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Convex Bodies of Minimal Volume, Surface Area and Mean Width with Respect to Thin Shells

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Abstract. Given $r > 1$, we consider convex bodies in $\mathbb{E}^n$ which contain a fixed unit ball, and whose extreme points are of distance at least r from the centre of the unit ball, and we investigate how well these convex bodies approximate the unit ball in terms of volume, surface area and mean width. As r tends to one, we prove asymptotic formulae for the error of the approximation, and provide good estimates on the involved constants depending on the dimension.

1 Notation

Let us introduce the notation used throughout the paper. For any notions related to convexity in this paper, consult R. Schneider [19]. We write o to denote the origin in $\mathbb{E}^n$, $\langle \cdot , \cdot \rangle$ to denote the scalar product, and $\| \cdot \|$ to denote the corresponding Euclidean norm. Moreover for non-collinear points u, v, w, the angle of the half lines vu and vw is denoted by $\angle uvw$. Given a set $X \subset \mathbb{E}^n$, the affine hull, the convex hull and the interior of X are denoted by $\operatorname{aff} X$, $\operatorname{conv} X$ and $\operatorname{int} X$, respectively. In addition if X is convex, then the relative boundary and the relative interior of X with respect to $\operatorname{aff} X$ are denoted by ∂X and $\operatorname{relint} X$, respectively. We write B^n to denote the unit ball centred at o, and S^{n-1} to denote ∂B^n.

The k-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$ (see [8, 16] for definition and main properties). We normalize it in a way such that it coincides with the Lebesgue measure in $\mathbb{E}^k$. If M is a bounded measurable subset of $\mathbb{E}^n$, then we call $\mathcal{H}^n(M)$ the volume $V(M)$ of M. As usual, we call a compact convex set in $\mathbb{E}^n$ with non-empty interior a convex body, and a two-dimensional compact convex set a convex disc. For a convex body C, we write $S(C) = \mathcal{H}^{n-1}(\partial C)$ to denote its surface area. When integrating on ∂C, we always do it with respect to $\mathcal{H}^{n-1}$. The two-dimensional Hausdorff measure of a two-dimensional convex compact set, or of a measurable subset X of the boundary of some convex body in $\mathbb{E}^3$, is also called the area $A(X)$ of X.

We recall that x is an extreme point of a convex compact set C if x does not lie in the relative interior of any segment contained in C. We write $\operatorname{ext} C$ to denote the family of extreme points, and note that $\operatorname{ext} C$ forms the minimal set whose convex hull is C.

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Given a compact convex set C in $\mathbb{E}^n$, its support function $h_C(u)$, $u \in \mathbb{E}^n$, is defined by $h_C(u) = \max_{x\in C}\langle x, u\rangle$. In particular, for any $u \in S^{n-1}$, the width of C in the direction u is $h_C(u) + h_C(-u)$. Therefore the mean width of C is

$$M(C) = \frac{2}{S(B^n)} \int_{S^{n-1}} h_C(u)\, du.$$

In particular $M(B^n) = 2$, and if C is a convex disc, then $M(C) = \frac{1}{\pi} S(C)$ according to the Cauchy formula (see [19]). We note that the volume, surface area, and the mean width of a compact convex set C in $\mathbb{E}^n$ can be expressed as the mixed volumes (quermassintegrals or normalized intrinsic volumes),

$$(1.1)\qquad \begin{aligned} V(C) &= V(C,\dots,C), \quad S(C) = n\,V(C,\dots,C,B^n),\\ M(C) &= \frac{2}{V(B^n)}, \quad V(C,B^n,\dots,B^n). \end{aligned}$$

2 Introduction

Let us define the main objects of study in this paper.

Definition Given $r > 1$, we write $\mathcal{F}_r^n$ to denote the family of convex bodies in $\mathbb{E}^n$ which contain B^n, and whose extreme points are of distance at least r from o. Moreover let P_r^n, Q_r^n and W_r^n be elements of $\mathcal{F}_r^n$ with minimal volume, surface area, and mean width, respectively.

The minima do exist according to the Blaschke selection theorem, and all extreme points of P_r^n, Q_r^n and W_r^n lie on rS^{n-1} by the monotonicity of the volume, surface area and the mean width. Unfortunately, we do not know whether the extremal convex bodies are polytopes. For example, if r is reasonably large, then it is conjectured that all extremal convex bodies are right cylinders whose bases are unit $(n-1)$-balls. Answering a question of J. Molnár [15], K. Böröczky and K. Böröczky, Jr. [3] proved that P_r^3 and Q_r^3 are regular octahedra when $r = \sqrt{3}$, and are regular icosahedra when $r = \sqrt{15 - 6\sqrt{5}}$. As discussed in [3], no regular polytope is extremal in its class if $n \geq 8$. Therefore in this paper we consider the case when r tends to 1 and there is no restriction on the dimension. Given real functions $f(r)$ and $g(r)$ of $r > 1$, we write $f(r) \sim g(r)$ if $\lim_{r\to 1} \frac{f(r)}{g(r)} = 1$. In addition we write $g(r) = O(f(r))$ if $|g(r)| \leq c\, f(r)$ for some constant c depending only on n.

Theorem 2.1 *If $n \geq 2$ and $r > 1$ tends to 1, then*

$$V(P_r^n \backslash B^n) \sim \theta_V(n) \cdot (r-1),$$
$$S(Q_r^n) - S(B^n) \sim \theta_S(n) \cdot (r-1),$$
$$M(W_r^n) - M(B^n) \sim \theta_M(n) \cdot (r-1),$$

where $\theta_V(n)$, $\theta_S(n)$, and $\theta_M(n)$ are positive constants depending only on n.

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Since all $V(rB^n) - V(B^n)$, $S(rB^n) - S(B^n)$, and $M(rB^n) - M(B^n)$ are of order $r-1$ if r is close to 1, it is not surprising that we have the factor $r-1$ in Theorem 2.1. We note that Theorem 2.1 is proved in [4] if $n \leq 3$. Additionally, it was determined in [4] that

$$\theta_V(2) = \frac{2\pi}{3}, \quad \theta_S(2) = \frac{4}{3}, \quad \theta_M(2) = \frac{4\pi}{3}$$

in the planar case, and

$$\theta_V(3) = \pi, \quad \theta_S(3) = 3\pi, \quad \theta_M(3) = \frac{7}{6}$$

in the three-dimensional case. Moreover [4] proved that if r is close to 1, then the typical faces of P_r^3, Q_r^3, and W_r^3 are asymptotically regular triangles.

For large n, combining Theorem 2.1 and Lemma 5.1 yields that there exist positive absolute constants c_1 and c_2 satisfying

$$\frac{c_1}{n} \cdot S(B^n) < \theta_V(n) < \frac{c_2 \ln n}{n} \cdot S(B^n), \tag{2.1}$$

$$c_1 \cdot S(B^n) < \theta_S(n) < c_2 \ln n \cdot S(B^n), \tag{2.2}$$

$$\frac{c_1}{n} < \theta_M(n) < \frac{c_2 \ln n}{n}, \tag{2.3}$$

where

$$S(B^n) = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

(see [19]). Next we state a theorem that is essential in proving (2.1), (2.2), and (2.3).

Theorem 2.2 *If $\rho > 0$ and C is a convex body in $\mathbb{E}^n$ with $\|x\| \geq \rho$ for all $x \in \operatorname{ext} C$, then*

$$\int_C \|x\|^2 \, dx \geq \frac{\rho^2}{9n} \cdot V(C).$$

Remark Theorem 2.2 is optimal up to an absolute constant factor, as is shown by the example of regular simplices inscribed into ρB^n.

A field closely related to our paper is polytopal approximation where a given smooth convex body C in $\mathbb{E}^n$ is approximated by polytopes of restricted number of vertices and facets. A typical problem is to consider the inscribed polytopes $P_{k,V}$ and $P_{k,M}$ with at most k vertices of maximal volume and of maximal mean width, respectively. As k tends to infinity, asymptotic formulae expressing $V(C) - V(P_{k,V})$ and $M(C) - M(P_{k,M})$ are known. In addition, the inscribed polytope $P_{k,H}$ with at most k vertices and minimizing the so-called Hausdorff distance from C (see (3.1)) is well investigated. It is also known that if C is a ball in $\mathbb{E}^3$, then the typical faces of the extremal polytopes are asymptotically regular triangles. For references, see the nice surveys by P. M. Gruber [12–14], and the recent manuscript by K. J. Böröczky, P. Tick, and G. Wintsche [6]. Various methods in this paper come from the field of polytopal approximation, in which small parts of S^{n-1} are approximated by paraboloids.

The paper is structured in the following way: Section 3 discusses polytopal approximation from our point of view, and Section 4 proves Theorem 2.2. Section 5 presents the basic statements and main idea of the proof for Theorem 2.1, and it proves the approximate version Lemma 5.1. The proof of Theorem 2.1 in the cases of volume and surface area in Section 9 is based on the properties of convex hypersurfaces discussed in Sections 6 and 7, and on Lemma 8.1 in Section 8 which describes how to transfer polytopal approximation into integration in $\mathbb{E}^{n-1}$. Theorem 2.1 in the case of the mean width is proved in Sections 10 and 11. We note that the case of mean width is substantially easier than the cases of volume or surface area.

3 Hausdorff Distance and Polytopal Approximation

We will frequently approximate convex bodies by polytopes (see [11–13] for general surveys). A natural measure of closeness between compact sets is the so-called *Hausdorff distance*. For a $x \in \mathbb{E}^n$ and a compact $X \subset \mathbb{E}^n$, we write $d(x, X)$ to denote the minimal distance between x and the points of X. If K and C are compact sets in $\mathbb{E}^n$, then their Hausdorff distance is

$$\delta_H(K, C) = \max\Bigl\{ \max_{x \in K} d(x, C), \max_{y \in C} d(y, K) \Bigr\}. \tag{3.1}$$

In the case when C and K are convex, the maximum of $d(x, C)$ among $x \in K$ is attained at some extreme point of K. We always consider the space of compact sets as the metric space induced by the Hausdorff distance that is readily a metric. In particular we say that a sequence $\{K_m\}$ of compact sets tends to a compact set C if $\lim_{m\to\infty} \delta_H(K_m, C) = 0$, and clearly C is convex if every K_m is convex. For the main properties of the Hausdorff distance, see [19]. For example, the Blaschke selection theorem says that if $\{K_m\}$ is a sequence of compact convex sets that are contained in a fixed ball, then $\{K_m\}$ has a subsequence $\{K_{m'}\}$ that tends to some compact convex set C. In addition the volume, surface area and the mean width are continuous functions of convex bodies. This latter property follows from the following fact: if the convex bodies K and C contain B^n, then

$$[1 + \delta_H(K, C)]^{-1} K \subset C \subset [1 + \delta_H(K, C)] \cdot K.$$

Let $\widetilde{\mathcal{F}}^n_r$ denote the family of all $C \in \mathcal{F}^n_r$ satisfying $\operatorname{ext} C \subset rS^{n-1}$. Then

$$P^n_r, Q^n_r, W^n_r \in \widetilde{\mathcal{F}}^n_r.$$

Lemma 3.1 *Let $1 < r < 2$ and let $0 < \mu < \frac{1}{4}\sqrt{r-1}$. If $C \in \widetilde{\mathcal{F}}^n_r$, then there exists a polytope $M \in \widetilde{\mathcal{F}}^n_r$ such that the distance between any two vertices of M is at least μ, and M satisfies $\delta_H(M, C) \le 4\mu\sqrt{r-1}$.*

Proof Let $x_1, \dots, x_k$ be a maximal system of points of rS^{n-1} such that $d(x_i, x_j) \ge \mu$ for $i \ne j$. Now the vertices of M are the x_i's whose distance from some extreme point of C is at most 2μ.

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First we show that M contains B^n. Let H^+ be any closed half space that avoids $\int B^n$, and whose bounding hyperplane touches B^n. Then H^+ contains some extreme point y of C, hence there exists a $z \in rS^{n-1}$ of distance at most μ from y satisfying that $rS^{n-1} \cap (z + \mu B^n) \subset H^+$. Since $z + \mu B^n$ contains some x_i that is then of distance at most 2μ from y, we conclude $B^n \subset M$.

Next we estimate the Hausdorff distance. If y is an extreme point of C, then its distance from some vertex x_i of M is at most 2μ, hence

$$d(y, M) \leq d(y, \operatorname{conv}\{x_i, B^n\}) \leq 2\mu \cdot \frac{\sqrt{r^2 - 1}}{r}.$$

The analogous argument for $d(x_i, C)$ where x_i is any vertex of M completes the proof of Lemma 3.1. ■

Let us remark that a result of R. Schneider [18] about approximation of smooth convex bodies by inscribed polytopes of restricted number of vertices with respect to the Hausdorff distance yields the following statement. If $N(r)$ is the minimal number of vertices of polytopes in $\widetilde{\mathcal{F}}_r^n$, then $N(r) \sim c(n) \cdot (r-1)^{-\frac{n-1}{2}}$ where $c(n)$ is an explicit constant depending on n.

4 Proof of Theorem 2.2

The proof of Theorem 2.2 will use Lemma 4.3, whose proof in turn is prepared by verifying Proposition 4.1.

Proposition 4.1 *Let $z_1, \ldots, z_{n+1}$ be the vertices of a regular simplex in $\mathbb{E}^n$ with $\|z_i\| = 1$, $i = 1, \ldots, n+1$, let $e_1, \ldots, e_n$ be an orthonormal basis in $\mathbb{E}^n$, and let $a_1, \ldots, a_n \in \mathbb{R}$. Then there exists i_0 such that*

$$\sum_{j=1}^{n} a_j^2 \langle z_{i_0}, e_j \rangle^2 \leq \frac{1}{n} \sum_{j=1}^{n} a_j^2.$$

Proof We think that $\mathbb{E}^n$ is embedded into $\mathbb{E}^{n+1}$ as a linear subspace, and let y be one of the unit normals to $\mathbb{E}^n$ in $\mathbb{E}^{n+1}$. In particular,

$$y_i = \sqrt{\frac{n}{n+1}}\, z_i + \sqrt{\frac{1}{n+1}}\, y, \quad i = 1, \ldots, n+1,$$

form an orthonormal basis of $\mathbb{E}^{n+1}$. For any e_j, we have $\langle e_j, y_i \rangle = \sqrt{\frac{n}{n+1}} \langle e_j, z_i \rangle$ for $i = 1, \ldots, n+1$, hence

$$1 = \|e_j\| = \sum_{i=1}^{n+1} \langle y_i, e_j \rangle^2 = \frac{n}{n+1} \sum_{i=1}^{n+1} \langle z_i, e_j \rangle^2.$$

It follows that

$$\sum_{i=1}^{n+1}\sum_{j=1}^{n} a_j^2\langle z_i, e_j\rangle^2 = \sum_{j=1}^{n} a_j^2 \sum_{i=1}^{n+1}\langle z_i, e_j\rangle^2 = \frac{n+1}{n}\sum_{j=1}^{n} a_j^2,$$

which in turn yields the existence of z_{i_0}. ■

For a linear map A, we recall a general fact that follows easily from the principal axis theorem applied to A^tA:

Fact 4.2 (Polar decomposition) Let $A\colon \mathbb{E}^n \to \mathbb{E}^n$ be a linear map. Then there are orthogonal maps $U, V\colon \mathbb{E}^n \to \mathbb{E}^n$ and a diagonal map $D\colon \mathbb{E}^n \to \mathbb{E}^n$ with diagonal elements $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ such that $A = UDV$. The diagonal elements of D are unique and called the singular numbers of A.

We recall that the centroid of a bounded measurable set M in $\mathbb{E}^n$ is the point

$$c = \frac{1}{V(M)}\int_M x\,dx,$$

and for any $y \in \mathbb{E}^n$, we have

$$\int_M \langle x - c, y\rangle\,dx = 0. \tag{4.1}$$

We note that if $S = \operatorname{conv}\{x_1, \dots, x_{n+1}\}$ and $T = \operatorname{conv}\{z_1, \dots, z_{n+1}\}$ are simplices in $\mathbb{E}^n$ whose centroids are the origin o, then $\sum_{i=1}^{n+1} x_i = o = \sum_{i=1}^{n+1} z_i$, hence there exists a unique linear map A with $A(z_i) = x_i$, $i = 1, \dots, n+1$.

Lemma 4.3 *For $r > 0$, let $S = \operatorname{conv}\{x_1, \dots, x_{n+1}\}$ be a simplex in $\mathbb{E}^n$ with $\|x_i\| \geq r$, $i = 1, \dots, n+1$ and with its centroid at the origin o, and let $T = \operatorname{conv}\{z_1, \dots, z_{n+1}\}$ be a regular simplex in $\mathbb{E}^n$ with $\|z_i\| = 1$, $i = 1, \dots, n+1$. If A is the linear map with $A(z_i) = x_i$, $i = 1, \dots, n+1$, and $a_1, \dots, a_n$ are the singular numbers of A, then*

$$\sum_{j=1}^{n} a_j^2 \geq nr^2.$$

Proof Applying orthogonal transformations to S and T changes neither the conditions on S and T nor the singular numbers of A, hence we may assume that $A = D$ where D is the diagonal matrix with diagonal elements $a_1, \dots, a_n$.

Let $e_1, \dots, e_n$ be the corresponding orthonormal basis in $\mathbb{E}^n$. By Proposition 4.1 there exists i_0 such that

$$\frac{1}{n}\sum_{j=1}^{n} a_j^2 \geq \sum_{j=1}^{n} a_j^2\langle z_{i_0}, e_j\rangle^2 = \sum_{j=1}^{n}\langle Dz_{i_0}, e_j\rangle^2 = \|Dz_{i_0}\|^2 = \|x_{i_0}\|^2 \geq r^2. \qquad ■$$

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Let T be a regular simplex whose centroid is the origin. Since the positive definite quadratic form $q_T(u) = \int_T \langle x, u\rangle^2 \, dx$ is invariant under the symmetries of T, we deduce that $q_T(u) = \lambda\langle u, u\rangle$ for suitable $\lambda > 0$ depending on T, namely, T is in isotropic position (see [10]). In particular if $e_1, \dots, e_n$ form an orthonormal basis of $\mathbb{E}^n$, then $\int_T \langle x, e_i\rangle^2 dx = \int_M \langle x, e_j\rangle^2 \, dx$ for $i \neq j$, therefore

$$\int_T \langle x, e_i\rangle^2 \, dx = \frac{1}{n} \int_T \|x\|^2 \, dx, \quad i = 1, \dots, n. \tag{4.2}$$

Proof of Theorem 2.2 We may assume that $\rho = 1$, and by approximation also that C is a polytope. Subdividing C into simplices shows that it is sufficient to prove Theorem 2.2 for an n-simplex $S = \operatorname{conv}\{x_1, \dots, x_{n+1}\}$ with $\|x_i\| \geq 1$, $i = 1, \dots, n+1$. We write c to denote the centroid of S, and we have

$$\int_S \|x\|^2 \, dx = \int_{S-c} \|x + c\|^2 \, dx = \int_{S-c} \|x\|^2 \, dx + 2 \int_{S-c} \langle x, c\rangle \, dx + \int_{S-c} \|c\|^2 \, dx.$$

Since o is the centroid of $S - c$, (4.1) yields $\int_{S-c} \langle x, c\rangle \, dx = 0$, hence

$$\int_S \|x\|^2 \, dx = \int_{S-c} \|x\|^2 \, dx + \int_{S-c} \|c\|^2 \, dx \geq \|c\|^2 V(S).$$

Now if $\|c\|^2 \geq \frac{1}{4n}$, then Theorem 2.2 readily follows. Therefore to prove Theorem 2.2, it is sufficient to verify that if $\|c\|^2 \leq \frac{1}{4n}$, then

$$\int_{S-c} \|x\|^2 \, dx \geq \frac{1}{9n} \cdot V(S). \tag{4.3}$$

It follows by the triangle inequality that the vertices $x_i - c$, $i = 1, \dots, n+1$ of $S - c$ satisfy

$$\|x_i - c\| \geq 1 - \frac{1}{2\sqrt{n}}.$$

Let $T = \operatorname{conv}\{z_1, \dots, z_{n+1}\}$ be a regular simplex with $\|z_i\| = 1$ and let A be the linear map defined by $A(z_i) = x_i - c$. We fix an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{E}^n$. Possibly after applying an orthogonal transformation to T, we may assume that the polar decomposition of A is of the form $A = UD$, where U is an orthogonal transformation, and D is a diagonal map whose diagonal elements are the singular numbers $a_1, \dots, a_n$ of A. After substituting $x = Uy$, we obtain

$$\int_{S-c} \|x\|^2 \, dx = \int_{DT} \|y\|^2 \, dy = \sum_{i=1}^n \int_{DT} \langle y, e_i\rangle^2 \, dy.$$

Next the substitution $y = Dw$ leads to

$$\int_{S-c} \|x\|^2 \, dx = \sum_{i=1}^n \int_T \langle Dw, e_i\rangle^2 \det(D) \, dw = \det(D) \sum_{i=1}^n a_i^2 \int_T \langle w, e_i\rangle^2 \, dw.$$

Since T is in isotropic position (see (4.2)), we deduce by $\left(1-\frac{1}{2\sqrt{n}}\right)^2 > \frac{1}{4}$ and by Lemma 4.3 that

$$\int_{S-c} \|x\|^2\, dx = \frac{\det(D)}{n} \cdot \int_T \|w\|^2\, dw \cdot \sum_{i=1}^{n} a_i^2 \geq \frac{\det(D)}{4} \cdot \int_T \|w\|^2\, dw.$$

In order to estimate $\int_T \|w\|^2\, dw$ from below, we recall the Stirling formula in the form (see [1])

$$\frac{t^t}{e^t}\sqrt{2\pi\, t} < \Gamma(t+1) < \frac{t^t}{e^t}\sqrt{2\pi(t+1)},$$

which in turn yields

$$V(T) = \frac{\sqrt{n+1}}{\Gamma(n+1)}\left(\frac{n+1}{n}\right)^{n/2} > \frac{3}{2}\cdot\frac{e^n}{n^n}\cdot\frac{1}{\sqrt{2\pi}};$$

$$V(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} < \frac{1}{\sqrt{2}}\cdot\frac{e^{n/2}\pi^{n/2}}{n^{n/2}}\cdot\frac{1}{\sqrt{2\pi}}.$$

It follows by considering the part of T outside $\sqrt{\frac{e}{\pi n}}\, B^n$ that

$$\int_T \|w\|^2\, dw \geq \left(1 - \frac{2}{3\sqrt{2}}\right)\frac{e}{\pi n}\cdot V(T) > \frac{4}{9n} V(T).$$

Therefore we conclude (4.3) by

$$\int_{S-c} \|x\|^2\, dx \geq \frac{\det D}{9n}\cdot V(T) = \frac{1}{9n} V(S),$$

which in turn completes the proof of Theorem 2.2. ∎

5 Some Preliminary Observations Concerning Theorem 2.1

We assume the dimension satisfies $n \geq 3$ for the whole section. The aim of the section is first to outline the basic idea of the proof of Theorem 2.1, and then to provide a "raw" form (see Lemma 5.1). Finally we will prove Lemma 5.2, which helps to find a suitable congruent copy of a given patch on S^{n-1}.

When determining the asymptotics of the volume, surface, and mean width difference, we will replace the optimal convex bodies in $\mathcal{F}_r^n$ by polytopes with the help of Lemma 3.1. After fixing a small $\varepsilon > 0$, we need estimates up to a factor $1 + O(\varepsilon)$ for any $r > 1$ very close to 1. Since any facet of the extremal bodies is of diameter at most $2\sqrt{r^2-1}$, we will consider patches of size $\frac{\sqrt{r-1}}{\varepsilon}$. A very useful property of configurations in $\mathbb{E}^{n-1}$ is that they can be dilated, hence we transfer the integrals over S^{n-1} to integrals over $\mathbb{E}^{n-1}$. In addition, in the cases of volume and surface area, we substitute the patches on S^{n-1} by patches on paraboloids because paraboloids better suit dilation in $\mathbb{E}^{n-1}$.

In this section we prove two auxiliary statements, Lemma 5.1, which is a raw form of Theorem 2.1, and Lemma 5.2, which allows choosing suitable patches on S^{n-1}. Let $r \in (1,2)$. We write $\pi_{S^{n-1}}$ to denote the radial projection into S^{n-1}, hence if $F \subset rB^n$ is a compact convex set with $\operatorname{aff} F \cap \operatorname{int} B^n = \emptyset$, then for any $x, y \in F$,

$$\|\pi_{S^{n-1}}(x) - \pi_{S^{n-1}}(y)\| \le \|x - y\| \le r^2 \cdot \|\pi_{S^{n-1}}(x) - \pi_{S^{n-1}}(y)\|.$$

Given a polytope $P \in \widetilde{\mathcal{F}}_r^n$, let $F_1, \dots, F_k$ be the facets of P. For $i = 1, \dots, k$, we write $x_i \in S^{n-1}$ to denote the unit exterior normal to F_i, and ν_i to denote the distance of $\operatorname{aff} F_i$ and B^n, moreover we define $z_i = (1+\nu_i)x_i \in \operatorname{aff} F_i$. If $y \in F_i$ and $x = \pi_{S^{n-1}}(y)$, then $\|y - x\| = \frac{1+\nu_i}{\langle x, x_i \rangle} - 1$, therefore the formula (6.3) proved by J. R. Sangwine-Yager [17] with $X = S^{n-1}$ and $Y = \partial P$ yields

$$\begin{aligned} V(P) - V(B^n) &= \frac{1}{n} \sum_{i=1}^{k} \int_{\pi_{S^{n-1}}(F_i)} \frac{(1+\nu_i)^n}{\langle x, x_i \rangle^n} - 1 \, dx \\ &= \sum_{i=1}^{k} \int_{F_i} \Big(\frac{1}{2} \|x - z_i\|^2 + \nu_i \Big) \, dx + O((r-1)^2). \end{aligned} \tag{5.1}$$

Concerning the mean width, let $v_1, \dots, v_l \in S^{n-1}$ be the points such that $rv_1, \dots, rv_l$ are the vertices of P. We write Q to denote the polytope determined by the tangent hyperplanes at $v_1, \dots, v_l \in S^{n-1}$, and G_j to denote the facet of Q containing v_j for $j = 1, \dots, l$. Thus

$$\begin{aligned} M(P) - M(B^n) &= \frac{2}{S(B^n)} \sum_{j=1}^{l} \int_{\pi_{S^{n-1}}(G_j)} \langle x, rv_j \rangle - 1 \, dx \\ &= 2(r-1) - \frac{1}{S(B^n)} \sum_{j=1}^{l} \int_{\pi_{S^{n-1}}(G_j)} \|x - v_j\|^2 \, dx \\ &\quad + O((r-1)^2). \end{aligned} \tag{5.2}$$

Let us show that the orders of $V(P_r^n) - V(B^n)$, $S(Q_r^n) - S(B^n)$, and $M(W_r^n) - M(B^n)$ are all $r - 1$.

Lemma 5.1 *If $1 < r < r_0$, then*

$$\frac{c_1}{n} \cdot S(B^n)(r-1) < V(P_r^n) - V(B^n) < \frac{c_2 \ln n}{n} \cdot S(B^n)(r-1),$$

$$c_1 \cdot S(B^n)(r-1) < S(Q_r^n) - S(B^n) < c_2 \ln n \cdot S(B^n)(r-1),$$

$$\frac{c_1}{n} \cdot (r-1) < M(W_r^n) - M(B^n) < \frac{c_2 \ln n}{n} \cdot (r-1),$$

where $c_1, c_2 > 0$ are absolute constants, and $r_0 > 1$ depends on n.

Proof To prove the upper bounds, we start with the mean width. Since $M(W_r^n) \le M(rB^n) = M(B^n) + 2(r-1)$, we may assume that n is large. For $v \in S^{n-1}$ and $\varphi \in (0, \pi/2)$, we define $B(v, \varphi) = \{x \in S^{n-1} : \langle v, x\rangle \ge \cos\varphi\}$. Projecting orthogonally to the tangent hyperplane at v shows that

$$(5.3) \qquad \mathcal{H}^{n-1}(B^{n-1}) \sin^{n-1}\varphi < \mathcal{H}^{n-1}(B(v,\varphi)) < \mathcal{H}^{n-1}(B^{n-1}) \frac{\sin^{n-1}\varphi}{\cos\varphi}.$$

Let $\psi = \arccos 1/r$. According to K. Böröczky, Jr. and G. Wintsche [7], there exists a covering of S^{n-1} by spherical balls $B(v_1, \psi), \dots, B(v_l, \psi)$ such that

$$(5.4) \qquad \sum_{j=1}^{l} \mathcal{H}^{n-1}(B(v_j, \psi)) < 400 n \ln n \cdot S(B^n) < n^2 \cdot S(B^n).$$

Let P be the convex hull of $rv_1, \dots, rv_l$. Since for any $x \in S^{n-1}$ there exists v_j with $\langle rv_j, x\rangle \ge 1$, we deduce that $B^n \subset P$, hence $P \in \mathcal{F}_r^n$. In the following we use the notation of (5.2), and define

$$\Omega = \bigcup_{j=1,\dots,l} B\Big(v_j, \Big(1 - \frac{4\ln n}{n}\Big)\psi\Big).$$

Since $(1 - \frac{4\ln n}{n})^{n-1} < 2/n^4$ for large n, we deduce by (5.3) and (5.4) that if r is close to 1, then

$$(5.5) \qquad \mathcal{H}^{n-1}(\Omega) < \frac{3}{n^2} \cdot S(B^n).$$

We have $\psi \sim \sqrt{2(r-1)}$. Therefore if r is close to 1, then

$$\begin{aligned} 1 - \cos\Big[(1 - \Big(\frac{4\ln n}{n}\Big)\psi\Big] &\ge \Big(1 - \frac{4\ln n}{n}\Big) \cdot \frac{1}{2}\Big[\Big(1 - \frac{4\ln n}{n}\Big)\psi\Big]^2 \\ &\ge \Big(1 - \frac{4\ln n}{n}\Big)^4 (r-1). \end{aligned}$$

In particular if $x \in \pi_{S^{n-1}}(G_j)\backslash\Omega$, then

$$\|x - v_j\|^2 \ge 2 \cdot \Big(1 - \frac{4\ln n}{n}\Big)^4 (r-1) \ge \Big(2 - \frac{32\ln n}{n}\Big) \cdot (r-1).$$

Therefore we conclude by (5.2) and (5.5) that if r is close to 1, then

$$M(P) - M(B^n) < 2(r-1) - \Big(1 - \frac{3}{n^2}\Big)\Big(2 - \frac{32\ln n}{n}\Big)(r-1) + \tfrac{1}{n}(r-1).$$

In particular, if n is large enough and $r \in (1, r_0)$ for suitable $r_0 > 1$ depending on n, then

$$M(P) - M(B^n) < \frac{33\ln n}{n} \cdot (r-1).$$

This settles the case of the mean width. In addition, the upper bounds of Lemma 5.1 in the cases of surface area and volume follow from the consequences

$$\frac{S(P)}{S(B^n)} \le \left(\frac{M(P)}{M(B^n)}\right)^{n-1} \quad \text{and} \quad \frac{V(P)}{V(B^n)} \le \left(\frac{M(P)}{M(B^n)}\right)^{n}$$

of the Alexander–Fenchel inequality for mixed volumes (see (1.1) and [19]).

To prove the lower bounds, we first consider the case of the volume. According to Lemma 3.1, it is sufficient to prove the lower bound for any polytope $P \in \mathcal{F}_r^n$ with $\operatorname{ext} P \subset rS^{n-1}$ where we use the notation of (5.1) for P. It is enough to show that for each F_i,

$$\int_{F_i} \left(\tfrac{1}{2}\|x - z_i\|^2 + \nu_i\right) dx > \frac{\tilde{c}}{n} \cdot \mathcal{H}^{n-1}(\pi_{S^{n-1}}(F_i)) \cdot (r-1) \tag{5.6}$$

where $\tilde{c}$ is a positive absolute constant. If $\nu_i \ge \frac{r-1}{n}$, then (5.6) readily holds. Otherwise $\operatorname{aff} F_i$ intersects B^n in an $(n-1)$-ball B_i of radius larger than $\sqrt{r-1}$. Since the vertices of F_i lie on ∂B_i, Theorem 2.2 completes the proof of (5.6), and in turn of the lower bound in Lemma 5.1 in the case of the volume. Finally the cases of surface area and the mean width follow from the Alexander–Fenchel inequality for mixed volumes (see (1.1) and [19]) in the form

$$\frac{S(Q_r^n)}{S(B^n)} \ge \left(\frac{V(Q_r^n)}{V(B^n)}\right)^{\frac{n-1}{n}} \quad \text{and} \quad \frac{M(W_r^n)}{M(B^n)} \ge \left(\frac{V(W_r^n)}{V(B^n)}\right)^{\frac{1}{n}},$$

and the inequalities $V(Q_r^n) \ge V(P_r^n)$ and $V(W_r^n) \ge V(P_r^n)$. ∎

An essential step of the arguments for all the three quermassintegrals is to find the right copy of a given patch on S^{n-1}. Let us recall that $SO(n)$ denotes the group of orientation preserving orthogonal transformations of $\mathbb{E}^n$ (see [19]).

Lemma 5.2 *If f is a bounded measurable function on S^{n-1} and $X \subset S^{n-1}$ is measurable with $\mathcal{H}^{n-1}(X) > 0$, then there exist $g_1, g_2 \in SO(n)$ such that*

$$\int_{g_1 X} f(x)\,dx \le \frac{\mathcal{H}^{n-1}(X)}{S(B^n)} \cdot \int_{S^{n-1}} f(x)\,dx \le \int_{g_2 X} f(x)\,dx.$$

Proof We write μ_n to denote the (invariant) Haar measure on $SO(n)$ normalized in a way such that $\mu_1(SO(1)) = 2\pi$, and for any measurable $Z \subset S^{n-1}$ and $x \in S^{n-1}$,

$$\mu_n\{g \in SO(n) : g^{-1}x \in Z\} = \mu_{n-1}(SO(n-1)) \cdot \mathcal{H}^{n-1}(Z).$$

In addition we write χ_Z to denote the characteristic function of a set $Z \subset S^{n-1}$. For $g \in SO(n)$, we define $h(g) = \int_{gX} f(x)\,dx = \int_{S^{n-1}} \chi_X(g^{-1}x) \cdot f(x)\,dx$. It follows by the Fubini theorem that

$$\begin{aligned}\int_{SO(n)} h(g)\,d\mu_n(g) &= \int_{S^{n-1}} \int_{SO(n)} \chi_X(g^{-1}x) \cdot f(x)\,d\mu_n(g)dx \\ &= \mu_{n-1}(SO(n-1)) \cdot \mathcal{H}^{n-1}(X) \cdot \int_{S^{n-1}} f(x)\,dx.\end{aligned}$$

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Therefore there exist $g_1, g_2 \in SO(n)$ satisfying

$$h(g_1) \leq \frac{\mathcal{H}^{n-1}(X)}{\mathcal{H}^{n-1}(S^{n-1})} \cdot \int_{S^{n-1}} f(x)\,dx \leq h(g_2). \qquad \blacksquare$$

6 Convex Hyper Surfaces

We will consider patches on the boundary of convex bodies. We say that an $X \subset \mathbb{E}^n$ is a *convex hypersurface* if $\operatorname{conv} X$ is closed with non-empty interior and contains X in its boundary, and X is the closure of its relative interior with respect to the boundary of $\operatorname{conv} X$. Moreover, the relative boundary $\operatorname{relbd} X$ of X is of $(n-1)$-measure zero.

We write $\operatorname{relint} X$ to denote the relative interior of X, and $u_X(x)$ to denote some exterior unit normal at $x \in \operatorname{relint} X$. We note that $u_X(x)$ is unique for all $x \in \operatorname{relint} X$ but of a set of $(n-1)$-measure zero. When integrating over X, we always do it with respect to $\mathcal{H}^{n-1}(\,\cdot\,)$. If the closest point x of $\operatorname{conv} X$ to some y lies in X, then we write $\pi_X(y) = x$. We note that if $\pi_X(y)$ and $\pi_X(y')$ are well defined, then

$$\|\pi_X(y) - \pi_X(y')\| \leq \|y - y'\|. \tag{6.1}$$

If the convex hypersurface $Y \subset \mathbb{E}^n$ is the union of $F_1, \dots, F_k$ such that each F_i is a Jordan measurable subset of some hyperplane and has positive $(n-1)$-measure, and $\operatorname{aff} F_1, \dots, \operatorname{aff} F_k$ are pairwise different, then we say that a Y is a *convex piecewise linear hypersurface*, and call $F_1, \dots, F_k$ the facets of Y.

For certain calculations it is useful to consider patches as graphs of functions. We think $\mathbb{E}^n$ as $\mathbb{E}^{n-1} \times \mathbb{R}$ where $x = (y, t)$ is the point of $\mathbb{E}^n$ corresponding to $y \in \mathbb{E}^{n-1}$ and $t \in \mathbb{R}$, and define $B^{n-1} = B^n \cap \mathbb{E}^{n-1}$. If $\Psi \subset \mathbb{E}^{n-1}$ has non-empty interior in $\mathbb{E}^{n-1}$, and $\theta\colon \Psi \to \mathbb{R}$ is any function, then the graph of θ is

$$\Gamma(\theta) = \{(y, \theta(y)) : y \in \Psi\} \subset \mathbb{E}^n.$$

In particular if Ψ and θ are convex, then $\Gamma(\theta)$ is a convex hypersurface.

We say that a convex hypersurface X is C^2 if any point of X has a relatively open neighbourhood on X that is congruent to the graph of some C^2 function. In order to define the principle curvatures at $x_0 \in \operatorname{relint} X$, we may assume that $\mathbb{E}^{n-1}$ is the tangent hyperplane to X at $x_0 = (y_0, 0)$, and a neighbourhood $X_0 \subset X$ of x_0 is the graph of a C^2 function θ on an open convex $\Psi \subset \mathbb{E}^{n-1}$. Then the *principle curvatures* $\kappa_1(x_0), \dots, \kappa_{n-1}(x_0)$ of X at x_0 are the eigenvalues of the symmetric matrix corresponding to the quadratic form representing the second derivative of θ at y_0. For $x \in X$, we define $\sigma_0(x) = 1$, and write $\sigma_j(x)$ to denote the j-th symmetric polynomial of the principal curvatures for $j = 1, \dots, n-1$; namely,

$$\sigma_j(x) = \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} \kappa_{i_1}(x) \cdots \kappa_{i_j}(x).$$

For the rest of the section, let X be a convex C^2 hypersurface, and let Y be a convex hypersurface such that π_X is defined on Y and is injective with $X = \pi_X(Y)$. Moreover, there exists $\eta > 0$ such that

$$\langle u_X(\pi_X(y)), u_Y(y)\rangle \geq \eta \text{ for any } y \in \operatorname{relint} Y. \tag{6.2}$$

It follows by (6.1) that $\pi_X(Y)$ is also a convex hypersurface with $\mathcal{H}^{n-1}(\pi_X(Y)) \leq \mathcal{H}^{n-1}(Y)$. In addition if $Z \subset \operatorname{relint} \pi_X(Y)$ is a convex hypersurface, then the subset Z' of Y satisfying $\pi_X(Z') = Z$ is a convex hypersurface by (6.2).

If $\pi_X(y) = x$ for $y \in \operatorname{relint} Y$, then we write $y = x_Y$ and define $r_{X,Y}(x) = \|y - x\|$. We define $\Omega(X, Y)$ to be the union of all segments $\operatorname{conv}\{y, \pi_X(y)\}$ for $y \in Y$, which satisfies

$$V(\Omega(X,Y)) = \sum_{j=1}^{n} \frac{1}{j} \int_X r_{X,Y}(x)^j \cdot \sigma_{j-1}(x)\, dx \tag{6.3}$$

according to J. R. Sangwine-Yager [17]. In addition the method of K. Böröczky, Jr. and M. Reitzner [5] yields the following formula for the difference of the $(n-1)$-measure of patches.

Lemma 6.1 *Using the notation as above,*

$$\mathcal{H}^{n-1}(Y) - \mathcal{H}^{n-1}(X) = \int_X \Big(\frac{1}{\langle u_X(x), u_Y(x_Y)\rangle} - 1 \Big)\, dx + \sum_{j=1}^{n-1} \int_X r_{X,Y}(x)^j \frac{\sigma_j(x)}{\langle u_X(x), u_Y(x_Y)\rangle}\, dx.$$

Proof For small $\mu > 0$, we write Ω_μ to denote the family of points $z \in \mathbb{E}^n$ such that the closest point of $\operatorname{conv} Y$ to z lies in $\operatorname{relint} Y$, and $\|\pi_Y(z) - z\| \leq \mu$. Next let X_μ be the family of points $x \in X$ with $d(x, \operatorname{relbd} X) \geq 2\mu$, and let $Y_\mu \subset Y$ satisfy $\pi_X(Y_\mu) = X_\mu$. For any $x \in X_\mu$, there exists a unique boundary point $z \in \Omega_\mu$ with $d(z, Y) = \mu$ and $\pi_X(z) = x$, and we write Z_μ to denote the family of all such z as x runs through X_μ. Now $\operatorname{relbd} X_\mu$ might be positive for some but only a countable family $\{\mu_i\}$ of $\mu > 0$. Therefore X_μ and Z_μ are convex hypersurfaces for $\mu > 0$, $\mu \notin \{\mu_i\}$, with $\pi_X(Z_\mu) = X_\mu$. In addition if $x_Y \in Y$ is a smooth point of Y for $x \in X_\mu$, then $r_{X_\mu, Z_\mu}(x) \leq r_{X,Y}(x) + \frac{\mu}{\eta}$ (*cf.* (6.2)), and

$$r_{X_\mu, Z_\mu}(x) = r_{X,Y}(x) + \frac{\mu}{\langle x, x_Y\rangle} + o(\mu) \text{ as } \mu \text{ tends to zero.}$$

Since the π_X image of singular points of Y are of $(n-1)$-measure zero, we deduce by (6.3) and as X and Y are Jordan measurable that

$$\mathcal{H}^{n-1}(Y) = \lim_{\substack{\mu \to 0 \\ \mu \notin \{\mu_i\}}} \frac{V(\Omega_\mu)}{\mu} = \lim_{\substack{\mu \to 0 \\ \mu \notin \{\mu_i\}}} \frac{V(\Omega(X_\mu, Z_\mu)) - V(\Omega(X_\mu, Y_\mu))}{\mu}$$
$$= \sum_{j=1}^{n} \int_X r_{X,Y}(x)^{j-1} \cdot \frac{\sigma_{j-1}(x)}{\langle x, x_Y\rangle}\, dx.$$

In turn we conclude Lemma 6.1. ∎

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7 Near Spherical Convex Hyper Surfaces

For $\varepsilon \in (0, \frac{1}{16})$, let $\rho \in (0, \varepsilon^2)$, and let $\Psi \subset \sqrt{\varepsilon} B^{n-1}$ be an $(n-1)$-dimensional convex body with $o \in \operatorname{relint} \Psi$. In addition let θ be a non-negative C^2 function defined on Ψ such that writing l_y to denote the linear form representing the derivative of θ, and q_y to denote the quadratic form representing the second derivative of θ at $y \in \Psi$, we have $\theta(o) = 0$, $l_o(z) = 0$, and $\|z\|^2 \leq q_y(z) \leq (1+\varepsilon) \cdot \|z\|^2$ for $z \in \mathbb{E}^{n-1}$. We define $X' = \Gamma(\theta)$, and write $\kappa_1(x), \dots, \kappa_{n-1}(x)$ to denote the principle curvatures at $x \in \operatorname{relint} X'$. We note that if $y \in \Psi$ and $x = (y, \theta(y))$, then

$$u_{X'}(x) = (1 + \|l_y\|^2)^{-1/2} \cdot (l_y, -1). \tag{7.1}$$

We deduce, by the Taylor formula for $y, z \in \Psi$, $x = (y, \theta(y))$,

$$\begin{aligned} \theta(z) - \theta(y) - l_y(z-y) &= \tfrac{1}{2} q_{y+t(z-y)}(z-y) \text{ for } t \in (0,1), \qquad (7.2)\\ &= \tfrac{1}{2}\|z-y\|^2 + O(\varepsilon)\|y-z\|^2, \\ \|l_z - l_y\| &= \|z-y\| + O(\varepsilon)\|z-y\|, \\ \kappa_i(x) &= 1 + O(\varepsilon),\ i = 1, \dots, n-1. \qquad (7.3) \end{aligned}$$

Now for any $x \in X'$, X' can be thought as the graph of a suitable C^2 function defined on the tangent hyperplane at x, hence the discussion above and (7.1) show that if $x, x' \in \operatorname{relint} X'$, then

$$\langle u_{X'}(x), u_{X'}(x') \rangle = 1 - \tfrac{1}{2}\|x - x'\|^2 + O(\varepsilon)\|x - x'\|^2. \tag{7.4}$$

Next let $X \subset X'$ be a convex hypersurface such that

$$d(x, \operatorname{relbd} X') \geq 4\sqrt{\rho} \text{ for } x \in X.$$

In addition let Y be a convex hypersurface such that $\pi_{X'}$ is defined on Y and is injective with $X = \pi_{X'}(Y)$, and $\inf_{y \in \operatorname{relint} Y} \langle u_X(\pi_X(y)), u_Y(y) \rangle > 0$. Therefore we may use the notation of Section 6. In particular we assume that

$$r_{X,Y}(x) \leq 2\rho \text{ for } x \in \operatorname{relint} X. \tag{7.5}$$

Naturally (6.3) and Lemma 6.1 are very general, and we provide three types of estimates based on them which will be useful in the later part of the paper. We write $\xi = (o, -1)$ to denote the *downwards* unit normal to $\mathbb{E}^{n-1}$. Since all eigenvalues of q_y are at most 2 for any $y \in \Psi$, there is a ball of radius $1/2$ touching X from inside at any $x \in X$ such that the ball intersects X only in x. It follows by (7.5) that

$$\langle u_X(x), u_Y(x_Y) \rangle^{-1} \leq 1 + 4\rho, \tag{7.6}$$

which in turn yields

$$\langle \xi, u_{X'}(x) \rangle = 1 + O(\varepsilon) \text{ for } x \in \operatorname{relint} X'. \tag{7.7}$$

The first type of estimate is a rather rough one; namely, (7.3), (7.5) and (7.6) imply

$$V(\Omega(X,Y)) = O(\rho)\cdot \mathcal{H}^{n-1}(X), \tag{7.8}$$

$$\mathcal{H}^{n-1}(Y) - \mathcal{H}^{n-1}(X) = O(\rho)\cdot \mathcal{H}^{n-1}(X). \tag{7.9}$$

The second type of estimate is needed when Y is a convex piecewise linear hypersurface. We write $F_1,\dots,F_k$ to denote the facets of Y, and $v_1,\dots,v_k$ to denote the corresponding exterior unit normals. We assume that for $i=1,\dots,k$, $v_i = u_{X'}(x_i)$ for some $x_i \in X'$, and $x_i + \nu_i v_i \in \operatorname{aff} F_i$ for some $\nu_i \geq 0$. Since there exists a ball of radius 2 that touches X at x_i and contains X if $x \in \pi_X(F_i)$, then the condition $r_{X,Y}(x) \leq 2\rho$ yields that $\|x - x_i\| \leq 4\sqrt{\rho}$, hence

$$r_{X,Y}(x) = \nu_i + \tfrac{1}{2}\,\|x-x_i\|^2 + O(\varepsilon\rho)$$

$$\langle u_X(x), u_Y(x_Y)\rangle^{-1} = \langle u_X(x), u_{X'}(x_i)\rangle^{-1} = 1 + \frac{1}{2}\|x-x_i\|^2 + O(\varepsilon\rho).$$

We conclude by (6.3) and Lemma 6.1 that

$$V(\Omega(X,Y)) = \sum_{i=1}^{k}\int_{\pi_X(F_i)}\Big(\nu_i + \frac{1}{2}\|x-x_i\|^2\Big)\,dx + O(\varepsilon\rho)\mathcal{H}^{n-1}(X),$$

$$\mathcal{H}^{n-1}(Y) - \mathcal{H}^{n-1}(X) = \sum_{i=1}^{k}\int_{\pi_X(F_i)}\big((n-1)\nu_i + \frac{n}{2}\|x-x_i\|^2\big)\,dx + O(\varepsilon\rho)\,\mathcal{H}^{n-1}(X). \tag{7.10}$$

Finally Lemma 7.2 provides the third type of estimate, which allows us to shift between patches on spheres and on paraboloids. Its proof uses the following statement.

Proposition 7.1 *Let $z_1, z_2 \in \mathbb{E}^{n-1}$ such that $\|z_2 - z_1\| \leq \tau$ for some $\tau > 0$, and let Y be the graph of a convex positive function on $z_1 + 2\tau B^{n-1}$ such that $\langle u_Y(y), \xi\rangle \geq \sqrt{3}/2$ for $y \in Y$ where $\xi = (o,-1)$ as above. If $y_1, y_2 \in Y$ satisfy that $\langle \frac{z_i - y_i}{\|z_i - y_i\|}, \xi\rangle \geq \sqrt{3}/2$ for $i = 1,2$ then*

$$\|y_1 - y_2\| \leq 2\cdot[\|z_1 - z_2\| + \|z_1 - y_1\|\cdot\angle(z_1 - y_1, o, z_2 - y_2)].$$

Proof We define $y_1' \in Y$ by the property that the vectors $z_1 - y_1'$ and $z_2 - y_2$ are parallel, and prove

$$\|y_1 - y_1'\| \leq 2\,\|z_1 - y_1\|\cdot \sin\angle(y_1, z_1, y_1'). \tag{7.11}$$

Let σ be the arc that is the intersection of the triangle $y_1 z_1 y_1'$ and Y, and let y be the point of σ farthest from the segment $y_1 y_1'$. Then the tangent line to σ at y is parallel to the line $y_1 y_1'$, hence $\langle u_Y(y), \xi\rangle \geq \sqrt{3}/2$ yields that the angle of $y_1' - y_1$ and ξ is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. Thus the angle of the triangle $z_1y_1y_1'$ at y_1' is between $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, therefore the law of sines implies (7.11).

Now an argument as above shows that $\|y_2 - y_1'\| \leq 2\,\|z_2 - z_1\|$, which in turn yields Proposition 7.1 by (7.11). ■

Lemma 7.2 *Given $\varepsilon \in (0, \varepsilon_0)$ and $\rho \in (0, \varepsilon^8)$ where $\varepsilon_0 \in (0, \frac{1}{16})$ depends only on n, let the convex functions h, f_1, f_2 on $\frac{(20\sqrt{\rho})}{\varepsilon} B^{n-1}$ satisfy that $f_2(o) = 0$, $f_2'(o) = 0$, f_1 and f_2 are C^2, and if $y \in \frac{(3\sqrt{\rho})}{\varepsilon} B^{n-1}$. Then on the one hand,*

$$h(y) \leq f_1(y) \leq f_2(y) \leq h(y) + 2\rho \quad \text{and} \quad f_1(y) \geq 0,$$

and on the other hand, writing $q_{i,y}$ to denote the quadratic form representing the second derivative of f_i at y for $i = 1, 2$, we have

$$\|z\|^2 \leq q_{i,y}(z) \leq (1 + \varepsilon^8) \cdot \|z\|^2 \quad \text{for } z \in \mathbb{E}^{n-1}.$$

We define $Y = \Gamma(h)$ and $X_i = \Gamma(f_i)$, $i = 1, 2$ (see Figure 1). For a compact convex $C \subset \mathbb{E}^{n-1}$ satisfying $\frac{\sqrt{\rho}}{4\varepsilon} B^{n-1} \subset C \subset \frac{2\sqrt{\rho}}{\varepsilon} B^{n-1}$ and for $i = 1, 2$, we write $\widetilde{X}_i = \pi_{X_i}(C)$ and Y_i to denote the subset of Y satisfying $\widetilde{X}_i = \pi_{X_i}(Y_i)$. Then

$$\mathcal{H}^{n-1}(\widetilde{X}_i) = \mathcal{H}^{n-1}(C) + O(\varepsilon) \cdot \mathcal{H}^{n-1}(C) \text{ for } i = 1, 2, \tag{7.12}$$

$$\mathcal{H}^{n-1}(Y_1) - \mathcal{H}^{n-1}(\widetilde{X}_1) = \mathcal{H}^{n-1}(Y_2) - \mathcal{H}^{n-1}(\widetilde{X}_2) + O(\varepsilon\rho)\mathcal{H}^{n-1}(C), \tag{7.13}$$

$$V(\Omega(\widetilde{X}_1, Y_1)) = V(\Omega(\widetilde{X}_2, Y_2)) + O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C). \tag{7.14}$$

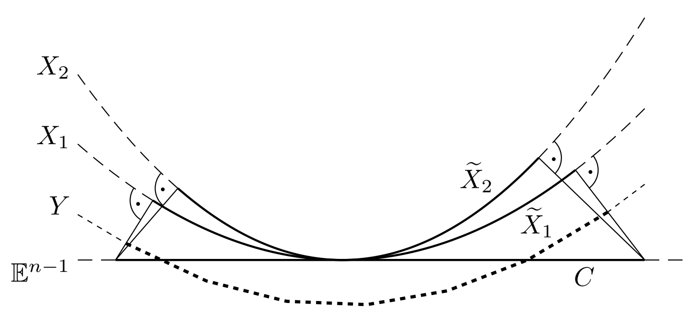

Figure 1

Proof It follows by (7.7) that if ε_0 is sufficiently small, then $\langle u_{X_i}(x), \xi\rangle \geq \frac{\sqrt{3}}{2}$ for any $x \in \operatorname{relint} X_i$. In addition if $y = (z, h(z))$ for $z \in C$ and u is an exterior unit normal to Y at y, then $d(y, X_1) \leq \rho$ and (7.2) yield that there exists $x \in X_1 \cap (y + 4\sqrt{\omega\rho}\, B^d)$ with $u = u_{X_1}(x)$, hence $\langle u, \xi\rangle \geq \frac{\sqrt{3}}{2}$, as well. In addition the conditions on h, f_1, f_2 and applying (7.2) to f_1, f_2 yield that

$$h(z) > 0 \quad \text{if } z \in C\backslash(\tfrac{1}{2}C), \tag{7.15}$$

$$f_2(z) \leq \frac{4\rho}{\varepsilon^2} \quad \text{if } z \in C, \tag{7.16}$$

$$f_2(z) - f_1(z) \leq 4\varepsilon^6\rho \quad \text{if } z \in C. \tag{7.17}$$

Therefore combining (7.9), (7.16), and $\rho/\varepsilon^2 < \varepsilon$ leads to (7.12). Moreover, writing $\gamma_1'(z) = X_1 \cap \operatorname{conv}\{z, \pi_{X_2}(z)\}$ for $z \in C$, we deduce by (7.9) and (7.17) that if ε_0 is small enough, then

$$\mathcal{H}^{n-1}(\gamma_1'(C)) - \mathcal{H}^{n-1}(\widetilde{X}_2) = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C). \tag{7.18}$$

Next we prove

$$\mathcal{H}^{n-1}(Y_1) - \mathcal{H}^{n-1}(Y_2) = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C). \tag{7.19}$$

Let $z \in \partial C$. For $i = 1, 2$, $\gamma_i(z) = Y \cap \operatorname{conv}\{z, \pi_{X_i}(z)\}$ exists by (7.15), hence the relative boundary of Y_i is $\gamma_i(\partial C)$. It follows by (7.16) that $\|z - \gamma_i(z)\| \leq \frac{4\rho}{\varepsilon^2}$, and the discussion above shows that

$$\left\langle \frac{z - \gamma_i(z)}{\|z - \gamma_i(z)\|}, \xi \right\rangle \geq \frac{\sqrt{3}}{2}.$$

Next we define $x_i = \pi_{X_i}(z)$. Since $d(\gamma_1'(z), X_2) \leq 4\varepsilon^6\rho$ by (7.17), and there exists a ball of radius $\frac{1}{2}$ touching X_2 from inside at x_2, we deduce that the angle α_2 of $u_{X_2}(x_2)$ and $u_{X_1}(\gamma_1'(z))$ is at most $12\varepsilon^3\sqrt{\rho}$. It follows that $\|\gamma_1'(z) - x_1\| = O(\varepsilon^3\sqrt{\rho})\|\gamma_1'(z) - z\| = O(\varepsilon\rho^{\frac{3}{2}})$, hence the angle α_1 of $u_{X_1}(x_1)$ and $u_{X_1}(\gamma_1'(z))$ is $O(\varepsilon\rho^{\frac{3}{2}})$ according to (7.4). Therefore choosing ε_0 small enough, we have

$$\angle(z - \gamma_1(z), o, z - \gamma_2(z)) \leq \alpha_1 + \alpha_2 = O(\varepsilon^3\sqrt{\rho}) < \tfrac{1}{8}\varepsilon^2\sqrt{\rho}.$$

In particular, it follows by Proposition 7.1 and $\|\gamma_1(z) - z\| \leq \frac{4\rho}{\varepsilon^2}$ that

$$\|\gamma_2(z) - \gamma_1(z)\| \leq \rho^{3/2}, \tag{7.20}$$

hence (7.19) is a consequence of

$$\mathcal{H}^{n-1}\left[Y \cap \left(\gamma_1(\operatorname{relbd} C) + \rho^{\frac{3}{2}}\, B^n\right)\right] = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C). \tag{7.21}$$

To prove (7.21), let $\tau = \frac{\sqrt{\rho}}{4\varepsilon}$, and let $z_1, \dots, z_k \in \partial C$ be a maximal family of points with the property that $\|z_i - z_j\| \geq 3\rho^{\frac{3}{2}}$ for $i \neq j$. Since $z_i + \rho^{\frac{3}{2}}\, B^{n-1}$ are pairwise

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disjoint for $i = 1, \dots, k$, and each is contained in the difference of $(1 + \frac{\rho^{3/2}}{\tau})C$ and $(1 - \frac{\rho^{3/2}}{\tau})C$, we deduce that

$$k = O\Big(\frac{\rho^{3/2}}{\tau}\Big) \cdot \mathcal{H}^{n-1}(C) \cdot (\rho^{3/2})^{-(n-1)} = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C) \cdot (\rho^{3/2})^{-(n-1)}. \tag{7.22}$$

Now let $y \in Y$ satisfy $\|y - \gamma_1(z)\| \le \rho^{3/2}$ for some $z \in \partial C$. There exists some z_i such that $\|z_i - z\| \le 3\rho^{3/2}$, hence $\|\pi_{X_1}(z_i) - \pi_{X_1}(z)\| \le 3\rho^{3/2}$. In particular (7.4) implies that the angle between $z_i - \gamma_1(z_i)$ and $z - \gamma_1(z)$, which is the angle between $u_{X_1}(\pi_{X_1}(z_i))$ and $u_{X_1}(\pi_{X_1}(z))$, is at most $4\rho^{3/2}$ (after choosing ε_0 small enough). Thus Proposition 7.1 yields that $\|\gamma_1(z_i) - \gamma_1(z)\| \le 7\rho^{3/2}$, hence $\|\gamma_1(z_i) - y\| \le 8\rho^{3/2}$. We deduce by (7.22) that

$$\begin{aligned}\mathcal{H}^{n-1}\big[Y \cap \big(\gamma_1(\partial C) + \rho^{\frac{3}{2}} B^n\big)\big] &\le \sum_{i=1}^{k} \mathcal{H}^{n-1}\big[Y \cap \big(\gamma_1(z_i) + 8\rho^{\frac{3}{2}} B^n\big)\big] \\ &\le k \cdot S(8\rho^{\frac{3}{2}} B^n) = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C).\end{aligned}$$

We conclude (7.21), and in turn (7.19).

Next applying the argument above to X_1 as Y, γ_1' as γ_2 and π_{X_1} as γ_1, we deduce first the analogue of (7.20), namely,

$$\|\gamma_1'(z) - \pi_{X_1}(z)\| \le \rho^{\frac{3}{2}}, \tag{7.23}$$

and secondly the analogue of (7.19), namely,

$$\mathcal{H}^{n-1}(\gamma_1'(C)) - \mathcal{H}^{n-1}(\widetilde{X}_1) = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C). \tag{7.24}$$

Therefore combining (7.18), (7.19) and (7.24) yields (7.13).

For (7.14), we observe that X_1 cuts $\Omega(\widetilde{X}_2, Y_2)$ into $\Omega' = \Omega(\widetilde{X}_2, \gamma_1'(C))$ and the closure Ω'' of $\Omega(\widetilde{X}_2, Y_2)\backslash\Omega'$. It follows by (7.8) and (7.17) that

$$V(\Omega') = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(\widetilde{X}_2) = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C). \tag{7.25}$$

We deduce by (7.23) and $f_1(y) - h(y) \le 2\rho$ that

$$[\Omega(\widetilde{X}_1, Y_1)\backslash\Omega''] \cup [\Omega''\backslash\Omega(\widetilde{X}_1, Y_1)] \subset \pi_{X_1}(\partial C) + 5\rho B^n.$$

Let $\tilde{z}_1, \dots, \tilde{z}_{\tilde{k}} \in \partial C$ be a maximal system of points in ∂C such that $\|\tilde{z}_i - \tilde{z}_j\| \ge 3\rho$ for $i \ne j$. We deduce using an argument as above

$$\tilde{k} = O\Big(\frac{\rho}{\tau}\Big) \cdot \mathcal{H}^{n-1}(C) \cdot \rho^{-(n-1)} = O(\varepsilon\rho^{3/2}) \cdot \mathcal{H}^{n-1}(C) \cdot \rho^{-n}.$$

Let $x \in \mathbb{E}^n$ satisfy $\|x - \pi_{X_1}(z)\| \le 5\rho$ for $z \in \partial C$. Now there exists $\tilde{z}_i \in \partial C$ such that $\|\tilde{z}_i - z\| \le 3\rho$, hence $\|x - \pi_{X_1}(\tilde{z}_i)\| \le 8\rho$. It follows that

$$\begin{aligned}\big|V(\Omega(\widetilde{X}_1, Y_1)) - V(\Omega'')\big| &\le \sum_{i=1}^{\tilde{k}} V(\pi_{X_1}(\tilde{z}_i) + 8\rho B^n) \\ &\le \tilde{k} \cdot V(8\rho B^n) = O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C).\end{aligned} \tag{7.26}$$

Since $V(\Omega(\widetilde{X}_2, Y_2)) = V(\Omega') + V(\Omega'')$, combining (7.25) and (7.26) completes the proof of Lemma 7.2. ■

8 Transfer Lemma for Paraboloids for the Cases of Surface Area and Volume

We will transfer integrals between patches on paraboloids and in $\mathbb{E}^{n-1}$ using Lemma 8.1 below. For given $\omega \in [1,2]$, we consider the paraboloid that is the graph of $\varphi_\omega(y) = \frac{\omega}{2}\,\|y\|^2$ on $\mathbb{E}^{n-1}$. The derivative satisfies

$$\|\partial\varphi_\omega(y)\| = \omega\|y\| \leq 2\|y\|, \tag{8.1}$$

hence if $x' = (y', \varphi_\omega(y'))$ and $x'' = (y'', \varphi_\omega(y''))$ satisfy $y', y'' \in tB^{n-1}$ for $t > 0$, then

$$\|y' - y''\| \leq \|x' - x''\| \leq (1 + 2t^2) \cdot \|y' - y''\|. \tag{8.2}$$

Next let $y_1, \dots, y_k \in \mathbb{E}^{n-1}$ and let $\nu_1, \dots, \nu_k \geq 0$. We observe that $l_i(z) = \langle \partial\varphi_\omega(y_i), z - y_i\rangle + \varphi_\omega(y_i)$ is the linear function whose graph is the tangent hyperplane to $\Gamma(\varphi_\omega)$ at $x_i = (y_i, \varphi_\omega(y_i))$, and define $\psi_i(z) = l_i(z) - \nu_i$. In particular for any $z \in \mathbb{E}^{n-1}$, the Taylor formula (see (7.2)) for φ_ω yields

$$\varphi_\omega(z) - \psi_i(z) = \tfrac{\omega}{2}(z - y_i)^2 + \nu_i. \tag{8.3}$$

Let $\Pi_1, \dots, \Pi_k$ be a family of pairwise non-overlapping convex polytopes in $\mathbb{E}^{n-1}$, which cover a convex body $C \subset \mathbb{E}^{n-1}$ in a way such that each $\Pi_i \cap C$ has non-empty interior, and satisfy

$$\tfrac{\omega}{2}\|z - y_i\|^2 + \nu_i \leq \tfrac{\omega}{2}\|z - y_j\|^2 + \nu_j \quad \text{for } z \in \Pi_i \text{ and } j = 1, \dots, k.$$

We define $\psi\colon \bigcup_{i=1}^k \Pi_i \to \mathbb{R}$ by $\psi(z) = \psi_i(z)$ for $z \in \Pi_i$, and observe that $Y = \Gamma(\psi)$ is a convex piecewise linear hypersurface. Let F_i be the graph of ψ above Π_i, hence $F_1, \dots, F_k$ are the facets of Y. We define $X = \pi_{\Gamma(\varphi_\omega)}(C)$, and assume that $i = 1, \dots, k'$ are the indices satisfying that $\pi_{\Gamma(\varphi_\omega)}(F_i)$ intersects X in a set of positive measure for some $k' \leq k$. Let ν_i' denote the distance of x_i from aff F_i for $i \leq k'$.

Lemma 8.1 *We use the notation as above. Let $\varepsilon \in (0, \varepsilon_0)$, and let $\rho \in (0, \varepsilon^{2n^2})$ where $\varepsilon_0 \in (0, \frac{1}{16})$ depends only on n. We assume $\frac{\sqrt{\rho}}{4\varepsilon}B^{n-1} \subset C \subset \frac{2\sqrt{\rho}}{\varepsilon}B^{n-1}$ and $\omega \in [1, 1+\varepsilon]$. Moreover,*

$$\tfrac{\omega}{2}\|z - y_i\|^2 + \nu_i \leq 2\rho \quad \text{if } i = 1, \dots, k \text{ and } z \in \Pi_i. \tag{8.4}$$

If in addition the family $\mathcal{V}$ of the vertices of all Π_i satisfies $\|y - z\| \geq \frac{1}{8}\varepsilon\sqrt{\rho}$ for $y \neq z \in \mathcal{V}$, then for $\eta \in [0,1]$ we have

$$\sum_{i=1}^{k'} \int_{X \cap \pi_{\Gamma(\varphi_\omega)}(F_i)} \eta\nu_i' + \tfrac{1}{2}\|x - x_i\|^2\,dx = \sum_{i=1}^{k} \int_{\Pi_i \cap C} \eta\nu_i + \tfrac{1}{2}\|z - y_i\|^2\,dz + O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C).$$

Moreover, $\mathcal{H}^{n-1}(C) = (1 + O(\varepsilon))\mathcal{H}^{n-1}(X)$, and for $z \in \Pi_i$ and $v = (z, \psi_i(z))$, $i = 1, \dots, k$ we have

$$(1+\varepsilon)^{-1} \cdot d(v, \Gamma(\varphi_\omega)) < \nu_i + \tfrac{1}{2}\,\|z - y_i\|^2 < (1+\varepsilon) \cdot d(v, \Gamma(\varphi_\omega)). \tag{8.5}$$

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Proof We write $\pi_{\mathbb{E}^{n-1}}(\,\cdot\,)$ to denote the orthogonal projection into $\mathbb{E}^{n-1}$. We observe that $\xi = (o, -1)$ is the exterior unit vector to $\Gamma(\varphi_\omega)$ at the origin, and

$$\pi_{\mathbb{E}^{n-1}}(X) \subset C.$$

Let $z \in \Pi_i$, $i = 1, \dots, k$, let $v = (z, \psi_i(z))$, and let $x = \pi_{\Gamma(\varphi_\omega)}(v)$. Combining (8.1), (8.3) and (8.4) yields that $\angle(u_X(x), o, \xi) = O(\frac{\sqrt{\rho}}{\varepsilon})$ and $\|\pi_{\mathbb{E}^{n-1}}(x) - z\| = O(\frac{\rho\sqrt{\rho}}{\varepsilon})$. Since $\rho/\varepsilon^2 < \varepsilon^2$, we deduce $\mathcal{H}^{n-1}(C) = (1 + O(\varepsilon))\mathcal{H}^{n-1}(X)$ and (8.5) for small ε_0. Writing C_i to denote the orthogonal projection of $\pi_{\Gamma(\varphi_\omega)}(F_i) \cap X$ into $\mathbb{E}^{n-1}$ for $i \le k'$, it also follows that $C_i \subset C$, and

$$\delta_H(C_i, \Pi_i \cap C) \le \gamma_0 \cdot \frac{\rho\sqrt{\rho}}{\varepsilon}$$

where $\gamma_0 > 0$ depends only on n. In addition (8.2) and (8.3), $\rho/\varepsilon^2 < \varepsilon$ and $\nu_i' = [1 + O(\rho/\varepsilon^2)] \cdot \nu_i$ imply that

$$\sum_{i=1}^{k'} \int_{\pi(F_i)\cap X} \eta\nu_i' + \tfrac{1}{2}\|x - x_i\|^2\, dx = \sum_{i=1}^{k'} \int_{C_i} \eta\nu_i + \tfrac{1}{2}\|z - y_i\|^2\, dz + O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C).$$

In particular Lemma 8.1 follows from the inequalities

$$\mathcal{H}^{n-1}\Big(C \backslash \big(\bigcup_{i=1}^{k'} C_i\big)\Big) = O(\varepsilon) \cdot \mathcal{H}^{n-1}(C),$$

$$\sum_{i=1}^{k'} \big[\mathcal{H}^{n-1}(\Pi_i \backslash C_i) + \mathcal{H}^{n-1}(C_i \backslash \Pi_i)\big] = O(\varepsilon) \cdot \mathcal{H}^{n-1}(C).$$

Since $d(x, \Gamma(\varphi_\omega)) = O(\rho/\varepsilon^2)$ for $x \in C$ according to (8.1) and (8.3), we deduce

$$C \backslash \big(\bigcup_{i=1}^{k'} C_i\big) \subset \partial C + \gamma_1 \frac{\rho\sqrt{\rho}}{\varepsilon^3} B^{n-1} \subset \partial C + \gamma_1 \sqrt{\rho} B^{n-1}$$

for some positive constant $\gamma_1 \ge 4$ depending only on n. In addition, the diameter of any Π_i is at most $4\sqrt{\rho} \le \gamma_1\sqrt{\rho}$. Therefore to prove Lemma 8.1, it is sufficient to verify the pair of inequalities

$$\mathcal{H}^{n-1}\big(\partial C + \gamma_1 \sqrt{\rho} B^{n-1}\big) = O(\varepsilon)\mathcal{H}^{n-1}(C), \tag{8.6}$$

$$\sum_{\Pi_i \subset \operatorname{relint} C} \mathcal{H}^{n-1}\Big(\Pi_i \cap \big(\partial \Pi_i + \gamma_0 \frac{\rho\sqrt{\rho}}{\varepsilon} B^{n-1}\big)\Big) = O(\varepsilon)\mathcal{H}^{n-1}(C). \tag{8.7}$$

Here (8.6) readily holds by $\frac{\sqrt{\rho}}{4\varepsilon} B^{n-1} \subset C$. We observe that if $x \in \operatorname{relint} \Pi_i$ and $d(x, \partial \Pi_i) = \omega$, then there exists an $(n-2)$-face F such that $x + \omega B^{n-1}$ touches aff F in a point of F. As $\frac{\sqrt{\rho}}{4\varepsilon} B^{n-1} \subset C$, (8.7) follows from the estimate

$$\sum_{\Pi_i \subset \operatorname{relint} C} \sum_{\substack{F \subset \Pi_i \\ (n-2)\text{-face}}} \mathcal{H}^{n-2}(F) \cdot \frac{\rho\sqrt{\rho}}{\varepsilon} = O(\varepsilon) \cdot \Big(\frac{\sqrt{\rho}}{\varepsilon}\Big)^{n-1}.$$

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We write $\mathcal{S}$ to denote the set of $(n-2)$-faces of any Π_i that lies in relint C, and observe that any $F \in \mathcal{S}$ is the $(n-2)$-face of exactly two Π_i. Since each $F \in \mathcal{S}$ is of diameter at most $4\sqrt{\rho}$, we have $\mathcal{H}^{n-2}(F) < O(\sqrt{\rho}^{n-2})$. Therefore writing $\#\mathcal{S}$ to denote the cardinality of $\mathcal{S}$, (8.7) follows if

$$\rho \cdot \#\mathcal{S} = O(\varepsilon^{-(n-3)}). \tag{8.8}$$

The condition on the family $\mathcal{V}$ of the vertices of $\Pi's$ yields that $\#\mathcal{V} = O(\varepsilon^{-2(n-1)})$. We choose $n-1$ vertices for each $F \in \mathcal{S}$ in such a way that the $n-1$ vertices do not lie in any affine $(n-3)$-plane. Thus $\#\mathcal{S}$ is the number of such $(n-1)$-tuples, which is $O(\varepsilon^{-2(n-1)^2})$. Therefore (8.8), and in turn Lemma 8.1 are the consequences of $\rho < \varepsilon^{2n^2}$. ∎

When comparing patches on paraboloids and on the sphere, we need to know how closely paraboloids approximate the sphere. The part of S^{n-1} below $\mathbb{E}^{n-1}$ is the graph of the function $\tilde{\varphi}(y) = -\sqrt{1-\|y\|^2}$ defined on B^{n-1}, and if $y \in \frac{1}{2}B^n$, then

$$-1 + \tfrac{1}{2}\|y\|^2 \leq \tilde{\varphi}(y) \leq -1 + \tfrac{1}{2}\|y\|^2 + \|y\|^4.$$

It follows that if $y \in \frac{1}{2}B^n$, then

$$-1 + \varphi_1(y) \leq \tilde{\varphi}(y) \leq -1 + \varphi_{1+2\,\|y\|^2}(y). \tag{8.9}$$

In addition writing q_y to denote the quadratic form representing the second derivative of $\tilde{\varphi}$ at y, if $y \in \frac{1}{3}\,B^n$ and $z \in \mathbb{E}^{n-1}$, then

$$\|z\|^2 \leq q_y(z) \leq (1 + 2\|y\|^2) \cdot \|z\|^2.$$

9 Proof of Theorem 2.1 in the Cases of Volume and Surface Area

We assume that $n \geq 4$, because if $n \leq 3$, then Theorem 2.1 is covered [4] in the cases of surface area and volume. The proofs of Theorem 2.1 in the cases of volume and surface area follow the very same pattern. We present the argument only in the case of the surface area, because it is the more involved case.

According to Lemma 5.1,

$$\liminf_{r\to 1^+} \frac{S(Q_r^n) - S(B^n)}{r-1} = \theta_S(n)$$

is finite and positive. Therefore Theorem 2.1 in the case of the surface area follows if, for any $\varepsilon \in (0, \tilde{\varepsilon})$ and $r \in (1, \tilde{r})$ where $\tilde{\varepsilon} > 0$ depends on n and $\tilde{r} > 1$ depends on n and ε, there exists $Q_{r,\varepsilon} \in \mathcal{F}_r^n$ such that

$$S(Q_{r,\varepsilon}) - S(B^n) \leq \theta_S(n) \cdot (r-1) + O(\varepsilon(r-1)). \tag{9.1}$$

Here $\tilde{\varepsilon}$ is at most the ε_0's of Lemma 7.2 and Lemma 8.1. First we define $\tilde{r}$. Namely, it follows by the definition of $\theta_S(n)$ that there exists $\tilde{r} \in (1, 1+\varepsilon^{2n^2})$ such that

$$S(Q_{\tilde{r}}^n) - S(B^n) \leq \theta_S(n) \cdot (\tilde{r}-1) + O(\varepsilon(\tilde{r}-1)).$$

Let $r \in (1, \tilde{r})$ which we fix for the rest of the proof of Theorem 2.1. We define now an auxiliary circumscribed polytope that will determine patches on S^{n-1} of size $\sqrt{r-1}/\varepsilon$. We choose a maximal family $s_1, \ldots, s_m \in S^{n-1}$ with the property that $\|s_i - s_j\| \geq \sqrt{r-1}/\varepsilon$ for $i \neq j$, and we write $G_1, \ldots, G_m$ to denote the facets of the circumscribed polytope whose facets touch B^n at $s_1, \ldots, s_m$. Writing B_j^{n-1} to denote the unit $(n-1)$-ball of centre o contained in the linear $(n-1)$-space parallel to aff G_j, we have

$$s_j + \frac{(1+\varepsilon)\sqrt{r-1}}{4\varepsilon} B_j^{n-1} \subset G_j \subset s_j + \frac{\sqrt{r-1}}{\varepsilon} B_j^{n-1}. \tag{9.2}$$

The $Q_{r,\varepsilon}$ in (9.1) will be defined as the convex hull of $\Gamma_1, \ldots, \Gamma_m$ constructed in Proposition 9.1 (see (9.16)).

Proposition 9.1 *Let $j = 1, \ldots, m$. Using the notation as above, there exists a convex piecewise linear surface Γ_j satisfying the following properties: Γ_j intersects G_j and the orthogonal projection of Γ_j into* aff G_j *covers G_j. In addition if F is a facet of Γ_j, then* aff F *does not intersect* int B^n*, the orthogonal projection of F into* aff G_j *intersects G_j, F is an $(n-1)$-polytope, and if v is a vertex of F, then*

$$r - 1 \leq d(v, B^n) \leq 2(r-1). \tag{9.3}$$

Moreover, if $X_j = \pi_{S^{n-1}}(G_j)$ and $Y_j \subset \Gamma_j$ satisfies $X_j = \pi_{S^{n-1}}(Y_j)$, then

$$\mathcal{H}^{n-1}(Y_j) - \mathcal{H}^{n-1}(X_j) \leq \frac{\mathcal{H}^{n-1}(X_j)}{S(B^n)} \cdot \theta_S(n)\,(r-1) + O(\varepsilon)\,(r-1) \cdot \mathcal{H}^{n-1}(X_j).$$

Proof We recall that $\widetilde{\mathcal{F}}_r^n$ denotes the family of all $C \in \mathcal{F}_r^n$ satisfying ext $C \subset rS^{n-1}$, and that $Q_r^n \in \widetilde{\mathcal{F}}_r^n$. Lemma 3.1 provides a polytope $\widetilde{Q}_\varepsilon \in \widetilde{\mathcal{F}}_{\tilde{r}}^n$ such that the distance between any two vertices of $\widetilde{Q}_\varepsilon$ is at least $\varepsilon\sqrt{\tilde{r}-1}$, and

$$S(\widetilde{Q}_\varepsilon) - S(B^n) \leq \theta_S(n) \cdot (\tilde{r}-1) + O(\varepsilon(\tilde{r}-1)). \tag{9.4}$$

We write $\widetilde{F}_1, \ldots, \widetilde{F}_{\tilde{l}}$ to denote the facets of $\widetilde{Q}_\varepsilon$. In addition we write $\tilde{w}_i$ to denote the exterior unit normal to $\widetilde{F}_i$, and define $\tilde{\rho}_i = d(\tilde{w}_i, \text{aff}\, \widetilde{F}_i)$. Let f be a bounded measurable function on S^{n-1} such that

$$f(x) = \frac{(1+\tilde{\rho}_i)^{n-1}}{\langle x, \tilde{w}_i \rangle^n} - 1$$

for $i = 1, \ldots, \tilde{l}$ and $x \in \pi_{S^{n-1}}(\text{relint}\, \widetilde{F}_i)$. Since

$$\|y - x\| = \frac{1+\tilde{\rho}_i}{\langle x, \tilde{w}_i \rangle} - 1$$

for any $y \in \widetilde{F}_i$ and $x = \pi_{S^{n-1}}(y)$, if $Y \subset \partial \widetilde{Q}_\varepsilon$ is a convex hypersurface and $X = \pi_{S^{n-1}}(Y)$ then Lemma 6.1 yields

$$\mathcal{H}^{n-1}(Y) - \mathcal{H}^{n-1}(X) = \int_X f(x)\, dx. \tag{9.5}$$

We define

$$\widetilde{G}_j = s_j + \lambda \cdot (G_j - s_j) \quad \text{for } \lambda = \frac{\sqrt{\tilde{r}-1}}{(1+\varepsilon)\sqrt{r-1}},$$

and let $\widetilde{X}_j = \pi_{S^{n-1}}(\widetilde{G}_j)$. Then Lemma 5.2 yields the existence of $g \in SO(n)$ such that

$$\int_{g\widetilde{X}_j} f(x)\,dx \le \frac{\mathcal{H}^{n-1}(\widetilde{X}_j)}{S(B^n)} \cdot \int_{S^{n-1}} f(x)\,dx. \tag{9.6}$$

We may assume that $\pi_{S^{n-1}}(\widetilde{F}_i)$ intersects $g\widetilde{X}_j$ in a set of positive $(n-1)$-measure if and only if $i \le \tilde{k}'$ for $\tilde{k}' \le \tilde{l}$. Let $\widetilde{Y}_j \subset \partial\widetilde{Q}_\varepsilon$ satisfy that $\pi_{S^{n-1}}(\widetilde{Y}_j) = g\widetilde{X}_j$. We deduce by (9.4), (9.5), and (9.6) that

$$\begin{aligned} \mathcal{H}^{n-1}(\widetilde{Y}_j) - \mathcal{H}^{n-1}(g\widetilde{X}_j) \le \frac{\mathcal{H}^{n-1}(\widetilde{X}_j)}{S(B^n)} \cdot \theta_S(n) \cdot (\tilde{r}-1) \\ + O(\varepsilon) \cdot (\tilde{r}-1) \cdot \mathcal{H}^{n-1}(\widetilde{G}_j). \end{aligned} \tag{9.7}$$

We may assume that $s_j = \xi = (o,-1)$ and g is the identity, hence aff $\widetilde{G}_j$ is parallel to $\mathbb{E}^{n-1}$. We write $\widetilde{C}_j$ to denote the orthogonal projection of $\widetilde{G}_j$ into $\mathbb{E}^{n-1}$, which satisfies (see (9.2)),

$$\frac{\tilde{r}-1}{4\varepsilon} B^{n-1} \subset \widetilde{C}_j \subset \frac{\tilde{r}-1}{\varepsilon} B^{n-1}.$$

Let us recall that $\tilde{\varphi}(y) = -\sqrt{1-\|y\|^2}$ and $\varphi_\omega(y) = \frac{\omega}{2}\|y\|^2$ for $y \in B^{n-1}$. It follows by (8.9) and $\tilde{r} < 1 + \varepsilon^{2n^2}$ that if $y \in \frac{2\sqrt{\tilde{r}-1}}{\varepsilon} B^{n-1}$, then

$$-1 + \varphi_1(y) \le \tilde{\varphi}(y) \le -1 + \varphi_{\tilde{\omega}}(y) \quad \text{for } \tilde{\omega} = 1 + \varepsilon^8.$$

In particular the graph $\widetilde{\Gamma}_{\tilde{\omega}}$ of $-1 + \varphi_{\tilde{\omega}}$ above $\frac{4\sqrt{\tilde{r}-1}}{\varepsilon} B^{n-1}$ satisfies $\widetilde{\Gamma}_{\tilde{\omega}} \subset B^n$. Therefore we define $\widetilde{Z}_j = \pi_{\widetilde{\Gamma}_{\tilde{\omega}}}(\widetilde{G}_j)$, and $\widetilde{Y}'_j \subset \partial\widetilde{Q}_\varepsilon$ by $\pi_{\widetilde{\Gamma}_{\tilde{\omega}}}(\widetilde{Y}'_j) = \widetilde{Z}_j$, and we deduce by Lemma 7.2 and (9.7) that

$$\begin{aligned} \mathcal{H}^{n-1}(\widetilde{Y}'_j) - \mathcal{H}^{n-1}(\widetilde{Z}_j) &= \mathcal{H}^{n-1}(\widetilde{Y}_j) - \mathcal{H}^{n-1}(\widetilde{X}_j) + O(\varepsilon(\tilde{r}-1)) \cdot \mathcal{H}^{n-1}(\widetilde{G}_j) \\ &\le \frac{\mathcal{H}^{n-1}(\widetilde{G}_j)}{S(B^n)} \cdot \theta_S(n) \cdot (\tilde{r}-1) \\ &\quad + O(\varepsilon) \cdot (\tilde{r}-1) \cdot \mathcal{H}^{n-1}(\widetilde{G}_j). \end{aligned} \tag{9.8}$$

We may assume that $i = 1, \dots, \bar{k}'$ are the indices satisfying that $\widetilde{F}_i$ intersects $\widetilde{Y}'_j$ in a set of positive measure for suitable $\bar{k}' \le \tilde{l}$. For $i \le \bar{k}'$, let $\tilde{x}_i \in \widetilde{Z}_j$ be the point where

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the tangent hyperplane to $\widetilde{Z}_j$ is parallel to aff $\widetilde{F}_i$, and let $\tilde{\nu}_i'$ denote the distance of $\tilde{x}_i$ from aff $\widetilde{F}_i$. We deduce by (7.10) and (9.8) that

$$\text{(9.9)}\quad \sum_{i=1}^{\tilde{k}'} \int_{\pi_{\widetilde{Z}_j}(\widetilde{F}_i)} (n-1)\tilde{\nu}_i' + \frac{n}{2}\|x-\tilde{x}_i\|^2\,dx \le \frac{\mathcal{H}^{n-1}(\widetilde{G}_j)}{S(B^n)}\cdot\theta_S(n)\,(\tilde{r}-1) + O(\varepsilon)(\tilde{r}-1)\cdot\mathcal{H}^{n-1}(\widetilde{G}_j).$$

Let $\widetilde{\Gamma}_j$ be the union of facets of $\widetilde{Q}_\varepsilon$ whose orthogonal projection into aff $\widetilde{G}_j$ intersects $\widetilde{G}_j$ in a set of positive $(n-1)$-measure. We assume that $\widetilde{F}_1,\dots,\widetilde{F}_k$ are the facets contained in $\widetilde{\Gamma}_j$ for suitable k, $\tilde{k}'\le k\le\tilde{l}$. For $i\le k$, we write $\widetilde{\Pi}_i$ and $\tilde{y}_i$ to denote the orthogonal projection of $\widetilde{F}_i$ and $\tilde{x}_i$, respectively, into $\mathbb{E}^{n-1}$, and define $\tilde{\nu}_i$ by the property that $(\tilde{y}_i,-1+\varphi_{\tilde{\omega}}(\tilde{y}_i)-\tilde{\nu}_i)\in$ aff $\widetilde{F}_i$. Writing $\widetilde{C}_j$ to denote the orthogonal projection of $\widetilde{G}_j$ into $\mathbb{E}^{n-1}$, combining (9.9) and Lemma 8.1, yields that

$$\text{(9.10)}\quad \sum_{i=1}^{k} \int_{\widetilde{C}_j\cap\widetilde{\Pi}_i} (n-1)\tilde{\nu}_i + \frac{n}{2}\|z-\tilde{y}_i\|^2\,dx \le \frac{\mathcal{H}^{n-1}(\widetilde{G}_j)}{S(B^n)}\cdot\theta_S(n)(\tilde{r}-1) + O(\varepsilon)\,(\tilde{r}-1)\cdot\mathcal{H}^{n-1}(\widetilde{G}_j).$$

In addition, if z is a vertex of $\widetilde{\Pi}_i$ for $i\le k$ and v is the corresponding vertex of $\widetilde{F}_i$, then

$$\text{(9.11)}\quad \tilde{\nu}_i+\tfrac{1}{2}\|z-\tilde{y}_i\|^2 \ge \frac{1}{1+\varepsilon}d(v,\widetilde{\Gamma}_{\tilde{\omega}}) \ge \frac{1}{1+\varepsilon}d(v,B^n) = \frac{1}{1+\varepsilon}(\tilde{r}-1),$$

$$\text{(9.12)}\quad \tilde{\nu}_i+\tfrac{1}{2}\|z-\tilde{y}_i\|^2 \le (1+\varepsilon)d(v,\widetilde{\Gamma}_{\tilde{\omega}}) \le (1+\varepsilon)^2(\tilde{r}-1).$$

Now we define $C_j=\lambda^{-1}\widetilde{C}_j$, hence $G_j=C_j+\xi$. Moreover, if $i\le k$, then let $\Pi_i=\lambda^{-1}\widetilde{\Pi}_i$, $y_i=\lambda^{-1}\tilde{y}_i$, and $\nu_i=\lambda^{-2}\tilde{\nu}_i$. We conclude by (9.10) that

$$\text{(9.13)}\quad \sum_{i=1}^{k} \int_{C_j\cap\Pi_i} (n-1)\nu_i + \tfrac{n}{2}\|z-y_i\|^2\,dx \le \frac{\mathcal{H}^{n-1}(G_j)}{S(B^n)}\cdot\theta_S(n)(r-1) + O(\varepsilon)\,(r-1)\cdot\mathcal{H}^{n-1}(G_j).$$

We define $\varphi(z)=-1+\|z\|^2$ for $z\in\frac{4\sqrt{r-1}}{\varepsilon}B^{n-1}$, and observe that $\Gamma(\varphi)\cap\operatorname{int}B^n=\varnothing$ according to (8.9). We write l_i to denote the linear function whose graph is the tangent hyperplane to $\Gamma(\varphi)$ at $x_i=(y_i,\varphi(y_i))$, and define $\psi_i(z)=l_i(z)-\nu_i$. In addition, we define $\psi\colon\bigcup_{i=1}^k\Pi_i\to\mathbb{R}$ by $\psi(z)=\psi_i(z)$ for $z\in\Pi_i$, and observe that $\Gamma_j=\Gamma(\psi)$ is a convex piecewise linear hypersurface. Let F_i be the graph of ψ above Π_i, hence

$F_1, \dots, F_k$ are the facets of Γ_j. If z is a vertex of Π_i for $i \le k$ and v is the corresponding vertex of F_i, then we deduce by Lemma 8.1, (9.11), and (9.12) that

$$\begin{aligned}
(9.14)\quad d(v,\Gamma(\varphi)) &\le (1+\varepsilon)(\nu_i + \tfrac{1}{2}\|z-y_i\|^2)\\
&= \lambda^{-2}(1+\varepsilon)(\tilde{\nu}_i + \tfrac{1}{2}\|\lambda^{-1}z - \tilde{y}_i\|^2)\\
&\le (1+\varepsilon)^5(r-1);
\end{aligned}$$

$$\begin{aligned}
(9.15)\quad d(v,\Gamma(\varphi)) &\ge \tfrac{1}{1+\varepsilon}(\nu_i + \tfrac{1}{2}\|z-y_i\|^2) = \tfrac{1}{1+\varepsilon}\lambda^{-2}(\tilde{\nu}_i + \tfrac{1}{2}\|\lambda^{-1}z - \tilde{y}_i\|^2)\\
&\ge r-1.
\end{aligned}$$

Now combining (9.14) and (9.15) yields (9.3).

We define $Z_j = \pi_{\Gamma(\varphi)}(G_j)$, assume that $i = 1, \dots, k'$ are the indices satisfying that $\pi_{\Gamma(\varphi)}(F_i)$ intersects Z_j in a set of positive measure for some $k' \le k$, and write ν_i' to denote the distance of x_i from aff F_i for $i \le k'$. We recall that $X_j = \pi_{S^{n-1}}(G_j)$, and write Y_j and Y_j' to denote the subset of Γ_j satisfying $X_j = \pi_{S^{n-1}}(Y_j)$ and $Z_j = \pi_{\Gamma(\varphi)}(Y_j')$, respectively. It follows first by Lemma 7.2, secondly by (7.10), and thirdly by Lemma 8.1 and (9.13) that

$$\begin{aligned}
\mathcal{H}^{n-1}(Y_j) - \mathcal{H}^{n-1}(X_j) &= \mathcal{H}^{n-1}(Y_j') - \mathcal{H}^{n-1}(Z_j) + O(\varepsilon(r-1))\cdot\mathcal{H}^{n-1}(G_j)\\
&= \sum_{i=1}^{k'} \int_{\pi_{\Gamma(\varphi)}(F_i)\cap Z_j} (n-1)\nu_i' + \tfrac{n}{2}\|x-x_i\|^2\,dx\\
&\qquad + O(\varepsilon(r-1))\cdot\mathcal{H}^{n-1}(G_j)\\
&\le \frac{\mathcal{H}^{n-1}(X_j)}{S(B^n)}\cdot\theta_S(n)\,(r-1) + O(\varepsilon)\,(r-1)\cdot\mathcal{H}^{n-1}(X_j).
\end{aligned}$$

In turn we conclude Proposition 9.1. ■

For the rest of the proof, we use the notation of Proposition 9.1. We define

$$(9.16)\qquad Q_{r,\varepsilon} = \operatorname{conv}\{\Gamma_1, \dots, \Gamma_m\}.$$

Then $Q_{r,\varepsilon} \in \mathcal{F}_r^n$ and $Q_{r,\varepsilon} \subset (1+2(r-1))B^n$. We define W_j to be the part of $\partial Q_{r,\varepsilon}$ satisfying $\pi_{S^{n-1}}(W_j) = X_j$, and prove that for some $\gamma > 0$ depending only on n and independent of j,

$$(9.17)\qquad \mathcal{H}^{n-1}(W_j) - \mathcal{H}^{n-1}(X_j) \le (1+\gamma\,\varepsilon)\cdot\frac{\mathcal{H}^{n-1}(X_j)}{S(B^n)}\cdot\theta_S(n)(r-1).$$

Let $X_j^0 = \pi_{S^{n-1}}(s_j + (1-96\varepsilon)(G_j - s_j))$, and let Y_j^0 be the part of Γ_j satisfying $\pi_{S^{n-1}}(Y_j^0) = X_j^0$. Now if $H \subset (1+2(r-1))B^n$ is a compact convex set whose affine hull avoids int B^n, then diam $H \le 6\sqrt{r-1}$. Therefore if F is a facet of $Q_{r,\varepsilon}$ such that F intersects Γ_t for some $t \ne j$ and $\pi_{S^{n-1}}(F) \cap X_j \ne \emptyset$, then $\pi_{S^{n-1}}(F) \subset$

relbd $X_j + 12\sqrt{r-1}\,B^n$. Since $\|\pi_{S^{n-1}}(x) - \pi_{S^{n-1}}(x')\| \geq \frac{1}{2}\|x - x'\|$ for $x, x' \in G_j$ and $s_j + \frac{\sqrt{r-1}}{4\varepsilon}\,B_j^{n-1} \subset G_j$, we deduce that $Y_j^0 \subset W_j$ and $\mathcal{H}^{n-1}(X_j \backslash X_j^0) = O(\varepsilon)\mathcal{H}^{n-1}(X_j)$. Therefore (7.9) yields

$$\mathcal{H}^{n-1}(W_j \backslash Y_j^0) - \mathcal{H}^{n-1}(X_j \backslash X_j^0) = O(\varepsilon)\,(r-1) \cdot \mathcal{H}^{n-1}(X_j). \tag{9.18}$$

In addition, Proposition 9.1 implies that

$$\begin{aligned} \mathcal{H}^{n-1}(Y_j^0) - \mathcal{H}^{n-1}(X_j^0) &\leq \mathcal{H}^{n-1}(Y_j) - \mathcal{H}^{n-1}(X_j) \\ &\leq (1 + O(\varepsilon)) \cdot \frac{\mathcal{H}^{n-1}(X_j)}{S(B^n)} \cdot \theta_S(n)(r-1), \end{aligned} \tag{9.19}$$

hence combining (9.18) and (9.19) leads to (9.17). Adding (9.17) for $j = 1, \dots, m$ proves (9.1), and in turn Theorem 2.1 in the case of the surface area. As we stated at the beginning, the proof in the case of the volume is quite analogous, thus we do not present it. ■

10 Transfer Lemma in the Case of Mean Width

We will transfer integrals between patches on the sphere and in $\mathbb{E}^{n-1}$ using Lemma 10.1. We recall that $\xi = (o, -1) \in \mathbb{E}^n$, and $\tilde{\varphi}(y) = -\sqrt{1 - \|y\|^2}$ parametrizes the lower hemisphere of S^{n-1} on B^{n-1}.

Lemma 10.1 *Let $\varepsilon \in (0, \varepsilon_0)$, and let $\rho \in (0, \varepsilon^4)$ where ε_0 is a suitable positive constant depending only on n. In addition let C be a compact convex set satisfying $\frac{\sqrt{\rho}}{4\varepsilon}B^{n-1} \subset C \subset \frac{2\sqrt{\rho}}{\varepsilon}B^{n-1}$, and let $y_1, \dots, y_k \in \mathbb{E}^{n-1}$ such that for any $z \in C + 2\sqrt{\rho}\,B^{n-1}$ there exists y_i satisfying $\frac{1}{2}\|z - y_i\|^2 \leq 2\rho$. Writing $X = \pi_{S^{n-1}}(C + \xi)$ and $x_i = (y_i, \tilde{\varphi}(y_i))$, we have $\mathcal{H}^{n-1}(C) = (1 + O(\varepsilon))\mathcal{H}^{n-1}(X)$, $d(C', X) \leq \sqrt{\rho}$ for the graph of $\tilde{\varphi}$ above C, and*

$$\int_X \min_i \{1 - \langle x, x_i \rangle\}\, dx = \int_C \min_i \tfrac{1}{2}\, \|z - y_i\|^2\, dz + O(\varepsilon\rho) \cdot \mathcal{H}^{n-1}(C).$$

Moreover, if $z \in C + 2\sqrt{\rho}B^{n-1}$, then $x = (z, \tilde{\varphi}(z))$ satisfies

$$(1 + \varepsilon)^{-1} \cdot \min_i \tfrac{1}{2}\|z - y_i\|^2 \leq \min_i \{1 - \langle x, x_i \rangle\} \leq (1 + \varepsilon) \cdot \min_i \tfrac{1}{2}\|z - y_i\|^2.$$

Proof The main observation is the following fact: if $z, z' \in tB^{n-1}$ for $t \in (0, \frac{1}{2})$, then $x = (z, \tilde{\varphi}(z))$ and $x' = (z', \tilde{\varphi}(z'))$ satisfies

$$1 - \langle x, x' \rangle = (1 + O(t^2)) \cdot \tfrac{1}{2}\|z - z'\|^2, \quad \text{and} \quad \|x - \pi_{S^{n-1}}(z)\| = O(t^3).$$

Since $\frac{\sqrt{\rho}}{\varepsilon} < \varepsilon_0\varepsilon$, choosing ε_0 small enough, we have the following properties: let $x = (z, \tilde{\varphi}(z))$ for $z \in C + 2\sqrt{\rho}\,B^{n-1}$, and let

$$\min_i \tfrac{1}{2}\|z - y_i\|^2 = \tfrac{1}{2}\|z - y_j\|^2 \quad \text{and} \quad \min_i \{1 - \langle x, x_i \rangle\} = 1 - \langle x, x_l \rangle.$$

Then first,

$$(1+\varepsilon)^{-1}\cdot \tfrac{1}{2}\|z-y_j\|^2 \le (1+\varepsilon)^{-1}\cdot \tfrac{1}{2}\|z-y_l\|^2 \le 1-\langle x,x_l\rangle \le 1-\langle x,x_j\rangle \le (1+\varepsilon)\cdot \tfrac{1}{2}\|z-y_j\|^2.$$

Secondly, writing X' to denote the orthogonal projection of X into $\mathbb{E}^{n-1}$, we have $\mathcal{H}^{n-1}(X')=(1+O(\varepsilon))\mathcal{H}^{n-1}(X)$ and $d_H(X',C)\le \frac{1}{2}\sqrt{\rho}$. Finally,

$$\int_X \min_i\{1-\langle x,x_i\rangle\}\,dx = \int_{X'} \min_i \tfrac{1}{2}\|z-y_i\|^2\,dz + O(\varepsilon\rho)\cdot \mathcal{H}^{n-1}(X').$$

In turn we conclude Lemma 10.1. ■

11 Proof of Theorem 2.1 in the Case of Mean Width

We assume that $n\ge 4$ because if $n\le 3$, then Theorem 2.1 in the case of the mean width is covered by [3] for $n=2$ (as mean width is proportional with the perimeter in this case), and by [4] for $n=3$.

First we present two formulae related to the difference of the mean width of a ball and a polytope. If P is a polytope with vertices $x_1,\dots,x_m\in S^{n-1}$, then

$$M(B^n)-M(P)=\frac{2}{S(B^n)}\int_{S^{n-1}}\min_i(1-\langle x,x_i\rangle)\,dx. \tag{11.1}$$

In addition if $\frac{1}{r}B^n\subset P$ and $\min_i(1-\langle x,x_i\rangle)=1-\langle x,x_l\rangle$ for $x\in S^{n-1}$ then

$$\|x-x_l\|\le \sqrt{2}\cdot\sqrt{1-\frac{1}{r}}. \tag{11.2}$$

In the case of mean width, it will be convenient to consider the family $\overline{\mathcal{F}}^n_r$ of all convex bodies that contain $\frac{1}{r}B^n$, and whose extreme points lie on S^{n-1}. In particular $\frac{1}{r}W^n_r\in\overline{\mathcal{F}}^n_r$. According to Lemma 5.1 and $M(rB^n)-M(B^n)=2(r-1)$,

$$\liminf_{r\to 1^+}\frac{M(W^n_r)-M(B^n)}{r-1}=\theta_M(n)$$

is positive and at most two. Therefore Theorem 2.1 in the case of the mean width follows if for any $\varepsilon\in(0,\varepsilon_0)$ and $r>\tilde r$ where ε_0 depends on n and $\tilde r$ depends on n and ε, there exists $W_{r,\varepsilon}\in\overline{\mathcal{F}}^n_r$ such that

$$M(B^n)-M(W_{r,\varepsilon})\ge(2-\theta_M(n))\cdot(r-1)+O(\varepsilon(r-1)). \tag{11.3}$$

Here ε_0 is at most the constant of Lemma 10.1. It follows by the definition of $\theta_M(n)$ that there exists $\tilde r\in(1,1+\varepsilon^4)$ such that

$$M(B^n)-M(W^n_{\tilde r})\ge(2-\theta_M(n))\cdot(\tilde r-1)+O(\varepsilon(\tilde r-1)). \tag{11.4}$$

Now let $r>\tilde r$. We choose a maximal family $s_1,\dots,s_m\in S^{n-1}$ with the property that $\|s_i-s_j\|\ge\sqrt{r-1}/\varepsilon$ for $i\ne j$, and we write $G_1,\dots,G_m$ to denote the facets of the circumscribed polytope whose facets touch B^n at $s_1,\dots,s_m$. In addition let $X_j=\pi_{S^{n-1}}G_j$.

Proposition 11.1 *Let $j = 1, \dots, m$. There exists a finite set $\mathcal{V}_j \subset S^{n-1}$ such that if $x \in X_j$, then $\langle x, v\rangle \geq 1/r$ for some $v \in \mathcal{V}_j$, if $v \in \mathcal{V}_j$, then $d(v, X_j) \leq 8\sqrt{r-1}$, and*

$$\frac{2}{S(B^n)} \int_{X_j} \min_{v \in \mathcal{V}_j} \{1 - \langle x, v\rangle\}\, dx \geq \frac{\mathcal{H}^{n-1}(X_j)}{S(B^n)} \cdot (2 - \theta_M(n))(r-1) + O(\varepsilon)\,(r-1) \cdot \mathcal{H}^{n-1}(X_j).$$

Proof The estimate (11.4) for $W^n_{\tilde r}$ and using polytopal approximation (Lemma 3.1) provide a polytope $\widetilde{W}_\varepsilon \in \overline{\mathcal{F}}^n_{\tilde r}$ such that

$$M(B^n) - M(\widetilde{W}_\varepsilon) \geq (2 - \theta_M(n)) \cdot (\tilde r - 1) + O(\varepsilon(\tilde r - 1)). \tag{11.5}$$

We write $\tilde x_1, \dots, \tilde x_l$ to denote the vertices of $\widetilde{W}_\varepsilon$.

We define

$$\widetilde{G}_j = s_j + \lambda \cdot (G_j - s_j) \quad \text{for } \lambda = \frac{(1+\varepsilon)^2\sqrt{\tilde r - 1}}{\sqrt{r-1}},$$

and $\widetilde{X}_j = \pi_{S^{n-1}}(\widetilde{G}_j)$. According to Lemma 5.2, (11.1), and (11.5), there exists a $g \in SO(n)$ such that after re-indexing $\tilde x_1, \dots, \tilde x_l$ in a way such that $d(\tilde x_i, g\widetilde{X}_j) \leq 6\sqrt{\tilde r - 1}$ if and only if $i \leq k$ for $k \leq l$, we have

$$\begin{aligned}\int_{g\widetilde{X}_j} \min_{i=1,\dots,k}\{1 - \langle x, \tilde x_i\rangle\}\, dx &= \int_{g\widetilde{X}_j} \min_{i=1,\dots,l}\{1 - \langle x, \tilde x_i\rangle\}\, dx \\ &\geq \frac{\mathcal{H}^{n-1}(\widetilde{X}_j)}{S(B^n)} \cdot \int_{S^{n-1}} \min_{i=1,\dots,l}\{1 - \langle x, \tilde x_i\rangle\}\, dx \\ &\geq \frac{2 - \theta_M(n)}{2} \cdot \mathcal{H}^{n-1}(\widetilde{X}_j)(\tilde r - 1) \\ &\quad + O(\varepsilon)\,(\tilde r - 1) \cdot \mathcal{H}^{n-1}(\widetilde{X}_j).\end{aligned}$$

We may assume that $s_j = \xi = (o, -1)$ and g is the identity. Then aff $\widetilde{G}_j$ is parallel to $\mathbb{E}^{n-1}$, and we write $\widetilde{C}_j$ to denote the orthogonal projection of $\widetilde{G}_j$ into $\mathbb{E}^{n-1}$. We write $\tilde y_i$ to denote the orthogonal projection of $\tilde x_i$ into $\mathbb{E}^{n-1}$ for $i \leq k$, and deduce by Lemma 10.1 that

$$\int_{\widetilde{C}_j} \min_{i=1,\dots,k} \frac{1}{2}\|z - \tilde y_i\|^2\, dz \geq \frac{2 - \theta_M(n)}{2} \cdot \mathcal{H}^{n-1}(\widetilde{C}_j)\,(\tilde r - 1) + O(\varepsilon)\,(\tilde r - 1) \cdot \mathcal{H}^{n-1}(\widetilde{C}_j).$$

Since $\frac{1}{\tilde r} B^n \subset \widetilde{W}_\varepsilon$, it also follows by Lemma 10.1 that if $z \in \widetilde{C}_j + 3\sqrt{\tilde r - 1}$, then

$$\min_{i=1,\dots,k} \tfrac{1}{2}\,\|z - \tilde y_i\|^2 \leq (1+\varepsilon) \cdot (\tilde r - 1).$$

We note that $d(\tilde{y}_i, \widetilde{C}_j) \leq 6\sqrt{\tilde{r}-1}$ for $i \leq k$. Now we define $C_j = \lambda^{-1}\widetilde{C}_j$ and $y_i = \lambda^{-1}\tilde{y}_i$ for $i \leq k$, hence $G_j = C_j + \xi$. We deduce that $d(y_i, C_j) \leq 6\sqrt{r-1}$ for $i \leq k$,

$$\int_{C_j} \min_{i=1,\dots,k} \tfrac{1}{2}\|z-y_i\|^2\, dz \geq \frac{2-\theta_M(n)}{2} \cdot \mathcal{H}^{n-1}(C_j)\,(r-1) + O(\varepsilon)\,(r-1)\cdot \mathcal{H}^{n-1}(C_j),$$

and if $z \in C_j + 2\sqrt{\tilde{r}-1}$, then

$$\min_{i=1,\dots,k} \tfrac{1}{2}\|z-y_i\|^2 \leq (1+\varepsilon)^{-3}\cdot(r-1) \leq (1+\varepsilon)^{-1}(1-\frac{1}{r}).$$

Therefore defining $\mathcal{V}_j = \{(y_1, \tilde{\varphi}(y_1)), \dots, (y_k, \tilde{\varphi}(y_k))\}$, Lemma 10.1 completes the proof of Proposition 11.1. ■

We define $W_{r,\varepsilon} = \operatorname{conv}\mathcal{V}$ for $\mathcal{V} = \bigcup_{j=1}^m \mathcal{V}_j$, and deduce by Proposition 11.1 that $W_{r,\varepsilon} \in \overline{\mathcal{F}}^n_r$. Let us observe that if $x, x' \in G_j$, then $\|\pi_{S^{n-1}}(x) - \pi_{S^{n-1}}(x')\| \geq \frac{1}{2}\|x-x'\|$, and $s_j + \frac{\sqrt{r-1}}{4\varepsilon} B_j^{n-1} \subset G_j$ where B_j^{n-1} denotes the unit $(n-1)$-ball of centre s_j in $\operatorname{aff} G_j$. We deduce for $X_j^0 = \pi_{S^{n-1}}(s_j + (1-80\varepsilon)(G_j - s_j))$ that if $x \in X_j^0$ and $x' \in X_t$ for $t \neq j$, then $\|x-x'\| \geq 10\sqrt{r-1}$, hence if $v \in \mathcal{V}_t$, then $\|x-v\| \geq 2\sqrt{r-1}$ according to Proposition 11.1. It follows by (11.2) that for any $x \in X_j^0$,

$$\min_{v\in\mathcal{V}}\{1-\langle x,v\rangle\} = \min_{v\in\mathcal{V}_j}\{1-\langle x,v\rangle\}.$$

Thus $\mathcal{H}^{n-1}(X_j) - \mathcal{H}^{n-1}(X_j^0) = O(\varepsilon)\cdot\mathcal{H}^{n-1}(X_j)$ and Proposition 11.1 yield

$$\begin{aligned}\text{(11.6)}\qquad \frac{2}{S(B^n)}\int_{X_j}\min_{v\in\mathcal{V}}\{1-\langle x,v\rangle\}\,dx &\geq \frac{2}{S(B^n)}\int_{X_j^0}\min_{v\in\mathcal{V}_j}\{1-\langle x,v\rangle\}\,dx\\ &\geq \frac{\mathcal{H}^{n-1}(X_j)}{S(B^n)}\cdot(2-\theta_M(n))(r-1)\\ &\quad + O(\varepsilon)\,(r-1)\cdot\mathcal{H}^{n-1}(X_j).\end{aligned}$$

Adding (11.6) for $j = 1, \dots, m$ implies (11.3), and completes the proof of Theorem 2.1. ■

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References

[1] E. Artin, *The Gamma Function.* Holt, Rinehart and Winston, New York, 1964.
[2] K. Mathéné Bognár and K. Böröczky, *Regular polyhedra and Hajós polyhedra.* Studia Sci. Math. Hungar. **35**(1999), no. 3-4, 415–426.

[3] K. Böröczky and K. Böröczky, Jr. *Polytopes of minimal volume with respect to a shell - another characterization of the octahedron and the icosahedron.* Disc. Comput. Geom., to appear. www.renyi.hu/˜carlos/radiusmain.pdf

[4] K. Böröczky, K. Böröczky, Jr., and G. Wintsche, *Typical faces of extremal polytopes with respect to a thin three-dimensional shell.* Periodica Math Hung. **53**(2006), no. 1-2, 83–102.

[5] K. Böröczky, Jr. and M. Reitzner, *Approximation of smooth convex bodies by random circumscribed polytopes.* Ann. Appl. Prob. **14**(2004), 239–273.

[6] K. J. Böröczky, P. Tick, and G. Wintsche, *Typical faces of best approximating three-polytopes*, preprint. www.renyi.hu/˜carlos/approxface.pdf

[7] K. Böröczky, Jr. and G. Wintsche: *Covering the sphere by equal spherical balls.* In: Discrete and Computational Geometry. Algorithms Combin. 25, Springer, Berlin, 2003, 237–253.

[8] K. J. Falconer, *The Geometry of Fractal Sets.* Cambridge Tracts in Mathematics 85, Cambridge University Press, Cambridge, 1985.

[9] L. Fejes Tóth, *Regular Figures.* Pergamon Press, New York, 1964.

[10] A. A. Giannopoulos and V. D. Milman, *Asymptotic convex geometry: short overview.* In: Different Faces of Geometry. Int. Math. Ser. (N.Y.) 3, Kluwer/Plenum, New York, 2004, pp. 87–162.

[11] P. M. Gruber, *Aspects of approximation of convex bodies.* In: Handbook of Convex Geometry. North-Holland, Amsterdam, 1993, pp. 319–345.

[12] ———, *Comparisons of best and random approximation of convex bodies by polytopes.* Rend. Circ. Mat. Palermo, **50**(1997), 189–216.

[13] ———, *Optimale Quantisierung.* Math. Semesterber. **49**(2002), no. 2, 227–251.

[14] ———, *Optimum quantization and its applications.* Adv. Math. **186**(2004), no. 2, 456–497.

[15] J. Molnár, *Alcune generalizzazioni del teorema di Segre-Mahler.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **30**(1961), 700–705.

[16] C. A. Rogers, Hausdorff Measure. Cambridge University Press, London, 1970.

[17] J. R. Sangwine-Yager, *A generalization of outer parallel sets of a convex set.* Proc. Amer. Math. Soc. **123**(1995), no. 5, 1559–1564.

[18] R. Schneider, *Zur optimalen Approximation konvexer Hyperflächen durch Polyeder.* Math. Ann. **256**(1981), no. 3, 289–301.

[19] R. Schneider. *Convex Bodies: the Brunn-Minkowski theory.* Encyclopedia of Mathematics and its Applications 44, Cambridge Univ. Press, 1993.

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