Can. J. Math., Vol. XXXI, No. 5, 1979, pp. 986–1004

COMBINATORIAL ORIENTED MAPS

W. T. TUTTE

1. Introduction. An orientable map is often presented as a realization of a finite connected graph G in an orientable surface so that the complementary domains of G, the "faces" of the map are topological open discs. This is not the definition to be used in the paper. But let us contemplate it for a while.

On each edge of G we can recognize two opposite directed edges, or "darts". Let θ be the permutation of the dart-set S that interchanges each dart with its opposite. The darts radiating from a vertex v occur in a definite cyclic order, fixed by a chosen positive sense of rotation on the surface. The cyclic orders at the various vertices are the cycles of a permutation P of S. The choice of P rather than P^{-1}, which corresponds to the other sense of rotation, makes the map "oriented".

Each face imposes a cyclic order on the darts having it to their right, and the cyclic orders around the various faces are the cycles of another permutation of S. But this permutation is found to be $P\theta$, the effect of applying first θ and then P.

Most introductory text-books on topology describe a construction for building 2-dimensional manifolds from polygons. It is based on work of H. R. Brahana [**1**]. Effectively it describes an arbitrary oriented map by giving $P\theta$ and θ. Edmonds' Theorem [**4**] does it by giving P and θ. It seems that the combinatorial properties of an oriented map can be abstracted as the properties of a pair (P, θ) of permutations of a set S of undefined elements called "darts", where the cycles of θ are all of length 2. Several writers have presented theories of maps in terms of these permutations (see [**2**], [**6**] and [**9**]). From now on we refer to oriented maps simply as "maps".

The present paper gives a theory of permutation-pairs (P, θ). In general such a pair is called a *premap*. It becomes a *map* if it is connected, that is if P and θ generate a group of permutations that is transitive on S. A map of this kind is called a *constellation* by A. Jacques [**6**]. However the theorems to be proved are those we are used to ascribing to structures called "maps", so we keep that word for our permutation-pairs. Special attention is given to *planar* maps, defined as maps of Euler characteristic 2.

The paper should be called "expository", neither its facts about maps nor its use of permutations being new. However it seems to the author that there is still a need for a reasonably concise theory of planar maps deduced rigorously from purely combinatorial axioms, free from any intrusion of topology,

Received March 29, 1978.

and going at least as far as the combinatorial equivalent of Jordan's Theorem. The paper tries to meet this need. It adopts the following philosophy. There is a theory of maps as permutation-pairs, there is a theory of topological realizations of such maps, and there is a theory of relationships between maps and graphs. All three theories are important, but the paper is concerned only with the first. The paper does indeed point out some graphs associated with a map, but these play only a minor role in the exposition.

2. Premaps and maps. Let S be a finite non-null set, of even cardinality, of elements called *darts*.

A *premap* on S is an ordered pair (P, θ) of permutations of S such that each cycle of θ is of length 2.

The cycles of P and θ are the *vertices* and *edges* of M respectively. A vertex and an edge are *incident* if they have a dart in common, and *doubly incident* if they have two. With this definition of incidence the vertices and edges of M are the vertices and edges respectively of a graph $G(M)$, the *graph of* M. The doubly incident edges are the loops of $G(M)$.

P and θ generate a permutation group $Y(M)$ acting on S. If $Y(M)$ is transitive on S we say that M is *connected*, or that M is a *map*.

In general $Y(M)$ partitions S into disjoint non-null subsets, the *equivalence classes* of M, with the following property: a dart D can be transformed into a dart E by a member of $Y(M)$ if and only if D and E belong to the same equivalence class. We denote the number of equivalence classes of M by $p(M)$. Thus $p(M) \geqq 1$, and $p(M) = 1$ if and only if M is a map.

If a dart D belongs to an equivalence class Z of M, then Z contains all the darts of the edge and vertex to which D belongs.

A member of $Y(M)$ is a product of permutations, each of which is P or θ. Hence two darts D and E belong to the same equivalence class if and only if there is an alternating sequence

$$U = (K_1, K_2, \ldots, K_m)$$

of edges and vertices of M such that D is in K_1, E is in K_m, and any two consecutive members of the sequence have a common dart. We call U a *connecting sequence* of D and E in M.

We can state the preceding observation as follows.

2.1. *Two darts D and E belong to the same equivalence class of M if and only if they belong to the same component of $G(M)$.*

3. Duality. Let $M = (P, \theta)$ be a premap on S. Then we denote the premap $(P\theta, \theta)$ on S by M^*, and call it the *dual* premap of M.

Since θ^2 is the identity permutation we find that M is the dual premap of M^*.

(3.1) $(M^*)^* = M$.

Now $P\theta$ and θ generate the same permutation group $Y(M)$ as do P and θ. Hence we have

(3.2) $Y(M^*) = Y(M)$, *and the equivalence classes of* M^* *are the equivalence classes of* M. *In particular* M^* *is a map if and only if* M *is a map.*

The vertices of M^* are called the *faces* of M, and correspondingly the vertices of M are the faces of M^*.

We denote the numbers of vertices, edges and faces of a premap M by $\alpha_0(M)$, $\alpha_1(M)$ and $\alpha_2(M)$ respectively. The *Euler characteristic* of M is an integer $N(M)$ defined as follows.

(3.3) $N(M) = \alpha_0(M) - \alpha_1(M) + \alpha_2(M)$.

Thus

(3.4) $N(M^*) = N(M)$.

Consider the case in which S has exactly two members. Then θ is uniquely determined, and we can write the two darts as D and θD. There are just two possibilities for P; it is either θ or the identical permutation I. The premaps (I, θ) and (θ, θ) are clearly maps. We call them the *link-map* and *loop-map* respectively on S. Each is the dual of the other. Since I has two cycles and θ only one we can assert the following proposition.

(3.5) *If* M *is a link-map or loop-map, then* $N(M) = 2$.

4. Subpremaps and submaps. Let $M = (P, \theta)$ be a premap on a set S. Let T be any non-null subset of S which with any dart D contains also θD. Let θ' be the restriction of θ to T. Let P' be the permutation of T defined as follows: if D is any dart of T then $P'D$ is the first member of the sequence

$$(PD, P^2D, P^3D, \ldots)$$

that is in T. We write M' for the premap (P', θ') on T.

We call M' the *subpremap of* M *on* T. If it is a map we call it also the *submap of* M *on* T.

If a vertex v of M is also a vertex of M', that is contains only darts of T, we call it an *inner* vertex of M'. Any other vertex of M that includes a dart of T is a *vertex of attachment* of M'.

Let v be a vertex of attachment of M'. Then there is a unique corresponding vertex v' of M'. It is obtained from v by striking out the darts not in T and keeping the cyclic order of those that remain. We call v' the *outer vertex* of M' in v. Since v must have at least one dart that is not in T we can form another cycle v'' by striking out the darts in T and keeping the cyclic order of the others. We call v'' the *complementary cycle* of v' in v.

We call M' a *proper* subpremap or submap of M if it is distinct from M, that is if T is a proper subset of S. In that case let M'' be the proper subpremap of

M on the non-null set $S - T$. We call M'' the *complementary subpremap* to M' in M. Evidently the vertices of attachment of M'' are those of M', and the outer vertices of M'' are the complementary cycles of the outer vertices of M'. Moreover M' is the complementary subpremap to M'' in M.

A *component* of M is a subpremap of M on an equivalence class of M. Thus each dart of S belongs to exactly one component of M.

4.1. *Let C be the component of M on an equivalence class Z. Then C is a map. Moreover the vertices, edges and faces of C are those vertices, edges and faces of M respectively that meet Z.*

Proof. Put $C = (P', \theta')$. By the definition of Z the cycles of P' and θ' are those cycles of P and θ respectively that meet Z. Hence the cycles of $P'\theta'$ are those cycles of $P\theta$ that meet Z. It remains only to verify that C is a map.

We can now assert that any connecting sequence in M of two darts D and E belonging to Z is a connecting sequence in C of D and E. Hence Z is the only equivalence class of C, and C is a map.

4.2. *The components of M^* are the dual maps of the components of M.*

Proof. Let Z be any equivalence class of M, and therefore of M^* by 3.2. Let C and C_1 be the components of M and M^* respectively on Z. Applying 4.1 to M and M^* we see that the vertices, edges and faces of C are the faces, edges and vertices respectively of C_1. Hence $C_1 = C^*$.

4.3. *$N(M)$ is the sum of the Euler characteristics of the components of M.*

Proof. See 4.1.

4.4. *If M' is a submap of M it is a submap of some component of M.*

Proof. Let D and E be any two darts of M'. They have a connecting sequence in M'. Replacing each outer vertex of M' in this sequence by the corresponding vertex of attachment we obtain a connecting sequence of D and E in M. The dart-set T of M' is thus contained in some equivalence class Z of M. By 4.1 M' is a submap of the component of M on Z.

4.5. *Let M' be a submap of M. Then it is a component of M if and only if it has no vertex of attachment.*

Proof. If M' is a component of M it has no vertex of attachment by 4.1.

Conversely suppose M' to have no vertex of attachment. Then M' is a submap of a component C of M, by 4.4. Let T be the set of darts of M', and U the set of all other darts of C.

Assume U non-null. Choose darts D in T and E in U. In any connecting sequence of D and E in C we can find a vertex v of C having a dart on T and a dart in U. But then v is a vertex of attachment of M' by 4.1, contrary to supposition.

We conclude that U is null. Hence M' and C are identical.

4.6. *Let M be a map. Then a subpremap M' of M is uniquely determined when its outer vertices are given.*

Proof. Let Q' be the given set of outer vertices, and Q'' the corresponding set of complementary cycles.

Suppose first that Q' is null. Any component N of M' is a submap of M with no vertices of attachment, and therefore a component of M, by 4.5. But then $M' = M$ since M is a map. Thus M' is uniquely determined.

Suppose next that Q' is non-null. Let M_1 be the premap obtained from M by replacing the vertices of attachment of M' by the members of Q' and Q'', so that each vertex of attachment splits into two disjoint dart-cycles, one made up of darts in M' and the other of darts in the complement of M' in M.

Let N be a component of M_1 having no vertex in Q' or Q''. Then N is a submap of M with no vertex of attachment. Hence $N = M$, by 4.5. But then $M_1 = M$ and we have the contradiction that Q' is null. We conclude that in fact each component of M_1 has a vertex in Q' or Q''.

Any component C' of M' is a submap of M_1, with no vertex of attachment in M_1. It is thus a component of M_1, by 4.5. We conclude that M' is uniquely determined; its components are those components of M_1 that include vertices of Q'.

5. Angles. Let $M = (P, \theta)$ be a premap on a set S. An *angle* of M is an ordered pair

(1) $\alpha = (D, PD)$,

where D is in S. We say that α is *at* the vertex v of M that contains D and PD.

Given α we can define a corresponding angle

(2) $\alpha^* = (\theta D, PD)$

of M^*. We call it the *dual* angle of α, and we note that α is the dual angle of α^*.

The angle α^* is at some vertex F of M^*. We say that F is the face of M *occupying* the angle α. Likewise v is the face of M^* occupying α^*.

It is convenient to set down here a notation for certain linear sequences of darts. Thus if α is given by (1) we write $[\alpha, \alpha]$ for the linear sequence

$$(PD, P^2D, \ldots, P^kD, D)$$

whose darts are those of v.

Let

(3) $\beta = (E, PE)$

be another angle at v in M. Then we write $[\alpha, \beta]$ for the linear sequence

$$(PD, P^2D, \ldots, E),$$

the subsequence of $[\alpha, \alpha]$ extending from PD to E. Likewise $[\beta, \alpha]$ is a linear sequence extending from PE to D. Each dart of v belongs to exactly one of $[\alpha, \beta]$ and $[\beta, \alpha]$.

Given a linear sequence L of darts we can *close* it, that is convert it into a cyclic sequence, by postulating that the first dart is the immediate successor of the last. We write the resulting cycle, the *closure* of L, as $K(L)$. Thus $K([\alpha, \alpha])$ is the vertex v.

Given two linear sequences L_1 and L_2 of darts, with no common member, we can combine them into a single linear sequence L_1L_2 by taking first the terms of L_1 in proper order and then those of L_2. Products of three or more linear sequences, no two with a common dart, can be defined analogously.

6. Splitting a vertex. In the course of Brahana's classification of surfaces some important combinatorial theorems are proved. For example the Euler characteristic of an oriented map is an even number, possibly negative, not exceeding 2. Moreover any two oriented maps with the same Euler characteristic can be transformed into one another by subdivisions of edges, subdivisions of faces, and the inverses of these operations.

In this section we experiment with another elementary operation, splitting a vertex. We use it to establish the limitations on the Euler characteristic.

Let $M = (P, \theta)$ be a premap on a dart-set S. We define the *valency* of a vertex or face of M as the number of its darts.

Let v be a vertex of M whose valency is at least 2. Let $\alpha = (D, PD)$ and $\beta = (E, PE)$ be distinct angles of M at v. Let P_1 be the permutation of S derived from P by replacing the cycle v by the two cycles $K([\alpha, \beta])$ and $K([\beta, \alpha])$. We say that the premap $M_1 = (P_1, \theta)$ is obtained from M by *splitting* the vertex v *between* α and β. We proceed to investigate the structure of M_1, noting that

$$(4) \qquad \alpha_0(M_1) = \alpha_0(M) + 1.$$

It is not possible to split a vertex w of valency 1. For if D is the single dart of w there is only one angle at w, namely (D, D). We can split the vertex of a loop-map and so obtain a link-map. But it is clear that the link-maps are the only maps in which each vertex is monovalent, and therefore they are the only maps in which no vertex can be split.

We return to the premaps M and M_1. Let us set out in detail the difference between P_1 and P, and between $P_1\theta$ and $P\theta$. We note that $P_1D = PE$ and $P_1E = PD$. But $P_1X = PX$ if X is not D or E. Equivalently we can say that $P_1\theta(\theta D) = PE$, $P_1\theta(\theta E) = PD$, and $P_1\theta(Y) = P\theta(Y)$ if Y is not θD or θE.

Suppose that α and β are occupied by distinct faces A and B respectively of M. Consider the sequences $[\alpha^*, \alpha^*]$ and $[\beta^*, \beta^*]$ of M^*. We note that $P_1\theta$ transforms the last dart θD of $[\alpha^*, \alpha^*]$ into the first dart PE of $[\beta^*, \beta^*]$, and the last dart θE of $[\beta^*, \beta^*]$ into the first dart PD of $[\alpha^*, \alpha^*]$. Otherwise $P_1\theta$ has the same effect as $P\theta$. We infer that one cycle of $P_1\theta$ is $K([\alpha^*, \alpha^*][\beta^*, \beta^*])$.

The others are the cycles of $P\theta$ other than A and B. In this case we deduce from (4) that $N(M_1) = N(M)$.

Suppose next that α and β are occupied by the same face A of M. Consider the sequences $[\alpha^*, \beta^*]$ and $[\beta^*, \alpha^*]$ of M^*. We note that for each one $P_1\theta$ transforms the last dart into the first, θE into PD in the first case and θD into PE in the second. Hence $K([\alpha^*, \beta^*])$ and $K([\beta^*, \alpha^*])$ are cycles of $P_1\theta$. The others are the cycles of $P\theta$ other than A. It now follows from (4) that $N(M_1) = N(M) + 2$.

A component of M_1 having neither $K([\alpha, \beta])$ nor $K([\beta, \alpha])$ as a vertex is a component of M, by 4.5. We deduce that

(5) $\quad p(M_1) = p(M)$ or $p(M) + 1$.

Moreover if $p(M_1) = p(M) + 1$ then $K([\alpha, \beta])$ and $K([\beta, \alpha])$ must be vertices of distinct components of M_1.

When α and β are occupied by distinct faces each of the vertices $K([\alpha, \beta])$ and $K([\beta, \alpha])$ has a dart in common with the face $K([\alpha^*, \alpha^*][\beta^*, \beta^*])$, PD in the first case and PE in the second. Hence the two vertices belong to the same component of M_1, and therefore $p(M_1) = p(M)$.

If α and β are occupied by the same face A of M it may happen that $p(M_1) = p(M) + 1$. Then the vertex $K([\alpha, \beta])$ and the face $K([\alpha^*, \beta^*])$ of M_1 belong to the same component of M_1 since they have the dart PD in common. The vertex $K([\beta, \alpha])$ and the face $K([\beta^*, \alpha^*])$ then belong to some other component of M_1.

We summarize the foregoing results in the next two theorems.

6.1. *Let α and β be occupied by distinct faces, A and B respectively, of M. Then $K([\alpha^*, \alpha^*][\beta^*, \beta^*])$ is a face of M_1. The remaining faces of M_1 are the faces of M other than A and B. Moreover $N(M_1) = N(M)$ and $p(M_1) = p(M)$.*

6.2. *Let α and β be occupied by the same face A of M. Then $K([\alpha^*, \beta^*])$ and $K([\beta^*, \alpha^*])$ are faces of M_1. The remaining faces of M_1 are the faces of M other than A. Moreover $N(M_1) = N(M) + 2$, and $p(M_1)$ is $p(M)$ or $p(M) + 1$.*

If $p(M_1) = p(M) + 1$ then $K([\alpha, \beta])$ and $K([\beta, \alpha])$ are vertices of distinct components, X and Y respectively, of M_1. Moreover $K([\alpha^*, \beta^*])$ is a face of X, and $K([\beta^*, \alpha^*])$ is a face of Y.

It should be noted that X and Y in 6.2 are submaps of M, each having v as its sole vertex of attachment.

By 6.1 and 6.2 we have in all cases

(6) $\quad N(M) - 2p(M) = N(M_1) - 2p(M_1) - 2\sigma(M_1, M)$,

where $\sigma(M_1, M)$ is 0 or 1.

Having got M_1 let us continue splitting vertices until the process terminates, as it must by (4) since no premap can have more vertices than darts. At termination we have a premap M_q in which each vertex is monovalent, and in which therefore each component is a link-map. By 3.5 and 4.3 we have

(7) $N(M_q) - 2p(M_q) = 0.$

By repeated application of (6) we can now deduce the following theorem.

6.3. *For any premap M the integer $N(M) - 2p(M)$ is even and non-positive.*

6.4. *If M is a map then $N(M)$ is an even integer not exceeding* 2.

Proof. See 6.3.

6.5. *Let M' be a submap of a map M. Then $N(M') \geqq N(M)$.*

Proof. If possible choose M and M' so that the theorem fails, so that $\alpha_1(M)$ has the least value consistent with this condition, and so that $\alpha_0(M)$ has the greatest value consistent with these requirements.

M' must be a proper submap of M. It has a vertex v of attachment, and the valency of v is at least 2. We can form a premap M_1 by splitting v between two distinct angles α and β so that all the darts of M' at v are in $[\alpha, \beta]$. We thus arrange that M' is a submap of M_1.

If M_1 is a map then $N(M') \geqq N(M_1) \geqq N(M)$, by 6.1, 6.2 and the choice of M and M'.

In the remaining case M_1 has exactly two components X and Y, by 6.1 and 6.2. We may suppose M' to be a submap of X, by 4.4. Then $N(M') \geqq N(X)$, by the choice of M and M'. But

(8) $N(M) - 2 = N(X) - 2 + N(Y) - 2,$

by 4.3, 6.1 and 6.2. Hence $N(X) \geqq N(M)$ by 6.4, and therefore $N(M') \geqq N(M)$.

In each case the choice of M and M' is contradicted. The theorem follows.

For original presentations in the literature of the theory of this section reference may be made to [**3**], [**5**] and [**8**].

7. Planar maps. A map M is *planar* if $N(M) = 2$. Thus link-maps and loop-maps are planar by 3.5.

7.1. *The dual of a planar map is a planar map.*

Proof. Use 3.2 and 3.4.

7.2. *Any submap of a planar map is a planar map.*

Proof. Use 6.4 and 6.5.

When we apply 6.1 or 6.2 to planar maps we should bear in mind the two following amendments.

7.3. *If the premap M of* 6.1 *is a planar map, then M_1 is a planar map.*

https://doi.org/10.4153/CJM-1979-091-3 Published online by Cambridge University Press

Proof. Use 6.1.

7.4. *If the premap M of* 6.2 *is a planar map then the alternative* $p(M_1) = p(M) + 1 = 2$ *must hold. The components X and Y of M_1 are planar submaps of M.*

Proof. Use 6.4 and 7.2.

8. Cuttings and belts. The results of this section constitute a somewhat generalized combinatorial form of Jordan's Theorem. They describe a partition of a planar map into a "cutting" and a complementary submap.

Let M be any premap. If α and β are distinct angles of M at the same vertex v we shall refer to the sequence $[\alpha, \beta]$ as a *sector* of M at v.

A *cutting* of M is a map C whose edges are edges of M and whose vertices are vertices of M and closures of disjoint sectors of M. Vertices of C of these two kinds are its *inner* and *outer* vertices respectively. We call C a *proper* cutting of M if it does not include all the darts of M.

A cutting of M is not necessarily a submap of M; it may have more than one outer vertex in a single vertex of M.

If C is any cutting of M let us write $M'(C)$ for the subpremap of M on the dart-set of C. Evidently each inner vertex of C is an inner vertex of $M'(C)$, and each outer vertex of C is contained in some vertex, inner or outer, of $M'(C)$. If vertices are considered simply as sets of darts we can deduce that the outer vertices of $M'(C)$ are unions of outer vertices of C, and that some inner vertices of $M'(C)$ may also be such unions. The other inner vertices of $M'(C)$ are inner vertices of C.

8.1. *If C is a cutting of a premap M, then $M'(C)$ is a submap of M and* $N(C) \geqq N(M'(C))$. *Hence C is planar if M is a planar map, by* 6.4 *and* 7.2.

To prove this we observe that C can be derived from $M'(C)$ by splitting vertices. Since $p(C) = 1$ we must have $p(M'(C)) = 1$ and $N(C) \geqq N(M'(C))$, by 6.1 and 6.2.

8.2. *A cutting C of a map M is uniquely determined when its outer vertices are given.*

For if we are given M and the outer vertices of C we can find the outer vertices of $M'(C)$, and $M'(C)$ itself by 4.6. Thence we get the inner vertices of C.

Let k be any positive integer. A *belt* of order k in the premap M is a cyclic sequence

(9) $$B = (\alpha_1, F_1, \beta_1, \alpha_2, F_2, \beta_2, \ldots, \alpha_k, F_k, \beta_k),$$

where the F_j are k distinct faces of M, and the α_i and β_j are $2k$ distinct angles of M, and which satisfies the following conditions.

(i) F_j occupies α_j and β_j, for each suffix j.

(ii) The angles β_j and α_{j+1} are at a common vertex v_j, for each j.

https://doi.org/10.4153/CJM-1979-091-3 Published online by Cambridge University Press

Addition in the suffixes is modulo k; e.g. $k + 1$ is identified with 1.

In this definition we do not require the k vertices v_j to be all distinct. If they are we call B a *circular* belt.

The *left sectors* of B are the sequences $[\alpha_{j+1}, \beta_j]$ and the *right sectors* the sequences $[\beta_j, \alpha_{j+1}]$. The *left segments* are the sequences $[\alpha_j^*, \beta_j^*]$ of M^*, and the *right segments* the sequences $[\beta_j^*, \alpha_j^*]$.

The *left swirl* of B is the cycle

$$K([\alpha_1^*, \beta_1^*][\alpha_2^*, \beta_2^*] \ldots [\alpha_k^*, \beta_k^*]),$$

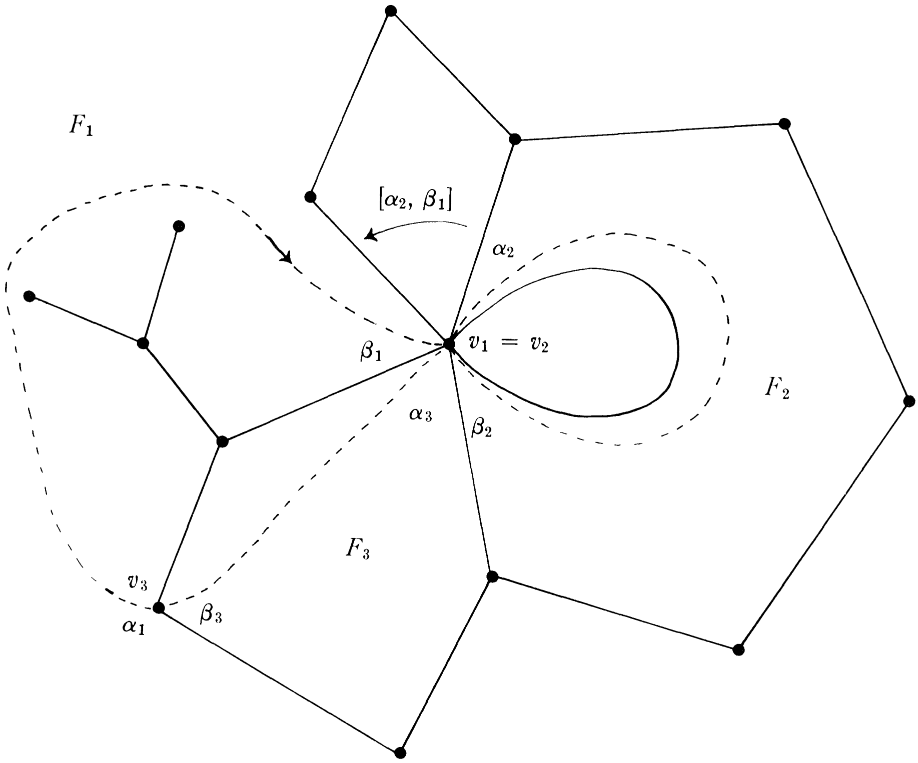
and the *right swirl* is the cycle

$$K([\beta_k^*, \alpha_k^*] \ldots [\beta_2^*, \alpha_2^*][\beta_1^*, \alpha_1^*]).$$

The belt B is *admissible* if its k left sectors are disjoint. Thus any circular belt is admissible. If B is admissible we define its *left sweep* as the cycle

$$K([\alpha_1, B_k][\alpha_k, \beta_{k-1}] \ldots [\alpha_3, \beta_2][\alpha_2, \beta_1]).$$

I had intended to eschew diagrams in this paper because of their topological flavour. (Shall we by Topology cast out Topology?) But I introduce one here in order to show how an admissible belt B looks in a planar map of the topological kind briefly described in Section 1.



In the Figure B is

$$(\alpha_1, F_1, \beta_1, \alpha_2, F_2, \beta_2, \alpha_3, F_3, \beta_3).$$

The broken line shows how we can split the map into two pieces by cutting round the belt. The assertion that we can indeed do this is a form of Jordan's Theorem.

We cut across the face F_1 from the angle α_1 into the angle β_1. We continue through the vertex v_1 into the angle α_2, and across F_2 into β_2, and so on. In passing through v_1 we snip off the left sector α_2, β_1 and leave it on the left. In cutting across F_3, or any other face of B, we partition its boundary into two pieces, representing its left and right segments.

In our Figure the cut passes through v_1 twice, severing the two left sectors α_2, β_1 and α_3, β_2. These are disjoint by the admissibility of B, and so the cut does not cross itself at v_1.

In the part of the map outside the broken curve we treat the various detached left sectors as separate vertices. Accordingly this part is a cutting of the original map M, corresponding to C in the following combinatorial proofs.

The residue of v_1 after the detachment of two left sectors is still treated as a single vertex. We can therefore say that the interior of the broken curve is occupied by a single submap of M (with a cut-vertex at v_1). This corresponds to the "right submap" $M''(C)$ of the following proofs.

It is easy to identify the combinatorial analogue of a cut around a belt. We have to perform the operation of vertex-splitting at v_j, for each suffix j of the belt. Theorem 8.3 investigates the effect of a single vertex-splitting. The belt B, if its order exceeds 1, is replaced by a shorter belt B_j in a simpler map M_j. Theorem 8.4 is concerned with the effect of the complete set of vertex-splittings, the partition of M into C and $M''(C)$.

Our semi-topological digression is finished; we resume the combinatorial story.

We go on to some definitions that are to be applied only when $k \geqq 2$. Then for each suffix j we define H_j as the cycle $K([\beta_j^*, \beta_j^*][\alpha_{j+1}^*, \alpha_{j+1}^*])$. This can be written at greater length as follows.

$$(10) \quad H_j = K([\alpha_j^*, \beta_j^*][\alpha_{j+1}^*, \beta_{j+1}^*][\beta_{j+1}^*, \alpha_{j+1}^*][\beta_j^*, \alpha_j^*]).$$

We write M_j for the premap derived from M by splitting v_j between β_j and α_{j+1}. Using 6.1 we see that the only change in the passage from M to M_j is that v_j is replaced by the two new vertices $K([\alpha_{j+1}, \beta_j])$ and $K([\beta_j, \alpha_{j+1}])$, and that the two faces F_j and F_{j+1} are replaced by the single new face H_j.

Finally we write B_j for the cyclic sequence derived from B be replacing the subsequence

$$(\alpha_j, F_j, \beta_j, \alpha_{j+1}, F_{j+1}, \beta_{j+1})$$

by the subsequence $(\alpha_j, H_j, \beta_{j+1})$.

8.3. *If B is an admissible belt of M of order $k \geqq 2$, then B_j is an admissible belt of M_j of order $k - 1$. Its left sectors are those of B, with the omission of $[\alpha_{j+1}, \beta_j]$. Its left and right swirls are those of B.*

Proof. The terms of B_j are $k - 1$ distinct faces of M_j and $2(k - 1)$ distinct angles of M_j. Moreover each face occupies the two adjacent angles.

Consider the angles α_{i+1} and β_i of M, where $i \neq j$. They are at the vertex v_i of M. They appear also as angles of M_j in B_j. If v_i is distinct from v_j then the two angles are at the vertex v_i in M_j. If $v_i = v_j$ neither angle can be at the vertex $K([\alpha_{j+1}, \beta_j])$ of M_j, by the admissibility of B. Hence both α_{i+1} and β_i must be at the vertex $K([\beta_j, \alpha_{j+1}])$ of M_j. Considering all possible values of i we see that B_j satisfies all the conditions for a belt, of order $k - 1$, of M_j.

It follows from the results of the preceding paragraph, and the admissibility of B, that the sector $[\alpha_{i+1}, \beta_i]$ of M appears also as the sector $[\alpha_{i+1}, \beta_i]$ of M_j, either at the vertex $K([\beta_j, \alpha_{j+1}])$ or at a vertex v_i distinct from v_j. We can therefore assert that the left sectors of B_j are those of B, with the omission of $[\alpha_{j+1}, \beta_j]$. Consequently B_j is an admissible belt of M_j.

The left segments of B_j are those of B except that $[\alpha_j^*, \beta_j^*]$ and $[\alpha_{j+1}^*, \beta_{j+1}^*]$, corresponding to F_j and F_{j+1}, are replaced by the single left segment $[\alpha_j^*, \beta_j^*][\alpha_{j+1}^*, \beta_{j+1}^*]$ corresponding to H_j. (See (10)). It follows that B and B_j have the same left swirl. Similarly they have the same right swirl.

8.4. *If B is an admissible belt of a planar map M then M has a cutting C with the following properties. The outer vertices of C are the closures of the left sectors of B. The left swirl of B is a face of C, and the other faces of C are faces of M. No dart of the right swirl of B belongs to C.*

Proof. Suppose first that the order k of B is 1. Then $B = (\alpha_1, F_1, \beta_1)$, and α_1 and β_1 are distinct angles at the same vertex v_1. Split v_1 between them to transform M into a new premap M_1. By 6.2 and 7.4 M_1 has exactly two components, one of which has the closure of the single left sector of B as its only outer vertex. By 6.2 this component of M_1 is a cutting of M having the properties required for C.

We now proceed by induction over k. Our inductive hypothesis asserts that the theorem is true when k is less than some integer $q \geqq 2$, and we consider the case $k = q$. Choosing a suffix j we construct M_j and B_j. We observe that M_j is a planar map, by 7.3.

By 8.3 and the inductive hypothesis M_j has a cutting C with the following properties. The outer vertices of C are the closures of the left sectors of B, with the omission of $K([\alpha_{j+1}, \beta_j])$. The left swirl of B is a face of C, and the other faces of C are faces of M_j. No dart of the right swirl of B belongs to C.

If a face F of C is a face of M_j it is also a face of M. For otherwise it would be H_j by 6.1, and H_j meets the right swirl of B.

Consider the inner vertices of the cutting C of M_j. One of them is $K([\alpha_{j+1}, \beta_j])$ since this has the second dart of α_{j+1} in common with the left swirl, and does not meet any outer vertex. But $K([\beta_j, \alpha_{j+1}])$ is not an inner vertex of C since it has the second dart of β_j in common with the right swirl. So the inner vertices of C, other than $K([\alpha_{j+1}, \beta_j])$, must be vertices of M. We deduce that C is a cutting of M, and that its outer vertices are the closures of the k left sectors of B. By the preceding considerations the cutting C of M has all the other properties required by the theorem.

The theorem follows, by induction.

8.5. *The cutting C of* 8.4 *is uniquely determined.*

Proof. See 8.2.

We call C the *left cutting* of B, and $M'(C)$ the *left submap* of B.

C is a proper cutting of M since it does not meet the right swirl of B. Hence $M'(C)$ is a proper submap of M and has a complementary subpremap $M''(C)$ in M. We call $M''(C)$ the *complementary subpremap of C in M.*

8.6. *If B is an admissible belt of a planar map M and if C is the left cutting of B in M, then $M''(C)$ is a submap of M having the right swirl of B as one face.*

Proof. We use the proof of 8.4, supplementing it as follows.

When $k = 1$ one of the two components of M_1 is identified as C. The other is $M''(C)$. It has the properties stated in the enunciation, by 6.2.

When $k = q$ we obtain C initially as the left cutting of B_j in M_j. Assume 8.6 to hold when $k < q$. Then if N is the complementary subpremap of C in M_j it is a submap of M_j, and by 8.3 it has the right swirl of B as one face. But N includes no dart from the inner vertex $K([\alpha_{j+1}, \beta_j])$ of C in M_j. Hence N is also a submap of M. We can thus deduce from our assumption that 8.6 holds also when $k = q$.

Considering successive values of q from 2 upwards we arrive at an inductive proof of 8.6.

We call $M''(C)$ the *right submap* of B.

We add one more to the list of maps associated with a belt. Let B be an admissible belt of a planar map M, with left cutting C. We can form a premap L from C by replacing its outer vertices by the left sweep of B as a single new vertex. We call L the *left contraction* of B.

We can think of L as being derived from M by contracting the whole right submap into a single vertex.

8.7. *Let L be the left contraction of an admissible belt B in a planar map M. Then L is a planar map. Its faces are the closures of the left segments of B and the inner faces of the left cutting C.*

Proof. We use the notation of equation (9). Let us write $C = (P', \theta)$ and $L = (P'', \theta')$, where θ' is a restriction of θ. Let D_j be the second dart of α_j and E_j the first dart of β_j, for each suffix j. Then P' and P'' have the same effect on each dart of C except for the last darts of the segments $[\alpha_{j+1}, \beta_j]$, that is the darts E_j. We have $P'E_j = D_{j+1}$ and $P''E_j = D_j$.

We deduce that $P'\theta'$ and $P''\theta'$ have the same effect on each dart of C except for the darts $\theta' E_j$. But $P''\theta'$ transforms $\theta' E_j$ into D_j, that is it maps the last dart of the sequence $[\alpha_j^*, \beta_j^*]$ onto the first. We conclude that the faces of L are as stated in the enunciation.

Any connecting sequence in C becomes one in L when each outer vertex of C in it is replaced by the left sweep of B. Hence L is a map. It has $k - 1$ fewer vertices than C, and $k - 1$ more faces. Hence $N(L) = N(C) = 2$, by 8.1.

As a consequence of 8.7 we can assert that L^* is a submap of M^*.

9. Circular belts. A cutting C of a premap M is *simple* if no vertex of M contains more than one outer vertex of C. Equivalently a simple cutting of M is a submap of M in which each outer vertex is the closure of a sector of M.

It may happen that a simple cutting C has a complementary subpremap which is also a simple cutting of M. If so we speak of *complementary* simple cuttings. Complementary simple cuttings of non-zero order can occur only in a map, by 4.4.

Consider a circular belt B of a planar map M. Let it be described by equation (9). We can write the terms of B in the reverse cyclic order, so obtaining a circular belt B^{-1} of M. The left and right sectors, segments and swirls of B^{-1} are the right and left sectors, segments and swirls of B respectively. We refer to the left cutting and left sweep of B^{-1} as the *right cutting* and *right sweep* of B. Similarly the left contraction of B^{-1} is the *right contraction* of B.

Using the definitions of Section 8 we can verify that the right and left cuttings of B are complementary simple cuttings of M, identical with the right and left submaps of B respectively.

B being given by (9) the cyclic sequence

$$(11) \quad B^* = (\alpha_1^*, v_k, \beta_k^*, \ldots, \alpha_3^*, v_2, \beta_2^*, \alpha_2^*, v_1, \beta_1^*)$$

is a circular belt of the planar map M^*. We call it the *dual* belt of B. Its right and left sectors, segments, sweeps and swirls are the right and left segments, sectors, swirls and sweeps of B respectively. Using 8.7 and 4.6 we obtain

9.1. *The dual of the left (right) cutting of B^* is the left (right) contraction of B.*

It is convenient to describe a planar map M as *separable* if it has a (necessarily circular) belt of order 1, that is if some face occupies two distinct angles at some vertex. If M is not separable we say it is 2-*connected*.

We say that a 2-connected planar map M is 2-*separable* if it has a circular belt of order 2 for which the right and left cuttings have each at least two edges, and is 3-*connected* otherwise.

It is not difficult to show that M is 2-connected or 3-connected if and only if its graph $G(M)$ is 2-connected or 3-connected respectively, according to the definitions of [**7**]. This result belongs to the third theory mentioned at the end of Section 1.

We conclude this section with two elementary theorems on connectivity.

9.2. *Let M be a planar map. Then M^* is separable, 2-connected or 3-connected according as M is separable, 2-connected or 3-connected respectively.*

This is because each circular belt B of M has a dual belt B^* of M^* of the same order, and because the left and right cuttings of B^* have the same numbers of edges as the left and right cuttings of B respectively, by 9.1.

9.3. *Let M be a 2-connected planar map. If a vertex or face of M contains two opposite darts D and θD, then M is a loop-map or link-map respectively.*

Proof. Suppose the vertex v of $M = (P, \theta)$ to contain both D and θD. Consider the angles (D, PD) and $(P^{-1}\theta D, \theta D)$ of M. They are both occupied by the face containing $\theta P^{-1}\theta D$, $P\theta(\theta P^{-1}\theta D) = \theta D$ and $P\theta(\theta D) = D$. They are both at v. Hence they are identical. Accordingly $\theta D = PD = P\theta(\theta D)$, and (θD) is a face of M. Similarly (D) is a face of M. Hence M is a loop-map.

Replacing M by M^* in this we complete the proof.

10. Rings. A *ring* of a planar map $M = (P, \theta)$ on a dart-set S is a cyclic sequence

(12) $\quad R = (D_1, H_1, D_2, H_2, \ldots, D_m, H_m),$

where the D_i are m distinct darts of M and the H_i are m distinct faces of M, and where D_i and θD_{i+1} belong to H_i for each suffix i. Addition in the suffixes is modulo the positive integer m. The term "ring" is taken from the theory of map-colourings.

We write $U(R)$ for the set of all darts D_i and θD_i, $(1 \leqq i \leqq m)$. We define M_R as the subpremap of M on $S - U(R)$, provided that that set is non-null. We refer to the cyclic sequence

$$(D_m, D_{m-1}, \ldots, D_2, D_1)$$

as the *sweep* $Sp(R)$ of R.

Let γ_i denote the angle $(P^{-1}D_i, D_i)$, and δ_i the angle (D_{i+1}, PD_{i+1}). Both these angles are occupied by H_i.

If γ_i and δ_i are distinct we say that H_i is *frontal* in R. In the remaining case we have

(13) $\quad P(D_{i+1}) = D_i.$

To begin with we will assume that R has at least one frontal face. We then enumerate the frontal faces of R as

(14) $\quad (F_1, F_2, \ldots, F_k)$

in the cyclic order of their appearance in R. We can put $F_j = H_{n(j)}$, where

$1 \leqq n(1) < n(2) < \ldots < n(k) \leqq m$, and then write

(15) $\alpha_j = \gamma_{n(j)}, \beta_j = \delta_{n(j)}$,

for each suffix j of (14). We note that the $2k$ angles α_j, β_j are all distinct.

Using (13) we see that the linear sequence

(16) $(D_{n(j+1)}, D_{n(j+1)-1}, \ldots, D_{n(j)+2}, D_{n(j)+1})$

is a sector of M at some vertex v_j. It is in fact the sector $[\alpha_{j+1}, \beta_j]$. We call it a *left sector* of R. We can now assert that the cyclic sequence

(17) $B(R) = (\alpha_1, F_1, \beta_1, \alpha_2, F_2, \beta_2, \ldots, \alpha_k, F_k, \beta_k)$

is an admissible belt of M whose left sectors are the left sectors of R. Its left sweep is $Sp(R)$.

We denote the right submap of $B(R)$ by $W(R)$, and the left contraction by $Q(R)$. One vertex of $Q(R)$ is $Sp(R)$. We call the others its *residual* vertices.

If we write the terms of (12) in the opposite cyclic order and then replace each dart D_i by θD_i we obtain a ring R^{-1} of M. We note that $(R^{-1})^{-1} = R$. If R^{-1} has a frontal face we can form $B(R^{-1})$, $W(R^{-1})$ and $Q(R^{-1})$. We propose to investigate the relation between $W(R)$ and $W(R^{-1})$. We need some preparatory definitions.

If D and E are distinct darts of M in the same vertex v we write $[D, E]$ for the linear sequence

$$(D, PD, P^2D, \ldots, P^rD = E)$$

of distinct darts of v. If instead D and E are distinct darts of the same face F we write $[D, E]^*$ for the linear sequence

$$(D, (P\theta)D, (P\theta)^2D, \ldots, (P\theta)^sD = E)$$

of distinct darts of F.

The *right vertices* of R are the vertices of M containing the darts D_i of R, and the *left vertices* of R are those containing the darts θD_i of R^{-1}. For each i the dart D_i belongs to a unique left sector $[\alpha_{j+1}, \beta_j]$ of R, and we have v_j as the corresponding right vertex of R.

10.1. *Let R and R^{-1} each have a frontal face. Then M_R has exactly two components, these being $W(R)$ and $W(R^{-1})$. The vertices of attachment of $W(R)$ and $W(R^{-1})$ in M are the right and left vertices of R respectively.*

Proof. Let C be the left cutting of $B(R)$.

Consider any right vertex v_j of R. It contains the second dart of B_j. But this belongs to the right segment $[\beta_j^*, \alpha_j^*]$ of $B(R)$ and therefore to $W(R)$, by 8.6. On the other hand v_j includes the darts of $[\alpha_{j+1}, \beta_j]$, which are in C. Hence v_j is a vertex of attachment of $W(R)$ in M.

Now let x be any vertex of attachment of $W(R)$ in M. By the definition of a right submap some outer vertex of C is contained in x. Hence x includes some dart D_i of R, and is a right vertex of R. Any dart of x that is not a dart of R can belong neither to an outer nor an inner vertex of C, and so must be a dart of $W(R)$. We infer that $W(R)$ has no vertex of attachment in M_R. It is a component of M_R, by 4.5.

Similarly $W(R^{-1})$ is a component of M_R having the left vertices of R as its vertices of attachment in M. Any component of M_R other than $W(R)$ and $W(R^{-1})$ would be a component of the map M, by 4.5, which is absurd.

It remains only to show that $W(R)$ and $W(R^{-1})$ are distinct.

Suppose a face H_i of R is frontal in both R and R^{-1}. Then H_i contains the right segment $[\delta_i^*, \gamma_i^*]$ of $B(R)$. Since

$$\delta_i^* = (\theta D_{i+1}, PD_{i+1}) \text{ and } \gamma_i^* = (\theta P^{-1} D_i, D_i)$$

this right segment is a subsequence of $[\theta D_{i+1}, D_i]^*$, actually containing all but its first and last members.

Similar reasoning with R^{-1} shows that the right segment of $B(R^{-1})$ in H_i is a subsequence of $[D_i, \theta D_{i+1}]^*$. But this is the left segment $[\gamma_i^*, \delta_i^*]$ of $B(R)$, and so lies in C by 8.4.

Now consider the case of a face H_i of R that is frontal in R^{-1} but not in R. It includes the dart D_i, which belongs to C but not to the left swirl of $B(R)$. Hence H_i is a face of C, by 8.4. Accordingly the right segment of $B(R^{-1})$ in H_i lies in C.

Combining these results we see that no dart of the right swirl of $B(R^{-1})$ belongs to $W(R)$. But this swirl is a face of $W(R^{-1})$ by 8.6. Hence $W(R^{-1})$ is distinct from $W(R)$ and the proof is complete.

COROLLARY. *No vertex of M is both a right and a left vertex of R.*

Theorem 10.1 is a dual form of Jordan's Theorem, showing that a ring divides a planar map into two parts. There is perhaps some slight advantage in stating it in terms of $Q(R)$ and $Q(R^{-1})$. We proceed to do this.

10.2. *Under the conditions of* 10.1 *the vertices of $Q(R)$ are $Sp(R)$, the left vertices of R and the inner vertices of $W(R^{-1})$. Its faces are the inner faces of C and the closures of those sequences $[D_i, \theta D_{i+1}]^*$ for which H_i is frontal in R. The corresponding propositions hold for the vertices and faces of $Q(R^{-1})$.*

It is sufficient to prove this for $Q(R)$. We observe that we can form a cutting of M from $W(R^{-1})$ by replacing its outer vertices by the left vertices of R, and then adjoining the closures of the left sectors of R as new vertices. This cutting is C, by 8.2. But $Q(R)$, by definition, is obtained from C by replacing the outer vertices by $Sp(R)$. This establishes the part of 10.2 dealing with vertices. The part concerning faces follows from 8.7.

We note that if H_i is not frontal in R the closure of $[D_i, \theta D_{i+1}]^*$ is simply H_i, by (13). We can eliminate the references to C and $W(R^{-1})$ in 10.2 by using the following variation.

10.3. *Let R and R^{-1} each have a frontal face. Then each dart of $S - U(R)$ belongs to just one of $Q(R)$ and $Q(R^{-1})$. Each vertex of M is a residual vertex of just one of $Q(R)$ and $Q(R^{-1})$, and each face of M not in R is a face of exactly one of them. The other faces of $Q(R)$ are the closures of the sequences $[D_i, \theta D_{i+1}]^*$, and the other faces of $Q(R^{-1})$ are the closures of the sequences $[\theta D_{i+1}, D_i]^*$.*

This theorem follows from 10.1 and 10.2.

Let us write $X(R)$ for the planar map whose vertices are $Sp(R)$ and $Sp(R^{-1})$, and whose faces are therefore the cycles $(D_i, \theta D_{i+1})$.

We can extend 10.3 to the case in which one or both of R and R^{-1} has no frontal face. For R to have no frontal face it is necessary and sufficient that $Sp(R)$ shall be a vertex of M, by (13). We then define $Q(R)$ as M and $Q(R^{-1})$ as $X(R)$. Similarly if R^{-1} has no frontal face we put $Q(R^{-1}) = M$ and $Q(R) = X(R)$. If neither R nor R^{-1} has a frontal face then $M = X(R)$ by 4.5. So, consistent with the above definitions, we write $Q(R) = Q(R^{-1}) = X(R)$.

With these definitions Theorem 10.3 extends trivially to a general R.

11. Jordan's theorem. In this section we put 10.3 into dual form. There is no real change of content. It may however be convenient to work with submaps rather than contractions, and to have a closer terminological analogy with other statements of Jordan's Theorem.

A *directed circuit* of a planar map $M = (P, \theta)$ is a cyclic sequence

(18) $$J = (D_1, v_1, D_2, v_2, \ldots, D_m, v_m),$$

where the D_i are $m > 1$ distinct darts of M and the v_i are m distinct vertices of M, and where v_i includes D_i and D_{i+1} for each i. The same sequence is a ring in M^*. We refer to the cyclic sequence

$$(D_m, D_{m-1}, \ldots, D_2, D_1)$$

as the *swirl* $Sl(J)$ of J.

11.1. *Let J be a directed circuit of M, given by* (18). *Then there is a unique submap $A(J)$ of M in which the closures of the sequences $[D_i, \theta D_{i+1}]$ are vertices, and in which all other vertices are vertices of M. One face of $A(J)$ is $Sl(J)$.*

Proof. Consider the contraction $Q(J)$ for the ring J of M^*. Its dual map is a submap of M having the vertices and face specified, by 10.2 and the extended form of 10.3. Uniqueness follows from 4.6.

Let us refer to the faces of $A(J)$ other than $Sl(J)$ as its *residual faces*.

Writing the terms of (18) in the reverse cyclic order, and replacing D_i by θD_i for each i, we obtain a directed circuit J^{-1} of M called the *opposite* of J. From the extended form of 10.3, applied to M^*, we obtain the following rule.

11.2. *Each dart or vertex of M not in J belongs to just one of $A(J)$ and $A(J^{-1})$. Each face of M is a residual face of exactly one of $A(J)$ and $A(J^{-1})$.*

One more fact should be pointed out. Let us say that a set U of vertices or faces of M, or any other premap, is *interconnected* if for each partition $\{U_1, U_2\}$ of U into two non-null subsets there exists a dart D in a member of U_1 such that θD is in a member of U_2. Then the following proposition holds.

11.3. *The set of residual faces of $A(J)$ (or of $A(J^{-1})$) is interconnected.*

Apart from the trivial case in which $A(J)$ is the dual of $X(J)$ this follows from the fact that $W(J)$ is a map.

References

1. H. R. Brahana, *Systems of circuits on two-dimensional manifolds*, Ann. Math., (2), *23* (1921), 144–168.
2. R. Cori, *Graphes planaires et systèmes de parenthèses*, Centre National de la Recherche Scientifique, Institut Blaise Pascal, (1969).
3. ——— *Un code pour les graphes pla—aires et ses applications*, Asterisk *27* (1975).
4. J. R. Edmonds, *A combinatorial representation for polyhedral surfaces*, Notices Amer. Math. Soc., *7* (1960), 646.
5. A. Jacques, *Sur le genre d'une paire de substitutions*, C. R. Acad. Sci. Paris *267* (1968), 625–627.
6. ——— *Constellations et graphes topologiques*, In Combinatorial Theory and its Applications II, Budapest (1970).
7. W. T. Tutte, *Connectivity in graphs* (University of Toronto Press, 1966).
8. T. R. S. Walsh, *Combinatorial enumeration of non-planar maps*, Thesis, U. of Toronto (1971).
9. T. R. S. Walsh and A. B. Lehman, *Counting rooted maps by genus I*, J. Combinatorial Theory *13* (1972), 192–218.

University of Waterloo,
Waterloo, Ontario