denotes the convex hull of the set A.

For an example of a natural quasi R^m_+-convex function, see Tanaka [6].

DEFINITION 1.4: A multifunction T from R^n into itself is called upper semicontinous if $\{x_n\}$ converging to x, and $\{y_n\}$, with $y_n \in T(x_n)$, converging to y, implies $y \in T(x)$.

In this paper, we establish an existence theorem for solutions for VSPP by making use of vector variational inequalities and convex functions.

The following theorem (KKM-Fan theorem, see Fan [3]) is important for the proof of our main result.

THEOREM 1.1. *Let E be a subset of topological vector space X. For each $x \in E$, let a closed set $F(x)$ in X be given such that $F(x)$ is compact for at least one $x \in E$. If the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of E is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$, then $\bigcap_{x \in E} F(x) \neq \emptyset$.*

2. EXISTENCE OF SOLUTIONS

First we prove the following Theorem.

THEOREM 2.1. *Let the sets K and C be convex and let the function $L : K \times C \longrightarrow R^m$ be R^m_+-convex in the first argument and R^m_+-concave in the second argument. Then any local weak R^m_+-saddle point of L is a global weak R^m_+-saddle point.*

PROOF: Let (x^*, y^*) be a local weak R^m_+-saddle point of $L(x,y)$ over $K \times C$. Then, for some neighbourhood V of (x^*, y^*),

$$\begin{aligned} L(x^*, y^*) - L(x, y^*) &\notin \text{int } R^m_+, \\ L(x^*, y) - L(x^*, y^*) &\notin \text{int } R^m_+, \quad \forall (x,y) \in V \cap (K \times C). \end{aligned}$$

Suppose, for contradiction, that (x^*, y^*) is not a global weak R^m_+-saddle point. Then, there is some $(x_1, y_1) \in K \times C$ for which

$$\begin{aligned} L(x^*, y^*) - L(x_1, y^*) &\in \text{int } R^m_+, \\ L(x^*, y_1) - L(x^*, y^*) &\in \text{int } R^m_+. \end{aligned}$$

Since the sets K and C are convex, for $0 < \alpha < 1$, $x^* + \alpha(x_1 - x^*) \in K$ and $y^* + \alpha(y_1 - y^*) \in C$. Since L is R^m_+-convex in the first argument and R^m_+-concave in the second argument,

$$\begin{aligned} L\big(x^* + \alpha(x_1 - x^*), y^*\big) - L(x^*, y^*) &\in -R^m_+ - \alpha\big(L(x^*, y^*) - L(x_1, y^*)\big) \\ &\in -R^m_+ - \text{int } R^m_+ \\ &\subseteq -\text{int } R^m_+ \end{aligned}$$

and

$$\begin{aligned} L\big(x^*, y^* + \alpha(y_1 - y^*)\big) - L(x^*, y^*) &\in R^m_+ - \alpha\big(L(x^*, y^*) - L(x_*, y_1)\big) \\ &\in R^m_+ + \text{int } R^m_+ \\ &\subseteq \text{int } R^m_+ \end{aligned}$$

which contradicts the local weak R^m_+-saddle point, since $\big(x^* + \alpha(x_1 - x^*), y^* + \alpha(y_1 - y^*)\big) \in V$ for sufficiently small positive α. ∎

Next, we establish the equivalence between the VSPP and the vector variational inequality problem (in short, VVIP) of finding $x^* \in K$, $y^* \in T(x^*)$ such that

$$\big\langle L'(x^*, y^*), x - x^* \big\rangle \notin -\text{int } R^m_+, \quad \forall x \in K, \tag{4}$$

where $T : K \longrightarrow C$ is a multifunction defined by

$$T(x^*) := \big\{ y \in C : L(x^*, z) - L(x^*, y) \notin \text{int } R^m_+, \quad \forall z \in C \big\}, \tag{5}$$

and $L'(x^*, y^*)$ denotes the Fréchet derivative of L at x^*.

Let $W := R^m \setminus \big(-\text{int } R^m_+\big)$.

Theorem 2.2. *Let the set K be convex and let each component L_i of the vector valued function L be R^m_+-convex and Fréchet differentiable in the first argument. Then the VSPP and VVIP have the same solution set.*

Proof: Let (x^*, y^*) be a solution of VSPP. If $x \in K$ and $0 \leqslant \alpha \leqslant 1$, then $x^* + \alpha(x - x^*) \in K$. Hence (1) becomes

$$\alpha^{-1}\big[L\big(x^* + \alpha(x - x^*), y^*\big) - L(x^*, y^*)\big] \in W, \quad \forall \alpha \in (0, 1].$$

Since W is closed and L is Fréchet differentiable in the first argument, it follows that

$$\big\langle L'(x^*, y^*), x - x^* \big\rangle \notin -\text{int } R^m_+,$$

and $y^* \in T(x^*)$ follows from (2).

Conversely, let (x^*, y^*) satisfy (1) and (2). Since L is R^m_+-convex then we have, for each $x \in K$,

$$L(x, y^*) - L(x^*, y^*) - \langle L'(x^*, y^*), x - x^* \rangle \in R^m,$$

and hence, by Chen [**1**, Lemma 2.1 (iv)] we have

$$L(x^*, y^*) - L(x, y^*) \notin R^m_+.$$

(2) follows from (5). ∎

Finally, we prove the main result of this paper.

THEOREM 2.3. *Let K be a nonempty closed convex set in R^n; let C be a nonempty compact set in R^p; let $L : K \times C \longrightarrow R^m$ be a continuously differentiable function which is R^m_+-convex in the first argument; let L' be a continuous function in both x and y; let $T : K \longrightarrow C$ be the multifunction defined by (5). Suppose that, for each fixed $(x, y) \in K \times C$, the function $\langle L'(x, y), z - x \rangle$ is a natural quasi R^m_+-convex function in $z \in K$. If there exists a nonempty compact subset B of R^n and $x_0 \in B \cap K$ such that for any $x \in K \setminus B$, there exists $y \in T(x)$ such that*

$$\langle L'(x, y), x_0 - x \rangle \in -\operatorname{int} R^m_+,$$

then VSPP has a global weak R^m_+-saddle point.

PROOF: In order to prove the theorem, it is sufficient to show that the VVIP has a solution $x^* \in K$, $y^* \in T(x^*)$. Define a multifunction $F : K \longrightarrow K$ by

$$F(z) = \{x \in K : \exists y \in T(x) \text{ such that } \langle L'(x, y), x - z \rangle \notin -\operatorname{int} R^m_+\}, \; z \in K.$$

We claim that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_m\}$ of K is contained in the corresponding union $\bigcup_{i=1}^{m} F(x_i)$, that is, $\text{Co}\{x_1, x_2, \ldots, x_m\} \subset \bigcup_{i=1}^{m} F(x_i)$. Indeed, let $\alpha_i \geqslant 0$, $1 \leqslant i \leqslant m$, with $\sum_{i=1}^{m} \alpha_i = 1$.

Suppose that $x = \sum_{i=1}^{m} \alpha_i x_i \notin \bigcup_{i=1}^{m} F(x_i)$. Then for any $y \in T(x)$,

$$\langle L'(x, y), x_i - x \rangle \in -\operatorname{int} R^m_+, \; \forall i.$$

Let

$$V := \left\{z \in K : \langle L'(x, y), z - x \rangle \in -\operatorname{int}\ R^m_+ \text{ for any } y \in T(x)\right\}$$

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for fixed $x \in K$. Let $z_1, z_2 \in V$ and $\alpha \in [0,1]$. Then we have

$$\langle L'(x,y), z_i - x\rangle \in -\operatorname{int}\ R^m_+,\ i = 1,2. \tag{6}$$

Since $\langle L'(x,y), z - x\rangle$ is natural quasi R^m_+-convex in $z \in K$ then there exists $\beta \in [0,1]$, such that

$$\langle L'(x,y), \alpha z_1 + (1-\alpha)z_2 - x\rangle \in \beta\langle L'(x,y), z_1 - x\rangle + (1-\beta)\langle L'(x,y), z_2 - x\rangle - R^m_+.$$

Using (6) we have

$$\langle L'(x,y), \alpha z_1 + (1-\alpha)z_2 - x\rangle \in -\operatorname{int} R^m_+ - \operatorname{int} R^m_+ - R^m_+ \subseteq -\operatorname{int} R^m_+.$$

Hence V is a convex subset of K for each fixed $x \in K$, and hence we have

$$\left\langle L'\left(\sum_{i=1}^m \alpha_i x_i, y\right), \sum_{i=1}^m \alpha_i x_i - \sum_{i=1}^m \alpha_i x_i\right\rangle \in -\operatorname{int} R^m_+.$$

Thus, $0 = -0 \in \operatorname{int}\ R^m_+$, which is a contradiction and our claim is then verified. Now, by the continuity of L and the closedness of $R^m \setminus (\operatorname{int} R^m_+)$, the set $T(x)$ is closed for each $x \in K$. Since $T(x)$ is a subset of compact set C, $T(x)$ turns out to be compact for each fixed $x \in K$. Let $\{x_n\}$ be a sequence in K such that $x_n \longrightarrow x \in K$ and let $\{y_n\}$ be a sequence such that $y_n \in T(x_n)$. Since $y_n \in T(x_n)$,

$$L(x_n, z) - L(x_n, y_n) \in R^m \setminus (\operatorname{int}\ R^m_+). \tag{7}$$

Since $\{y_n\} \subset C$ and C is compact, without loss of generality, we can assume that there exists $y \in C$ such that $y_n \longrightarrow y$. Now the continuity of L and the closedness of W gives that

$$L(x,z) - L(x,y) \in R^m \setminus (\operatorname{int} R^m_+),$$

which implies that $y \in T(x)$. Thus the multifunction T is upper semicontinuous.

Next, we claim that $F(z)$ is closed for each $z \in K$. Indeed, let $\{x_n\} \subset F(z)$ such that $x_n \longrightarrow x \in K$. Since $x_n \in F(z)$ for all n, there exists $y_n \in T(x_n)$ such that

$$\langle L'(x_n, y_n), z - x_n\rangle \in W,\ \forall z \in K.$$

As $\{y_n\} \subset C$, without loss of generality, we can assume that there exists $y \in C$ such that $y_n \longrightarrow y$.

Since L' is continuous, T is upper semicontinous and W is closed, we have

$$\langle L'(x_n, y_n), z - x_n\rangle \longrightarrow \langle L'(x,y), z - x\rangle \in W$$

or

$$\langle L'(x,y), (z - x)\rangle \notin -\operatorname{int} R^m_+.$$

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Hence $x \in F(z)$.

Finally, we claim that for $x_0 \in B \cap K$, $F(x_0)$ is compact. Indeed, suppose that there exists $\overline{x} \in F(x_0)$ such that $\overline{x} \notin B$. Since $\overline{x} \in F(x_0)$, there exists $\overline{y} \in T(\overline{x})$ such that

$$\langle L'(\overline{x},\overline{y}), x_0 - \overline{x} \rangle \notin -\operatorname{int} R^m_+. \tag{8}$$

Since $\overline{x} \notin B$, by hypothesis, there exists $\overline{y} \in T(\overline{x})$ such that

$$\langle L'(\overline{x},\overline{y}), x_0 - \overline{x} \rangle \in -\operatorname{int} R^m_+,$$

which contradicts (8). Hence $F(x_0) \subset B$. Since B is compact and $F(x_0)$ is closed, $F(x_0)$ is compact. By Theorem 1.1, it follows that $\bigcap_{z \in K} F(z) \neq \emptyset$. Thus, there exists $x^* \in K, y^* \in T(x^*)$ such that

$$\langle L'(x^*, y^*), z - x^* \rangle \notin -\operatorname{int} R^m_+, \ \forall z \in K. \qquad \square$$

REMARK.

(i) If $L(x,y)$ depends upon x only, then VVIP reduces to the problem considered by Chen and Craven [2]. See also Kazmi [4].

(ii) If $L(x,y)$ is a scalar valued function, VSPP reduces to the scalar saddle point problem studied by Parida and Sen [5] by making use of the Kakutani fixed point theorem.

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