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RIGHT-ORDERED GROUPS

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1. Introduction. A group G is right-ordered if it can be totally ordered so that for any a, b, c in G, $a < b$ implies that $ac < bc$. Right-ordered groups, considered as order preserving automorphisms of ordered sets, were studied by Cohn in [**4**]; but the first systematic study of the structure of these groups was made by Conrad in [**5**] where he gave several natural characterizations of right-ordered groups. We mention here that the class of right-ordered groups is precisely the subgroup closure of the class of lattice ordered groups (see [**6**], [**7**], [**9**] or [**10**]).

Conrad was particularly successful in the study of the structure of groups G which can be right-ordered in such a way that

(*) for each pair of positive elements a, b in G there exists a positive integer n such that $a^n b > a$.

The class of such groups coincides with the class of groups having a normal system with torsion-free abelian factors. If G is a finitely generated group in this class then G/G' is necessarily infinite. We still do not know whether every finitely generated right-ordered group G has the property that G/G' is infinite. We can only prove the following result.

THEOREM 1. *A finitely generated group $G \neq \{e\}$ can not be right-ordered if G/G' is finite and G has a nilpotent subgroup of finite index in G.*

The above question is significant because it is related to the problem of deciding whether the integral group-ring of a torsion-free group can have zero divisors. For any class $\mathscr{X}$ of groups, define the class of *locally $\mathscr{X}$-indicable* groups to consist of those groups G in which every finitely generated non-trivial subgroup has a non-trivial homomorphic image in $\mathscr{X}$. The terminology is derived essentially from that of Higman [**8**] who proved that the integral group-ring of a locally $\mathscr{Z}$-indicable group has no zero divisors. (Here $\mathscr{Z}$ denotes the class of infinite cyclic groups.) It was shown in [**12**] that the integral group-ring of a right-ordered group has no zero divisors. Burns and Hale [**3**, Theorem 2] observed that a locally RO-indicable group is an RO-group, where RO denotes the class of right-ordered groups. In particular locally $\mathscr{Z}$-indicable groups are RO-groups. We know of the existence of finitely generated torsion-free groups G with G' nilpotent and G/G' finite (see, for example, [**11**, p. 250] or [**2**]). Thus Theorem 1 effectively tells us that a different

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approach is needed to determine whether the integral group-rings of such groups have zero divisors.

Let $\overline{\text{RO}}$ denote the class of those RO-groups in which every right order has the property (*).

THEOREM 2. *Every torsion-free locally nilpotent group is an* $\overline{\text{RO}}$-*group.*

THEOREM 3. *A finitely generated group* $G \neq \{e\}$ *can not be right-ordered if* G/G' *is finite and* G *has a subgroup* K *of finite index in* G *with* $K \in \overline{\text{RO}}$.

It is easy to see that Theorem 1 is a consequence of Theorems 2 and 3. In § 4 we produce an example of a metabelian RO-group that is not an $\overline{\text{RO}}$-group. We do not know if every polycyclic RO-group is an $\overline{\text{RO}}$-group.

We now turn our attention to the following question raised by Ault in [**1**]. Can every partial right-order be extended to a full right-order in a torsion-free nilpotent group? Ault proved the result in the special case when the group is nilpotent of class two. A sub-semigroup P of a group G defines a partial right-order on G if $P \cap P^{-1} = \phi$. The partial right-order is obtained by putting $x < y$ if and only if $yx^{-1} \in P$. A sub-semigroup Q is called an extension of the partial right-order P if $Q \supseteq P$ and $Q \cap Q^{-1} = \phi$, and Q is a (full) right-order if in addition $Q \cup Q^{-1} \cup \{e\} = G$.

THEOREM 4. *Every partial right-order can be extended to a right-order in a torsion-free nilpotent group.*

In § 4 we give an example of a metabelian RO-group in which not every partial right-order can be extended to a right-order.

2. Proofs of Theorems 2 and 3.

LEMMA 2.1 (B. H. Neuman). *Let* G *be a locally nilpotent group,* X *a subset of* G *and* S *the semigroup generated by* X. *Then given* u, v *in* S, *there exists* z, t *in* S *such that* $zu = tv$.

Proof. We use induction on the nilpotency class of $L = \text{group}\,\langle u, v\rangle$. If L is abelian then take $z = v$ and $t = u$. If L is nilpotent of class $r > 1$, then $M = \text{Group}\langle uv, vu\rangle$ is nilpotent of class $r - 1$ since $vu = uv[v, u]$. Thus there exists a, b in $\text{Semigroup}\langle uv, vu\rangle$ satisfying $avu = buv$. Take $z = av$ and $t = bu$.

Let G be a torsion-free locally nilpotent group, P the positive cone of a given right-order on G and a, b in P. Suppose, if possible, that $a^n b < a$ for all integers $n > 0$. Let $S = \text{Semigroup}\langle ab, a\rangle$. By Lemma 2.1, there exists z, t in S such that $zab = ta$. Since a, b are both in P, $S \subseteq P$ so that $t > e$ and $ta > a$. Now $zab = a^{\alpha_1}ba^{\alpha_2}b \ldots a^{\alpha_r}b$ with $\alpha_i > 0$, $i = 1, \ldots, r$. Since $a^n b < a$ for all $n > 0$, $a^{\alpha_1}ba^{\alpha_2}b \ldots a^{\alpha_r}b < a^{\alpha_2+1}b \ldots a^{\alpha_r}b < \ldots < a^{\alpha_r+1}b < a$, a contradiction. This completes the proof of Theorem 2.

We prove Theorem 3 by way of contradiction. Let $<$ be a right-order on G. By hypothesis G is finitely generated torsion-free, G/G' is finite and G has a normal subgroup K of finite index in G with $K \in \overline{\mathrm{RO}}$. Since K is a finitely generated $\overline{\mathrm{RO}}$-group, K/K' contains elements of infinite order. Let I be the isolator of K' in K. Then K/I is torsion-free abelian. Choose coset representatives $e = x_1 < x_2 < \ldots < x_n$ of K in G and take the transfer map $\tau : G \to K/I$ given by

$$\tau(g) = \left(\prod_{i=1}^{n} x_i g(\overline{x_i g})^{-1}\right) I,$$

where $\overline{x_i g}$ is the coset representative of $x_i g$. τ is the trivial map since K/I is torsion-free abelian and G/G' is finite. Observe that if $e < x < y$ then $e < yx^{-1}$. For any $y \geqq x_n$, $y^n \in K$ and $y^n > x_n$. Thus if $g > x_n$ and $g \in K$, then $g^{x_2} \ldots g^{x_n} g > g$ and $g^{x_2} \ldots g^{x_n} g \in I$. Since K is a finitely generated $\overline{\mathrm{RO}}$-group, there exists $g > x_n$ in K such that the convex subgroup generated by g is K (see § 4 of [**5**]). Let C be the union of convex subgroups of K not containing g. Then K/C is isomorphic to a subgroup of the additive group of reals. Hence $I \leqq C$. Also $g > x$ for all $x \in C$. This gives the required contradiction.

3. Proof of Theorem 4. Let G be a torsion-free nilpotent group and let P be the positive cone of a partial right-order $<$ on G. Assume, by way of contradiction, that P is maximal but not a full right-order. Then for some $x \neq e$, $x \notin P \cup P^{-1}$. Since P is maximal we conclude that

$$e \in \text{Semigroup}\langle P, x\rangle \cap \text{Semigroup}\langle P, x^{-1}\rangle.$$

More specifically,

$$x^{\alpha_1} p_1 \ldots x^{\alpha_m} p_m = e = x^{-\beta_1} q_1 \ldots x^{-\beta_n} q_n \tag{1}$$

where $p_1, \ldots, p_m, q_1, \ldots, q_n$ lie in P and $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are all positive integers.

If $x \in Z(G)$, the centre of G, then (1) reduces to $x^\alpha p = e = x^{-\beta} q$ for some $\alpha, \beta > 0$ and p, q in P. This is not possible since it implies $x^{\alpha\beta} = e$ and $\alpha\beta \neq 0$. Thus $Z(G) \leqq P \cup P^{-1} \cup \{e\}$. Assume that $Z_i(G) \nleqq P \cup P^{-1} \cup \{e\}$ but $z_{i-1}(G) \leqq P \cup P^{-1} \cup \{e\}$ for some integer $i > 1$. ($Z_j(G)$ denotes the jth term of the upper central series of G.) Thus (1) holds for some $x \in Z_i(G) \backslash Z_{i-1}(G)$. We now investigate the consequence of the left-hand side of equation (1).

Let W be the set of all words $x^{\alpha_1} p_1 \ldots x^{\alpha_m} p_m$ that are equal to e with α_i's positive, $m \geqq 1$ and p_i's in P. Define a function μ from P to the set of non-negative integers by the rule

$$\mu(p) = \begin{cases} 0 \text{ if } [p, x] = e \\ j \text{ if } [p, x] \in Z_j(G) \backslash Z_{j-1}(G), j = 1, 2, \ldots . \end{cases}$$

For any $w = x^{\alpha_1} p_1 \ldots x^{\alpha_m} p_m$ in W, let $|w| = \max\{\mu(p_i) : i = 1, 2, \ldots, m\}$. Note that $|w| = 0$ implies that $[x, p_i] = e$ for $i = 1, \ldots, m$ so that $x^{\alpha_1} p_1 \ldots x^{\alpha_m} p_m = x^\alpha p$ with $\alpha = \sum_{i=1}^{n} \alpha_i > 0$ and $p = p_1 \ldots p_m \in P$. Hence $x^{-\alpha} \in P$. We will show that there exists w in W with $|w| = 0$. Suppose, if possible, that $|w| > 0$ for every $w \in W$. Let W_1 be the subset of W consisting of those words w with $|w|$ minimal. We call p_i a *dominant component* of $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ if $\mu(p_i) = |w|$. Since $w = e$, there are at least two dominant components in w. Let W_2 be the set of those words in W_1 with the least number of dominant components. Let W_3 be the set of those words $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_2 with $\mu(p_1) = |w|$. Let $j > 1$ be the smallest integer such that for all $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_3, $\mu(p_i) < \mu(p_1)$ for all i satisfying $1 < i < j$, but $\mu(p_j) = \mu(p_1)$ for some w in W_3. Let W_4 be the set of those $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_3 with $\mu(p_j) = \mu(p_1)$. Finally, let W_5 be the set of those $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_4 with m minimal. Of course we are assuming, by way of contradiction, that $m > 1$.

Pick any $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_5. Since $x^{\alpha_j}p_j = p_j(x[x, p_j])^{\alpha_j}$, we must have $[x, p_j] < e$, for otherwise

$$x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m = x^{\alpha_1}p_1 \ldots x^{\alpha_{j-1}}p_{j-1}p_j(x[x, p_j])^{\alpha_j}x^{\alpha_{j+1}} \ldots p_m \equiv w' \in W$$

and $|w| = |w'| = \mu(p_1) \geqq \mu(p'_{j-1})$ where $p'_{j-1} = p_{j-1}p_j$. If $j = 2$ or $\mu(p_1) > \mu(p'_{j-1})$ then we contradict the choice of W_2 and if $j \neq 2$ and $\mu(p_1) = \mu(p'_{j-1})$ then we contradict the choice of W_4. Thus $[p_j, x] > e$ and

$$w_1 = x^{\alpha_1}p_1 \ldots x^{\alpha_{j-1}}p_{j-1}x^{\alpha_j+1}p_{j1}x^{\alpha_{j+1}-1} \ldots p_m \in W_5,$$

where $p_{j1} = p_j[p_j, x] \in P$ and $\mu(p_{j1}) = \mu(p_j)$. Repeated application of the above argument yield w_i, $i = 1, \ldots, \alpha_{j+1}$ where

$$w_i = x^{\alpha_1}p_1 \ldots p_{j-1}x^{\alpha_j+i}p_{ji}x^{\alpha_{j+1}-i} \ldots p_m,$$

$p_{ji} = p_{ji-1}[p_{ji-1}, x] \in P$ and $\mu(p_1) = \mu(p_{ji})$. Now $w_i \in W_5$ for $i < \alpha_{j+1}$, but

$$w_{\alpha_{j+1}} = x^{\alpha_1}p_1 \ldots p_{j-1}x^{\alpha_j+\alpha_{j+1}}p'x^{\alpha_{j+2}} \ldots p_m$$

where $p' = p_{j\alpha_{j+1}}p_{j+1} \in P$ and $\mu(p') \leqq \mu(p_1)$. Now $\mu(p') < \mu(p_1)$ contradicts the choice of W_2 and $\mu(p') = \mu(p_1)$ contradicts the choice of W_5. Note that should $j = m$, we take x^{α_1} for $x^{\alpha_{j+1}}$ and in this case we have

$$w_{\alpha_1} = x^{\alpha_m+\alpha_1}(p_{m\alpha_1}p_1)x^{\alpha_2}p_2 \ldots x^{\alpha_{m-1}}p_{m-1}$$

which contradicts the choice of W_2.

We have thus established that $x^\alpha p = e$ for some integer $\alpha > 0$ and some $p \in P$. Similarly we obtain $x^{-\beta}q = e$ for some $\beta > 0$ and some $q \in P$ as a consequence of the right-hand side of (1). These two equations imply that $x^{\alpha\beta} = e$ with $\alpha\beta \neq 0$, a contradiction.

4. Examples. We give two examples to show the limitations of Theorems 2 and 4.

Let G be the group generated by two permutations, α and τ, of the real line given by:

$$x\alpha = x + 1$$
$$x\tau = x/2.$$

Thus $x(\tau\alpha\tau^{-1}) = x/2(\alpha\tau^{-1}) = ((x+2)/2)\tau^{-1} = x\alpha^2$. In fact it is easy to verify that G is isomorphic to Group$\langle \alpha, \tau; \tau\alpha\tau^{-1} = \alpha^2\rangle$ which is metabelian. G is a subgroup of the group of order-preserving permutations of the real line in the sense that $x < y$ implies $x\theta < y\theta$ for all $\theta \in G$. Thus G is an RO-group and we can order it in the fashion described by Conrad in [**4**], by well-ordering the set **R** of real numbers in any appropriate way and then, for any $\theta \in G$, look at the first $r \in \mathbf{R}$ in the well-ordering for which $r\theta \neq r$. Put $\theta > e$ if $r\theta > r$ and $\theta < e$ otherwise. In particular, by well-ordering **R** so that 0 is the first element and -1 is the second, we make $\alpha > e$, $\tau > e$ and $\alpha\tau > e$. But $(\alpha\tau)^n\tau(\alpha\tau)^{-1} < e$ for all $n > 0$ since 0 is mapped to $(2^{n-1}/2^n) - 1$ under $(\alpha\tau)^n\alpha^{-1}$. Thus the right-order on G described above does not satisfy the property (*). This example is basically similar to Example III [**5**] of Conrad, except that Conrad's example is more complicated and not metabelian.

Our second example is $G = $ Group$\langle a, b; a^{-1}ba = b^{-1}\rangle$. It is a metacyclic RO-group. $P = $ Semigroup$\langle a^2, b, ba^{-2}\rangle$ defines a partial right-order on G. This is easily verified since $[a^2, b] = e$. Under any extension of P to a full right-order on G we must have $a > e$ since $a^2 \in P$. Also $ba^{-2} \in P$ implies $ba^{-1} > a > e$ and hence $aba^{-1} = b^{-1} > e$, a contradiction.

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