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FLUID LIMIT OF GENERALIZED JACKSON QUEUEING NETWORKS WITH STATIONARY AND ERGODIC ARRIVALS AND SERVICE TIMES

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Abstract

We use a sample-path technique to derive asymptotics of generalized Jackson queueing networks in the fluid scale; that is, when space and time are scaled by the same factor n. The analysis only presupposes the existence of long-run averages and is based on some monotonicity and concavity arguments for the fluid processes. The results provide a functional strong law of large numbers for stochastic Jackson queueing networks, since they apply to their sample paths with probability 1. The fluid processes are shown to be piecewise linear and an explicit formulation of the different drifts is computed. A few applications of this fluid limit are given. In particular, a new computation of the constant that appears in the stability condition for such networks is given. In a certain context of a rare event, the fluid limit of the network is also derived explicitly.

Keywords: Generalized Jackson network; sample-path technique; fixed point; fluid limit

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1. Introduction

In this paper, we consider a (single-class) generalized Jackson network and its fluid limit. Such networks have been considered by, among others, Jackson [12] and Gordon and Newell [10]. In [6], Chen and Mandelbaum derived the fluid approximation for generalized Jackson networks. The queue-length, busy-time, and workload processes are obtained from the input processes through the oblique reflection mapping due to Skorokhod [16] in a one-dimensional setting, and to Harrison and Reiman [11] in the context of open networks. Using this fluid approach and assuming that service times and interarrival times are independent and identically distributed (i.i.d.), Dai showed in [7] that generalized Jackson networks are stable (i.e. positive Harris recurrent) when the nominal load is less than 1 at each station. The first stability result for generalized Jackson networks under ergodicity assumptions can be found in the paper of Foss [9]. In [13], Majewski derived a unified formalism that allows for discrete and fluid customers. The inputs for the model are the cumulative service times, the cumulative external arrivals, and the cumulative routeing decisions of the queues. A path space fixed-point equation characterizes the corresponding behavior of the network.

The framework that we use here is that of Baccelli and Foss [1], where only stationarity and ergodicity of the data are assumed. We denote by X^n_0 the time taken to empty the system

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when n customers enter the network simultaneously from the outside world. On the basis of a subadditive argument, the following limit is shown to hold in [1]:

$$\lim_{n\to\infty} \frac{X_0^n}{n} = \gamma(0) \quad \text{almost surely (a.s.).} \tag{1.1}$$

The constant $\gamma(0)$ corresponds to the maximal throughput capacity of the network. In fact, the saturation rule [2] makes this intuition rigorous and ensures that the network is stable if $\rho := \lambda\gamma(0) < 1$, where λ is the intensity of the arrival process. In this paper, we provide a new proof of (1.1), using fluid approximations, which gives an explicit formula for the constant $\gamma(0)$. One contribution of this paper is to provide a connection between the fluid approximation of a generalized Jackson network and the stability condition for this network under stationarity and ergodicity assumptions on the data. In particular, no i.i.d. assumptions are needed (on interarrival times or service times) and we can consider more general routeing mechanisms than Bernoulli routeing.

The other application of this paper will be linked, in a companion paper [3], to the calculation of tails in generalized Jackson networks with subexponential service distributions. Here, we are able to give the behavior (in the fluid scale) of the network in a 'rare' event. (We refer to [3] for an exact notion of what we mean by 'rare' event.)

The results of [6] and [13] will be of minor help to us since a lot of work would be required to obtain our explicit result from them; for these reasons, we have taken a different approach. For each time t, we are able to give an explicit formulation of the fluid limit. The simplicity of the result is due to the concavity of the processes in the fluid scale – a property that, to the best of our knowledge, has not been proved before. In other words, given some drifts for the input processes, when a queue becomes empty it remains empty forever. It seems that this basic fact has not been exploited yet. It allows us to reduce the computation of the fluid limits (which are solutions of a fixed-point network equation in a functional space, as described in [13]) to the computation of a certain traffic intensity for a simplified network that evolves in time. Hence, for a fixed time, we only have to compute a fixed-point solution of some traffic equations (see Section 3). Proposition 3.3 gives the fluid approximation of generalized Jackson networks. To obtain the time to empty the system, we simply observe that if the network is processing fluid, then one of the queues has been working since the initial time. This gives us a very compact way of obtaining the constant $\gamma(0)$ (Theorem 4.1 of Section 4.1). Proposition 4.1 is a slight extension of the main Theorem 4.1, and will be needed in the computation of the fluid picture of a generalized Jackson network in the specific case of a 'single big event' (see [3]).

The paper is structured as follows. In Section 2, we introduce notation for single-server queues and generalized Jackson networks. The fluid limits are established in Section 3. Then, the computation of constant $\gamma(0)$ is given in Section 4 with connections with the stability condition of such networks. In Section 5, we give the fluid picture of the network in the single-big-event framework.

2. General setting and notation

We will use the following notation.

1. $\mathbb{A}_1$ is the set of nonnegative sequences $u = \{u_i\}_{1\le i\le n}$ such that $n \le \infty$ and $u_i \ge 0$ for all $i \le n$. $\mathbb{A}_1^*$ is the set of such sequences such that, rather, $u_i > 0$.

2. $\mathbb{A}_2$ is the set of nondecreasing sequences $U = \{U_i\}_{1\le i\le n}$ such that $n \le \infty$ and $0 \le U_i \le U_{i+1}$ for all $i \le n-1$. $\mathbb{A}_2^*$ is the set of such sequences such that, rather, $0 < U_i < U_{i+1}$.

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We will denote by $\mathbb{A}$ and $\mathbb{A}^*$ the sets of discrete measures on $\mathbb{R}_+$ such that, for each member $\mathrm{d}\mathfrak{U}$ of either set, there exists a $U \in \mathbb{A}_2$ or, respectively, a $U \in \mathbb{A}_2^*$ with $\mathrm{d}\mathfrak{U} = \sum_{1\le i\le n} \delta_{U_i}$. Here, δ_x is the Dirac measure at $x \in \mathbb{R}_+$. To such a measure we can associate a sequence $u \in \mathbb{A}_1$ or $u \in \mathbb{A}_1^*$ in the following manner: $u_i = U_i - U_{i-1}$ for $i \ge 1$, with the convention that $U_0 = 0$. $\mathbb{A}_3$ and $\mathbb{A}_3^*$ will denote the sets of counting functions $\mathcal{U}\colon \mathbb{R}_+ \to \mathbb{N}$ such that $\mathcal{U}(t) = \sum_{1\le i\le n} \mathbf{1}_{\{U_i\le t\}} = \int_0^t \mathrm{d}\mathfrak{U}$ with $\mathrm{d}\mathfrak{U} \in \mathbb{A}$ or, respectively, $\mathrm{d}\mathfrak{U} \in \mathbb{A}^*$. Clearly, the spaces $\mathbb{A}$, $\mathbb{A}_1$, $\mathbb{A}_2$, and $\mathbb{A}_3$ are isomorphic, and the same holds for $\mathbb{A}^*$, $\mathbb{A}_1^*$, $\mathbb{A}_2^*$, and $\mathbb{A}_3^*$.

2.1. Single-server queues

A single-server queue will be defined by $\mathbf{Q} = (\tau^A, \sigma)$, where $\tau^A = \{\tau_i^A\}_{1\le i\le n}$ and $\sigma = \{\sigma_i\}_{1\le i\le n}$ belong to $\mathbb{A}_2$ and $\mathbb{A}_1$, respectively. The interpretations are the following: customer i arrives in the queue at time τ_i^A and its service time is σ_i.

Associated to a queue $\mathbf{Q}$, we define the departure process $\{\tau_i^D\}_{1\le i\le n} \in \mathbb{A}_2$, where τ_i^D is the departure time of customer i, by

$$\tau_1^D = \tau_1^A + \sigma_1,$$
$$\tau_i^D = \max\{\tau_i^A, \tau_{i-1}^D\} + \sigma_i, \qquad 2 \le i \le n.$$

Expanding this recursion yields

$$\tau_i^D = \max_{j=1,\dots,i} (\tau_j^A + \sigma(j,i)), \qquad 1 \le i \le n, \tag{2.1}$$

with the notation $\sigma(j,i) := \sigma_j + \cdots + \sigma_i$. Hence, we define a mapping $\Phi\colon \mathbb{A}\times\mathbb{A} \to \mathbb{A}$ such that

$$\tau^D = \{\tau_i^D\}_{1\le i\le n} = \Phi(\mathbf{Q}). \tag{2.2}$$

We will use the following notation for the different counting functions:

- $A(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i^A \le t\}}$;
- $\Sigma(t) = \sum_{n=1}^{\infty} \mathbf{1}_{\{\sigma(1,n)\le t\}}$;
- $D(t) = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i^D \le t\}}$.

For any nondecreasing function F, we denote by $F^{\leftarrow}(x) = \inf\{t, F(t) \ge x\}$ the pseudo-inverse of F (which is left continuous). We have $F^{\leftarrow}(x) \le u \Leftrightarrow x \le F(u)$. Moreover, we use the notation '$\wedge$' for 'min' and '$\vee$' for 'max'. The following lemma gives a new description of the departure process in terms of counting functions.

Lemma 2.1. *Given a queue $\mathbf{Q} \in \mathbb{A}^*\times\mathbb{A}$, let $D = \Phi(\mathbf{Q})$, where Φ is the mapping defined by (2.1) and (2.2). In terms of counting functions, we have*

$$D(t) = A(t) \wedge \inf_{0\le s\le t} \Sigma(t - s + \Sigma^{\leftarrow}(A(s))). \tag{2.3}$$

The proof is postponed to Appendix A.

Remark 2.1. Equations (2.1) and (2.3) give two equivalent definitions of the mapping $\Phi\colon \mathbb{A}^*\times\mathbb{A} \to \mathbb{A}$. However, for $\tau^A \in \mathbb{A}$ only (2.1) gives the correct definition of Φ. In particular, notice that we always have $\tau_i^D \ge \sigma(1,i) \vee \tau_i^A$, from which we derive that $D(t) \le \Sigma(t) \wedge A(t)$.

2.2. Generalized Jackson networks

We recall here the notation introduced in [1] to describe a generalized Jackson network with K nodes. The networks we consider are characterized by the fact that service times and routeing decisions are associated with stations and not with customers. This means that the jth service at station k takes $\sigma_j^{(k)}$ units of time, where $\{\sigma_j^{(k)}\}_{j\geq 1}$ is a predefined sequence. In the same way, when this service is completed the departing customer is sent to station $\nu_j^{(k)}$ (or leaves the network if $\nu_j^{(k)} = K+1$) and is put at the end of the queue at this station, where $\{\nu_j^{(k)}\}_{j\geq 1}$ is also a predefined sequence, called the routeing sequence. The sequences $\{\sigma_j^{(k)}\}_{j\geq 1}$ and $\{\nu_j^{(k)}\}_{j\geq 1}$, where k ranges over the set of stations, are called the driving sequences of the net. A generalized Jackson network will be defined by

$$\mathbf{JN} = \{\{\sigma_j^{(k)}\}_{j\geq 1}, \{\nu_j^{(k)}\}_{j\geq 1}, n^{(k)}, 0 \leq k \leq K\},$$

where $(n^{(0)}, n^{(1)}, \dots, n^{(K)})$ describes the initial condition. The interpretation is as follows: for $k \neq 0$, at time $t = 0$ and in node k there are $n^{(k)}$ customers with service times $\sigma_1^{(k)}, \dots, \sigma_{n^{(k)}}^{(k)}$ (if appropriate, $\sigma_1^{(k)}$ may be interpreted as a residual service time).

Node 0 models the external arrival of customers in the network. The following statements then hold.

- If $n^{(0)} = 0$, there is no external arrival.
- If $\infty > n^{(0)} \geq 1$ then, for all $1 \leq j \leq n^{(0)}$, the arrival time of the jth customer in the network takes place at $\sigma_1^{(0)} + \cdots + \sigma_j^{(0)}$ and it joins the end of the queue at station $\nu_j^{(0)}$. Hence, $\sigma_j^{(0)}$ is the jth interarrival time. Note that, in this case, there may be a finite number of customers passing through a given station, so that the network is actually well defined once finite sequences of routeing decisions and service times are given on this station.
- If $n^{(0)} = \infty$ then, if we assume that the sequence $\{\sigma_j^{(0)}\}_{j\geq 1}$, say, is i.i.d., the arrival process is a renewal process.

To each node of a generalized Jackson network, we can associate the following counting functions in $\mathbb{A}$:

1. $K+1$ functions associated to the service times $\sigma^{(k)}$ (as in the single-server queue);
2. $K(K+1)$ functions that count the number of customers routed from one node in $\{0, \dots, K\}$ to another node in $\{1, \dots, K\}$;
3. $K+1$ functions associated to $n^{(k)}$.

Hence, a generalized Jackson network with K nodes is an object in $\mathbb{A}^{(K+1)(K+2)} =: \mathbb{A}^{\mathbf{JN}}$. We will use the following notation for each of these counting functions:

- $N = (n^{(0)}, \dots, n^{(K)})$, with $n^{(i)} \geq 0$;
- $\sigma^{(k)} = \{\sigma_j^{(k)}\}_{j\geq 1}$ and $\sigma^{(k)}(1,n) = \sum_{j=1}^{n} \sigma_j^{(k)}$ for $0 \leq k \leq K$;
- $\Sigma^{(i)}(t) = \sum_{n\geq 1} \mathbf{1}_{\{\sigma^{(i)}(1,n)\leq t\}}$ for $0 \leq i \leq K$;
- $P_{i,j}(n) = \sum_{l\leq n} \mathbf{1}_{\{\nu_l^{(i)}=j\}}$ for $0 \leq i \leq K$, $1 \leq j \leq K+1$.

We denote the arrival and departure processes of each queue k of the network by $A^{(k)}$ and $D^{(k)}$, respectively, with the following notation: $\mathbf{A} = (A^{(1)}, \dots, A^{(K)})$ and $\mathbf{D} = (D^{(1)}, \dots, D^{(K)})$. A procedure to construct the processes $\mathbf{A}$ and $\mathbf{D}$ is given in Appendix B. Given a departure process $\Sigma^{(0)}$ for queue 0, departure processes $\mathbf{X} = \{X^{(i)}\}_{1\le i\le K}$ for the queues $i \in [1, K]$, and an initial number of customers $n^{(i)}$ in each queue, we construct the following arrival processes $\mathbf{Y} = \{Y^{(i)}\}_{1\le i\le K}$:

$$Y^{(i)}(t) = n^{(i)} + P_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} P_{j,i}(X^{(j)}(t)).$$

We denote this by $\mathbf{Y} = \Gamma(\mathbf{X}, \mathbf{JN})$.

Finally, given an arrival process $\mathbf{Y}$ for each queue, we define the corresponding departure process $\mathbf{X}$ and denote it by $\mathbf{X} = \Phi(\mathbf{Y}, \mathbf{JN})$. Hence, we have $X^{(i)} = \Phi(Y^{(i)}, \Sigma^{(i)})$, where Φ was defined for the single-server queue in (2.1).

Proposition 2.1. $\mathbf{A}$ *and* $\mathbf{D}$*, the arrival and departure processes of the generalized Jackson network, are the unique solutions of the fixed-point equation*

$$\begin{cases} \mathbf{A} = \Gamma(\mathbf{D}, \mathbf{JN}), \\ \mathbf{D} = \Phi(\mathbf{A}, \mathbf{JN}). \end{cases} \tag{2.4}$$

We will denote by Ψ the mapping from $\mathbb{A}^{\mathbf{JN}}$ to $\mathbb{A}^2$ that to any Jackson network $\mathbf{JN}$ *associates the corresponding couple* $(\mathbf{A}, \mathbf{D})$.

The proof is postponed to Appendix B.

Remark 2.2. This proposition gives the connection between two possible descriptions of a generalized Jackson network. One of these descriptions was given in words at the beginning of this section and is presented with greater precision in Appendix B. The other description is in terms of fixed-point equation (2.4), which was introduced by Majewski in [13]. These two descriptions are equivalent in the special case of discrete inputs and an empty network at time $t = 0-$.

3. Fluid limit and bottleneck analysis

3.1. Fluid limit for single-server queues

For any sequence of functions $\{f^n\}$, we define the corresponding scaled sequence $\{\hat{f}^n\}$ by $\hat{f}^n(t) = f^n(nt)/n$. We say that $f^n \to f$ uniformly on compact sets (u.o.c.) if, for each $t > 0$,

$$\sup_{0\le u\le t} |f^n(u) - f(u)| \to 0 \quad \text{as } n \to \infty.$$

Recall the following lemma, known as Dini's theorem.

Lemma 3.1. *Let $\{f^n\}$ be a sequence of nondecreasing functions on $\mathbb{R}_+$ and let f be a continuous function on $\mathbb{R}$. Assume that $f^n(t) \to f(t)$ for all t (weak convergence is denoted by $f^n \to f$). Then $f^n \to f$ u.o.c.*

The following Lemma can be found in Billingsley [4, p. 287]:

Lemma 3.2. *If f_n are nondecreasing functions and $f_n \to f$, then $f_n^{\leftarrow} \to f^{\leftarrow}$.*

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Proposition 3.1. *Consider a sequence of single-server queues* $\{\mathbf{Q}^n\} = \{\tau^{A,n}, \sigma^n\} \in (\mathbb{A}\times\mathbb{A})^{\mathbb{N}}$ *with associated arrival process* $\tau^{A,n}$ *such that* $\hat{A}^n(t) \to \hat{A}(t)$ *for all* $t > 0$*, with* $\hat{A}$ *concave on* $\mathbb{R}_+$*, and associated service-time process* σ^n *such that* $\hat{\Sigma}^n(t) \to \mu t$ *for all* $t \geq 0$*, with* $\mu \geq 0$*. For such a sequence,* $\hat{D}^n \to \hat{D}$ *u.o.c, with* $\hat{D}(t) = \mu t \wedge \hat{A}(t)$*.*

Proof. First observe that $D^n(t) \leq \Sigma^n(t) \wedge A^n(t)$ (recalling Remark 2.1). Hence, making the fluid scaling and taking the limit in n, we have $\hat{D}(t) \leq \mu t \wedge \hat{A}(t)$. Proposition 3.1 follows, in the case $\mu = 0$, by Lemma 3.1. In the case $\mu > 0$, we assume that $\mathbf{Q}^n \in \mathbb{A}^* \times \mathbb{A}$ for all n and $\hat{A}(0) = 0$.

Since $\hat{A}(0) = 0$, $\hat{A}$ is continuous on $\mathbb{R}_+$ and Lemma 3.1 gives $\hat{A}^n \to \hat{A}$ u.o.c. Moreover, by Lemma 3.2, the sequences $\hat{\Sigma}^n$ and $\hat{\Sigma}^{n\leftarrow}$ converge u.o.c. to the respective functions $t \mapsto \mu t$ and $t \mapsto t/\mu$.

For fixed $t \geq 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{D^n(nt)}{n} &= \lim_{n\to\infty} \left\{ \inf_{0\leq u\leq t} \left(\frac{1}{n} \Sigma^n[n(t-u) + (\Sigma^n)^{\leftarrow}(A^n(nu))] \right) \wedge \frac{A^n(nt)}{n} \right\} \\
&= \inf_{0\leq u\leq t} \left(\lim_{n\to\infty} \frac{1}{n} \Sigma^n[n(t-u) + (\Sigma^n)^{\leftarrow}(A^n(nu))] \right) \wedge \lim_{n\to\infty} \frac{A^n(nt)}{n} \\
&= \inf_{0\leq u\leq t} (\mu(t-u) + \hat{A}(u)) \wedge \hat{A}(t) \\
&= \mu t \wedge \hat{A}(t),
\end{aligned}$$

by uniformity on compact sets, where the last equality follows from concavity of $\hat{A}$. Now, by Lemma 3.1, the result follows in this case. To extend the result to the case of $\mathbf{Q}^n \in \mathbb{A} \times \mathbb{A}$, we consider the sequence $\tau_i^{B,n} = \tau_i^{A,n} + 1/i$, which belongs to $\mathbb{A}^*$. For any $\varepsilon > 0$, we have $A^n(n(t-\varepsilon)) \leq B^n(nt) \leq A^n(nt)$ for $n \geq 1/\varepsilon$. Hence, $\hat{A}(t-\varepsilon) \leq \hat{B}(t) \leq \hat{A}(t)$ and, since $\hat{A}$ is continuous, we have $\hat{B} = \hat{A}$. Moreover, since $\tau_i^{B,n} \geq \tau_i^{A,n}$, we have $D_B^n = \Phi(B^n, \Sigma^n) \leq \Phi(A^n, \Sigma^n)$, and we can apply the first part of the proof to $\hat{B}$. Hence, $D_B^n(t) \to \hat{A}(t) \wedge \mu t$ and the result follows in this case.

The case $\hat{A}(0) \neq 0$ can be dealt with using the same monotonicity argument. For any $\varepsilon > 0$, consider the sequence $\tau_i^{C,n} = \tau_i^{B,n} \vee i\varepsilon$: we have $\hat{C}(t) = (t/\varepsilon) \wedge \hat{A}(t)$ and $\tau_i^{C,n} \geq \tau_i^{A,n}$. We can apply the first part of the proof to $\hat{C}$; hence, $D_C^n(t) \to \hat{C}(t) \wedge \mu t$. For $\varepsilon \leq \mu^{-1}$, we find that $\hat{D}(t) \geq \mu t \wedge \hat{A}(t)$.

3.2. Bottleneck analysis

We first define the noncapture condition, as follows.

Condition 3.1. (Noncapture (NC).) *We say that the* $K \times K$ *matrix* $\boldsymbol{P} = (p_{i,j})_{1\leq i,j\leq K}$ *satisfies the NC condition if* $\boldsymbol{P}$ *is a substochastic matrix such that the stochastic matrix*

$$\boldsymbol{R} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,K} & 1-\sum_i p_{1,i} \\ \vdots & \ddots & \vdots & \vdots \\ p_{K,1} & \cdots & p_{K,K} & 1-\sum_i p_{K,i} \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

has only $K+1$ *as absorbing state, i.e. such that the Markov chain with transition matrix* $\boldsymbol{R}$ *almost surely equals* $K+1$*, eventually.*

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The following lemma is proved in Appendix C.

Lemma 3.3. *Let $\boldsymbol{P}$ be a $K \times K$ substochastic matrix. The following properties are equivalent:*

1. *$\boldsymbol{P}$ satisfies the NC condition.*
2. *The Perron–Frobenius eigenvalue of $\boldsymbol{P}$ is $r < 1$.*
3. *$(I - \boldsymbol{P}^\top)$ is invertible, where $\boldsymbol{P}^\top$ denotes the matrix transpose of $\boldsymbol{P}$.*

For $\boldsymbol{x}$ and $\boldsymbol{y}$ two vectors of $\mathbb{R}^K$, we will write $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all i. For any matrix $\boldsymbol{P}$ and any vectors $\boldsymbol{\alpha}, \boldsymbol{y} \in \mathbb{R}_+^K$, we define $\boldsymbol{F_\alpha} : \mathbb{R}_+^K \to \mathbb{R}_+^K$ and $\boldsymbol{G_y} : \mathbb{R}_+^K \to \mathbb{R}_+^K$ by

$$(\boldsymbol{F_\alpha})_i(x_1, \ldots, x_K) = \alpha_i + \sum_{j=1}^{K} p_{j,i} x_j,$$

$$(\boldsymbol{G_y})_i(x_1, \ldots, x_K) = x_i \wedge y_i.$$

Proposition 3.2. *If the matrix $\boldsymbol{P}$ satisfies the NC condition, the fixed-point equation*

$$\boldsymbol{F_\alpha} \circ \boldsymbol{G_y}(\boldsymbol{x}) = \boldsymbol{x}$$

has a unique solution $\boldsymbol{x}(\boldsymbol{\alpha}, \boldsymbol{y})$. Moreover, $(\boldsymbol{\alpha}, \boldsymbol{y}) \mapsto \boldsymbol{x}(\boldsymbol{\alpha}, \boldsymbol{y})$ is a continuous, nondecreasing function.

Remark 3.1. These relations have already appeared in Massey [14] and Chen and Mandelbaum [6, Section 3.1]. In fact, as pointed out in [6], we can use Tarski's fixed-point theorem [17] to show the existence of this fixed point (called 'inflow' in [6]). However, here we give a self-contained proof that illustrates the continuity and monotonicity properties of the solution.

Proof of Proposition 3.2. The existence of a solution to the fixed-point equation is a simple consequence of monotonicity. Since $\boldsymbol{F_\alpha}$ and $\boldsymbol{G_y}$ are nondecreasing functions and $\boldsymbol{F_\alpha} \circ \boldsymbol{G_y}(\boldsymbol{0}) \geq \boldsymbol{0}$, we see that $(\boldsymbol{F_\alpha} \circ \boldsymbol{G_y})^n(\boldsymbol{0}) \nearrow \boldsymbol{b}$, say. Hence, $\boldsymbol{b} \leq \boldsymbol{F}_\alpha(\boldsymbol{y})$ and $\boldsymbol{F}_\alpha \circ \boldsymbol{G_y}(\boldsymbol{b}) = \boldsymbol{b}$.

For a given subset Δ of $[1, K]$ and $\boldsymbol{y} \in \mathbb{R}_+^K$, we define $\boldsymbol{F}_{\boldsymbol{\alpha},\boldsymbol{y}}^{\Delta} : \mathbb{R}_+^K \to \mathbb{R}_+^K$ by

$$(\boldsymbol{F}_{\boldsymbol{\alpha},\boldsymbol{y}}^{\Delta})_i(x_1, \ldots, x_K) = \alpha_i + \sum_{j \in \Delta} p_{j,i} y_j + \sum_{j \in \Delta^c} p_{j,i} x_j.$$

Therefore, $F_{\cdot,\cdot}^{\Delta}(\cdot)$ depends only on $\{x_i, i \in \Delta^c\}$, and $\boldsymbol{F_\alpha} = \boldsymbol{F}_{\boldsymbol{\alpha},\boldsymbol{y}}^{\varnothing}$.

We fix $\boldsymbol{y} \in \mathbb{R}_+^K$ and first study the case $\boldsymbol{F}_{\boldsymbol{\alpha},\boldsymbol{y}}^{\Delta}(\boldsymbol{x}) = \boldsymbol{x}$. This equation is

$$\begin{cases} x_1 = \alpha_1 + \sum_{j \in \Delta} p_{j,1} y_j + \sum_{j \in \Delta^c} p_{j,1} x_j, \\ \vdots \quad \vdots \qquad\qquad \vdots \\ x_K = \alpha_K + \sum_{j \in \Delta} p_{j,K} y_j + \sum_{j \in \Delta^c} p_{j,K} x_j. \end{cases}$$

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In fact, we only have to calculate $\{x_i, i \in \Delta^c\}$ in order to obtain $\{x_i, i \in \Delta\}$ also. Renumbering the indices of $\boldsymbol{x}$, and taking into account only those in Δ^c, we have

$$\begin{cases} x_1 = \lambda_1(\boldsymbol{\alpha}, \boldsymbol{y}) + \sum_{j=1}^n p_{j,1}^{\Delta} x_j, \\ \vdots \quad \vdots \qquad\qquad \vdots \\ x_n = \lambda_n(\boldsymbol{\alpha}, \boldsymbol{y}) + \sum_{j=1}^n p_{j,n}^{\Delta} x_j. \end{cases} \tag{3.1}$$

$\boldsymbol{P}^{\Delta} = (p_{i,j}^{\Delta}, i, j = 1, \dots, n)$ is a substochastic matrix and $\boldsymbol{I} - \boldsymbol{P}^{\Delta}$ is invertible (even for $\Delta = \varnothing$ – see Lemma 3.3). Hence, if $\boldsymbol{\lambda}(\boldsymbol{\alpha}, \boldsymbol{y}) = (\lambda_1(\boldsymbol{\alpha}, \boldsymbol{y}), \dots, \lambda_n(\boldsymbol{\alpha}, \boldsymbol{y}))$, (3.1) has only one solution: $\tilde{\boldsymbol{x}}^{\Delta} = \boldsymbol{\lambda}(\boldsymbol{\alpha}, \boldsymbol{y}) + \tilde{\boldsymbol{x}}^{\Delta}\boldsymbol{P}^{\Delta} \Leftrightarrow \tilde{\boldsymbol{x}}^{\Delta} = \boldsymbol{\lambda}(\boldsymbol{\alpha}, \boldsymbol{y})(\boldsymbol{I} - \boldsymbol{P}^{\Delta})^{-1}$.

We now return to our fixed-point problem $\boldsymbol{x} = \boldsymbol{F}_{\boldsymbol{\alpha}} \circ \boldsymbol{G}_{\boldsymbol{y}}(\boldsymbol{x})$. To show uniqueness of the solution, take any solution $\boldsymbol{z} = \boldsymbol{F}_{\boldsymbol{\alpha}} \circ \boldsymbol{G}_{\boldsymbol{y}}(\boldsymbol{z})$. We have $\boldsymbol{z} \geq \boldsymbol{0}$ and, hence, $\boldsymbol{F}_{\boldsymbol{\alpha}} \circ \boldsymbol{G}_{\boldsymbol{y}}(\boldsymbol{z}) \geq \boldsymbol{F}_{\alpha} \circ \boldsymbol{G}_{\boldsymbol{y}}(\boldsymbol{0})$ and $\boldsymbol{z} \geq \boldsymbol{b}$. Let $A = \{i : z_i > y_i\}$ and $B = \{i : b_i > y_i\}$. Then, we have $B \subset A$ and $\boldsymbol{b} = \tilde{\boldsymbol{x}}^B$ since $\boldsymbol{F}_{\boldsymbol{\alpha},\boldsymbol{y}}^B(\boldsymbol{b}) = \boldsymbol{F}_{\boldsymbol{\alpha}} \circ \boldsymbol{G}_{\boldsymbol{y}}(\boldsymbol{b}) = \boldsymbol{b}$. Moreover,

$$z_i = \alpha_i + \sum_{j \in B} p_{j,i} y_j + \sum_{j \in A \setminus B} p_{j,i} y_j + \sum_{j \notin A} p_{j,i} z_j,$$
$$(\boldsymbol{F}_{\boldsymbol{\alpha},\boldsymbol{y}}^B)_i(\boldsymbol{z}) = \alpha_i + \sum_{j \in B} r_{j,i} y_j + \sum_{j \in A \setminus B} p_{j,i} z_j + \sum_{j \notin A} p_{j,i} z_j,$$

and, hence, we have $\boldsymbol{F}_{\boldsymbol{\alpha},\boldsymbol{y}}^B(\boldsymbol{z}) \geq \boldsymbol{z}$. However, since $(\boldsymbol{F}_{\boldsymbol{\alpha},\boldsymbol{y}}^B)^n(\boldsymbol{z}) \nearrow \tilde{\boldsymbol{x}}^B = \boldsymbol{b}$, this implies that $\boldsymbol{b} \geq \boldsymbol{z}$. We are forced to conclude that $\boldsymbol{z} = \boldsymbol{b}$.

For any Δ, $(\boldsymbol{\alpha}, \boldsymbol{y}) \mapsto \tilde{\boldsymbol{x}}^{\Delta}(\boldsymbol{\alpha}, \boldsymbol{y}) = \boldsymbol{\lambda}(\boldsymbol{\alpha}, \boldsymbol{y})(\boldsymbol{I} - \boldsymbol{P}^{\Delta})^{-1}$ is a continuous, nondecreasing function. Fix any $(\boldsymbol{\alpha}, \boldsymbol{y})$, and define $A = \{i : x_i(\boldsymbol{\alpha}, \boldsymbol{y}) \geq y_i\}$ and $B = \{i : x_i(\boldsymbol{\alpha}, \boldsymbol{y}) > y_i\}$. Then $\boldsymbol{x}(\boldsymbol{\alpha}, \boldsymbol{y}) = \tilde{\boldsymbol{x}}^A(\boldsymbol{\alpha}, \boldsymbol{y}) = \tilde{\boldsymbol{x}}^B(\boldsymbol{\alpha}, \boldsymbol{y})$ and, for $(\boldsymbol{\beta}, \boldsymbol{z})$ in a neighborhood of $(\boldsymbol{\alpha}, \boldsymbol{y})$, we have $\boldsymbol{x}(\boldsymbol{\beta}, \boldsymbol{z}) \in \{\tilde{\boldsymbol{x}}^A(\boldsymbol{\beta}, \boldsymbol{z}), \tilde{\boldsymbol{x}}^B(\boldsymbol{\beta}, \boldsymbol{z})\}$, and the continuity of $(\boldsymbol{\alpha}, \boldsymbol{y}) \mapsto \boldsymbol{x}(\boldsymbol{\alpha}, \boldsymbol{y})$ follows from that of $(\boldsymbol{\alpha}, \boldsymbol{y}) \mapsto \tilde{\boldsymbol{x}}^{\Delta}(\boldsymbol{\alpha}, \boldsymbol{y})$. Now, to see that this function is nondecreasing, take $(\boldsymbol{\beta}, \boldsymbol{z}) \geq (\boldsymbol{\alpha}, \boldsymbol{y})$. We then have

$$\boldsymbol{F}_{\boldsymbol{\beta}} \circ \boldsymbol{G}_{\boldsymbol{z}}(\boldsymbol{x}(\boldsymbol{\alpha}, \boldsymbol{y})) \geq \boldsymbol{F}_{\boldsymbol{\alpha}} \circ \boldsymbol{G}_{\boldsymbol{y}}(\boldsymbol{x}(\boldsymbol{\alpha}, \boldsymbol{y})) = \boldsymbol{x}(\boldsymbol{\alpha}, \boldsymbol{y})$$

and the sequence $\{(\boldsymbol{F}_{\boldsymbol{\beta}} \circ \boldsymbol{G}_{\boldsymbol{z}})^n(\boldsymbol{x}(\boldsymbol{\alpha}, \boldsymbol{y}))\}_{n \geq 0}$ increases to $\boldsymbol{x}(\boldsymbol{\beta}, \boldsymbol{z})$.

3.3. Fluid limit for generalized Jackson networks

We consider the following sequence of Jackson networks:

$$\mathbf{JN}^n = \{\sigma^n, \nu^n, N^n\} \quad \text{with}$$
$$\lim_{n \to \infty} \frac{N^n}{n} = (n^{(0)}, n^{(1)}, \dots, n^{(K)}), \qquad n^{(0)} \leq \infty,\ n^{(i)} < \infty, i \neq 0.$$

Thanks to Procedure 2, given in Appendix B, we can construct the corresponding arrival and departure processes $\mathbf{A}^n$ and $\mathbf{D}^n$. We assume that the driving sequences satisfy

$$\hat{\Sigma}^{(0),n}(t) \to \Sigma^{(0)}(t), \quad \text{where } t \mapsto \Sigma^{(0)}(t) \wedge n^{(0)} \text{ is a concave function;}$$
$$\hat{\Sigma}^{(k),n}(t) \to \mu^{(k)} t \quad \text{for all } k \geq 1 \text{ and all } t \geq 0\ (\mu^{(k)} \geq 0)\text{; and}$$
$$\hat{P}_{i,j}^n(t) \to p_{i,j} t \quad \text{for all } t \geq 0.$$

We suppose that the routeing matrix $\boldsymbol{P} = (p_{i,j})_{1 \leq i,j \leq K}$ satisfies the NC condition.

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Proposition 3.3. *The processes* $\hat{\mathbf{A}}^n$ *and* $\hat{\mathbf{D}}^n$ *converge uniformly on compact sets to a fluid limit defined by*

$$\hat{A}^{(i)}(t) = n^{(i)} + p_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} p_{j,i}\hat{D}^{(j)}(t), \tag{3.2}$$

$$\hat{D}^{(i)}(t) = \hat{A}^{(i)}(t) \wedge \mu^{(i)}t. \tag{3.3}$$

Remark 3.2. 1. The existence and the uniqueness of solutions to (3.2) and (3.3) follow directly from Proposition 3.2, as shown in the proof. Moreover, it easily follows from the proof that each component of $\hat{\mathbf{A}}$ and $\hat{\mathbf{D}}$ is concave and that if $\Sigma^{(0)}$ is piecewise linear then so are the processes $\hat{\mathbf{A}}$ and $\hat{\mathbf{D}}$.

2. Theorem 7.1 of [6] gives the fluid approximation of a generalized Jackson network. If we take a linear function for $\Sigma^{(0)}$ then, from $(\hat{\mathbf{A}}, \hat{\mathbf{D}})$, we can explicitly calculate the solution of the equations of this theorem.

Proof of Proposition 3.3. For any fixed $n \geq 1$, we define the sequences of processes $\{\mathbf{A}^n_{\mathrm{t}}(k), \mathbf{D}^n_{\mathrm{t}}(k)\}_{k\geq 0}$ and $\{\mathbf{A}^n_{\mathrm{b}}(k), \mathbf{D}^n_{\mathrm{b}}(k)\}_{k\geq 0}$ with the same recurrence equation

$$\begin{cases} \mathbf{A}^n(k+1) = \Gamma(\mathbf{D}^n(k), \mathbf{JN}^n), \\ \mathbf{D}^n(k+1) = \Phi(\mathbf{A}^n(k+1), \mathbf{JN}^n), \end{cases}$$

but with different initial conditions $\mathbf{D}^n_{\mathrm{t}}(0) = (\Sigma^{(1),n}, \dots, \Sigma^{(K),n})$ and $\mathbf{D}^n_{\mathrm{b}}(0) = (0, \dots, 0)$.

We recall the notation

$$\Gamma_i(\mathbf{X}, \mathbf{JN}^n)(t) = n^{(i),n} + P^n_{0,i}(\Sigma^{(0),n}(t) \wedge n^{(0),n}) + \sum_{j=1}^{K} P^n_{j,i}(X_j(t)),$$

$$\Phi_i(\mathbf{X}, \mathbf{JN}^n)(t) = \Phi(X_i, \sigma^{(i),n})(t),$$

and we will use the scaled sequences $\hat{\mathbf{A}}^n(k)(t) = \mathbf{A}^n(k)(nt)/n$ and $\hat{\mathbf{D}}^n(k)(t) = \mathbf{D}^n(k)(nt)/n$. We introduce the mappings $\Gamma^s\colon C(\mathbb{R}_+)^K \to C(\mathbb{R}_+)^K$ and $\Phi^s\colon C(\mathbb{R}_+)^K \to C(\mathbb{R}_+)^K$ (where $C(\mathbb{R}_+)$ is the set of continuous functions on $\mathbb{R}_+$):

$$\Gamma^s_i(x_1, \dots, x_K)(t) = n^{(i)} + p_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} p_{j,i}x_j(t),$$

$$\Phi^s_i(x_1, \dots, x_K)(t) = x_i(t) \wedge \mu^{(i)}t.$$

(These appear implicitly in (3.2) and (3.3).) The following lemma holds for both top and bottom sequences; hence, we omit the subscripts 't' and 'b'.

Lemma 3.4. *Assume that, for a fixed k,* $\hat{\mathbf{D}}^n(k) \to \hat{\mathbf{D}}(k)$ *u.o.c. and that each component of* $\hat{\mathbf{D}}(k)$ *is a concave function. Then we have*

$$\hat{\mathbf{A}}^n(k+1) \xrightarrow{n\to\infty} \Gamma^s(\hat{\mathbf{D}}(k)) = \hat{\mathbf{A}}(k+1) \quad \textit{u.o.c.},$$

$$\hat{\mathbf{D}}^n(k+1) \xrightarrow{n\to\infty} \Phi^s(\hat{\mathbf{A}}(k+1)) = \hat{\mathbf{D}}(k+1) \quad \textit{u.o.c.},$$

and the components of $\hat{\mathbf{A}}(k+1)$ *and* $\hat{\mathbf{D}}(k+1)$ *are concave functions.*

Proof. For any fixed t, we have

$$\frac{A^{(i),n}(k+1)(nt)}{n} = \frac{n^{(i),n}}{n} + \frac{P^n_{0,i}(\Sigma^{(0),n}(nt) \wedge n^{(0),n})}{n} + \sum_{j=1}^{K} \frac{P^n_{i,j}(D^{(j),n}(k)(nt))}{n}.$$

Hence, by Lemma 3.1, we have $\hat{\mathbf{A}}^n(k+1) \xrightarrow{n\to\infty} \Gamma^s(\hat{\mathbf{D}}(k))$ u.o.c. and each component of $\hat{\mathbf{A}}(k+1) = \Gamma^s(\hat{\mathbf{D}}(k))$ is clearly a concave function. The result then follows from Proposition 3.1.

We now return to the proof of Proposition 3.3. We have $\hat{\mathbf{A}}(k+1) = \Gamma^s \circ \Phi^s(\hat{\mathbf{A}}(k))$. This equation gives the relation between two functions of a real parameter t. However, we can fix this parameter and so obtain, for any fixed t, an equation between real numbers that we write as $\hat{\mathbf{A}}(k+1)(t) = \Gamma^s \circ \Phi^s(\hat{\mathbf{A}}(k)(t))$ (even if $\Gamma^s \circ \Phi^s$ is supposed to act on functions). Moreover, as a consequence of Proposition 3.2, we know that the fixed-point equation $\Gamma^s \circ \Phi^s(\boldsymbol{\zeta}(t)) = \boldsymbol{\zeta}(t)$ has a unique solution, namely $\boldsymbol{\zeta}(t) = \boldsymbol{x}(\boldsymbol{\alpha}, \mu^{(1)}t, \dots, \mu^{(K)}t)$, with

$$\boldsymbol{\alpha} = (n^{(1)} + p_{0,1}(\Sigma^{(0)}(t) \wedge n^{(0)}), \dots, \\ n^{(i)} + p_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}), \dots, n^{(K)} + p_{0,K}(\Sigma^{(0)}(t) \wedge n^{(0)})).$$

For any t, the sequence $\{\hat{\mathbf{A}}_b(k)(t)\}_{k\geq 1}$ is nondecreasing and $\{\hat{\mathbf{A}}_t(k)(t)\}_{k\geq 1}$ is nonincreasing. We have

$$\hat{\mathbf{A}}_{\mathrm{b}}(k)(t) \xrightarrow{k\to\infty} \boldsymbol{\zeta}(t) \quad \text{and} \quad \hat{\mathbf{A}}_{\mathrm{t}}(k)(t) \xrightarrow{k\to\infty} \boldsymbol{\zeta}(t),$$

and

$$\hat{\mathbf{D}}_{\mathrm{b}}(k)(t) \xrightarrow{k\to\infty} \Phi^s(\boldsymbol{\zeta}(t)) \quad \text{and} \quad \hat{\mathbf{D}}_{\mathrm{t}}(k)(t) \xrightarrow{k\to\infty} \Phi^s(\boldsymbol{\zeta}(t)).$$

Moreover, if we fix any $n \geq 1$, the mappings $\cdot \mapsto \Gamma(\cdot, \mathbf{JN}^n)$ and $\cdot \mapsto \Phi(\cdot, \mathbf{JN}^n)$ are nondecreasing, and

$$\begin{cases} \mathbf{A}^n = \Gamma(\mathbf{D}^n, \mathbf{JN}^n), \\ \mathbf{D}^n = \Phi(\mathbf{A}^n, \mathbf{JN}^n). \end{cases}$$

Hence, for all $k \geq 0$, we have

$$\mathbf{A}^n_{\mathrm{b}}(k) \leq \mathbf{A}^n \leq \mathbf{A}^n_{\mathrm{t}}(k), \\ \mathbf{D}^n_{\mathrm{b}}(k) \leq \mathbf{D}^n \leq \mathbf{D}^n_{\mathrm{t}}(k).$$

Furthermore,

$$\frac{\mathbf{A}^n_{\mathrm{b}}(k)(nt)}{n} \leq \frac{\mathbf{A}^n(nt)}{n} \leq \frac{\mathbf{A}^n_{\mathrm{t}}(k)(nt)}{n}, \\ \hat{\mathbf{A}}_{\mathrm{b}}(k)(t) \leq \liminf_n \frac{\mathbf{A}^n(nt)}{n} \\ \leq \limsup_n \frac{\mathbf{A}^n(nt)}{n} \leq \hat{\mathbf{A}}_{\mathrm{t}}(k)(t),$$

and, hence, we have

$$\lim_n \frac{\mathbf{A}^n(nt)}{n} = \zeta(t) \quad \text{for all } t.$$

The result follows from Lemma 3.1.

4. Maximal dater asymptotic

4.1. Motivation

We first recall the definition of a simple Euler network from Section 4.1 of [1]. Consider a route $\boldsymbol{p} = (p_1, \dots, p_L)$ with $1 \le p_i \le K$ for $i = 2, \dots, L-1$. Such a route is successful if $p_1 = 0$ and $p_L = K+1$. We can associate to such a route a routeing sequence ν and a vector $\boldsymbol{\phi}$ as follows ('$\oplus$' means concatenation).

Procedure 1.

−1− **for** $k = 0, \dots, K$ **do**
  $\nu^{(k)} := \varnothing$;
  $\phi^{(k)} := 0$;
od

−2− **for** $i = 1, \dots, L-1$ **do**
  $\nu^{(p_i)} := \nu^{(p_i)} \oplus p_{i+1}$;
  $\phi^{(p_i)} := \phi^{(p_i)} + 1$;
od

Note that $\phi^{(j)}$ is the number of visits to node j in such a route.

A simple Euler network is a generalized Jackson network $E = \{\sigma, \nu, N\}$, with

$$N = (1, \underbrace{0, \dots, 0}_{d}).$$

The routeing sequence $\nu = \{\nu_i^{(k)}\}_{i=1,\dots,\phi^{(k)}}$ is generated by a successful route, and $\sigma = \{\sigma_i^{(k)}\}_{i=1,\dots,\phi^{(k)}}$ is a sequence of real-valued nonnegative numbers representing service times.

Now consider a sequence of simple Euler networks, say $\{E(l)\}_{l=1,\dots,\infty}$, where $E(l) = \{\sigma(l), \nu(l), (1, 0, \dots, 0)\}$ (here, $\sigma(l)$ and $\nu(l)$ are the driving sequences of $E(l)$). We define σ and ν to be the infinite concatenations of $\{\sigma(l)\}_{l=1,\dots,\infty}$ and $\{\nu(l)\}_{l=1,\dots,\infty}$, respectively. Denote by σ_c the sequence obtained from σ in the following manner: $\sigma_c = (c\sigma^{(0)}, \sigma^{(1)}, \dots, \sigma^{(K)})$. We consider the corresponding sequence of Jackson networks $\{\mathbf{JN}_c^n\}_n = \{\{\sigma_c, \nu, N^n\}\}_n$, with $N^n = (n, 0, \dots, 0)$. The Jackson network $\mathbf{JN}_c^n$ corresponds to an empty network with n customers in node 0 at time $t = 0$. We will denote by X_c^n the time to empty the system $\mathbf{JN}_c^n$, called the maximal dater of the network. By the Euler property of $\{E(i)\}_{i\ge 1}$, we know that $X_c^n < \infty$ for all n (see [1]). We suppose that

$$\lim_{n\to\infty} \frac{\sigma_c^{(0)}(1,n)}{n} = \frac{c}{\lambda},$$

$$\lim_{n\to\infty} \frac{\sigma^{(k)}(1,n)}{n} = \frac{1}{\mu^{(k)}}, \qquad \mu^{(k)} > 0, \qquad 1 \le k \le K, \tag{4.1}$$

$$\lim_{n\to\infty} \frac{P_{i,j}(n)}{n} = p_{i,j}, \qquad 0 \le i \le K,\ 1 \le j \le K+1. \tag{4.2}$$

We assume that $\boldsymbol{P} = (p_{i,j})_{1\le i,j\le K}$ satisfies the NC condition, and denote by π_i the solution to the following system of equations:

$$\pi_i = p_{0,i} + \sum_{j=1}^{K} p_{j,i}\pi_j, \qquad i = 1, \dots, K. \tag{4.3}$$

The constant π_i is the expected number of visits to site i for the Markov chain with transition matrix $\boldsymbol{P}$ and with initial distribution $p_{0,i}$ (see the proof of Lemma 3.3). We will prove the following theorem.

Theorem 4.1. *Under the previous conditions, we have*

$$\lim_{n\to\infty}\frac{X_c^n}{n} = \max_{1\le i\le K}\frac{\pi_i}{\mu^{(i)}} \vee \frac{c}{\lambda} \quad \textit{for all } c \ge 0.$$

4.2. Proof of Theorem 4.1

Given a routeing matrix $\boldsymbol{P} = (p_{i,j}, i, j = 1, \dots, K)$ that satisfies the NC condition, and a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_K) \in \mathbb{R}_+^K$, we denote by $\pi_i^{\boldsymbol{\alpha}}$ the solution of the following system of equations (see Lemma 3.3):

$$\pi_i^{\boldsymbol{\alpha}} = \alpha_i + \sum_{j=1}^{K} p_{j,i}\pi_j^{\boldsymbol{\alpha}}, \qquad i = 1, \dots, K.$$

Proposition 4.1. *Consider a sequence of Jackson networks, as in Proposition 3.3, such that* $\mu^{(k)} > 0$ *for all* k, $\Sigma^{(0)}(t) = \lambda t/c$ *with* $\lambda > 0$ *and* $c \ge 0$ *(with the convention that division by 0 yields* ∞*), and* $X^n < \infty$ *for all* n*. For such a sequence,*

$$\lim_{n\to\infty}\frac{X_c^n}{n} = \max_{1\le i\le K}\frac{\pi_i^{\boldsymbol{\alpha}}}{\mu^{(i)}} \vee \frac{cn^{(0)}}{\lambda},$$

where $\boldsymbol{\alpha} = (n^{(1)} + n^{(0)}p_{0,1}, \dots, n^{(K)} + n^{(0)}p_{0,K})$.

Proof. To prove the lower bound, consider the auxiliary Jackson network $\tilde{\mathbf{JN}}^n = \{0, \nu^n, N^n\}$, and the associated vector $\boldsymbol{Y}(n)$, where $Y^{(i)}(n)$ is the total number of customers that pass through node i in this network. We have

$$Y^{(i)}(n) = n^{(i),n} + P_{0,i}^n(n^{(0),n}) + \sum_{j=1}^{K} P_{j,i}^n(Y^{(j)}(n))$$

and, hence, $\lim_n Y^{(i)}(n)/n = \pi_i^{\boldsymbol{\alpha}}$, by the NC condition on $\boldsymbol{P}$.

Now consider the original network $\mathbf{JN}_c^n$. The number of customers that pass through node i is still $Y^{(i)}(n)$. Hence, we have the following inequality for the maximal dater of node $i \ge 1$: $X^{(i),n} \ge \sigma^{(i),n}(1, Y^{(i)}(n))$. For node 0, the analogous inequality is $X_c^{(0),n} \ge \sigma_c^{(0),n}(1, n^{(0),n})$. Therefore,

$$\liminf_{n\to\infty}\frac{X^{(i),n}}{n} \ge \lim_{n\to\infty}\frac{\sigma^{(i),n}(1, Y^{(i)}(n))}{n} = \frac{\pi_i^{\boldsymbol{\alpha}}}{\mu^{(i)}},$$
$$\liminf_{n\to\infty}\frac{X_c^{(0),n}}{n} \ge \lim_{n\to\infty}\frac{\sigma_c^{(0),n}(1, n^{(0),n})}{n} = \frac{cn^{(0)}}{\lambda}.$$

Since $X_c^n = \max_{1\le i\le K} X^{(i),n} \vee X_c^{(0),n}$, the lower bound follows.

To prove the upper bound, we consider the original Jackson network. From Proposition 3.3, we know that the corresponding arrival and departure processes $\mathbf{A}^n$ and $\mathbf{D}^n$ converge to the fluid

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limits $\hat{A}$ and $\hat{D}$, respectively. Let $T^{(i)} = \inf\{t > 0 \colon \hat{A}^{(i)}(t) = \hat{D}^{(i)}(t)\}$, $T = \max_{i\in[1,K]} T^{(i)}$, and $M = T \vee cn^{(0)}/\lambda$. Then

$$\hat{A}^{(i)}(t) = n^{(i)} + p_{0,i}n^{(0)} + \sum_{j=1}^{K} p_{j,i}\hat{A}^{(j)}(t) \quad \text{for all } t \geq M$$

and, hence, we have

$$\hat{A}^{(i)}(t) = \hat{D}^{(i)}(t) = \pi_i^{\boldsymbol{\alpha}} \quad \text{for all } t \geq M. \tag{4.4}$$

Writing $i_0 = \arg\max\{T^{(i)}\}$, we have $\hat{A}^{(i_0)}(T) = \hat{D}^{(i_0)}(T) = \mu^{(i_0)}T$ by concavity of $\hat{A}^{(i_0)}$ and, hence, $T = \pi_{i_0}^{\boldsymbol{\alpha}}/\mu^{(i_0)}$. Moreover, (4.4) implies that, for all $t \geq M$,

$$\frac{Y^{(i)}(n) - D^{(i),n}(nt)}{n} \xrightarrow{n\to\infty} 0,$$

where $Y^{(i)}(n)$ is the total number of customers that pass through node i. Since $X_c^n < \infty$, we know that

$$X_c^n \leq nt + \sum_{i=1}^{K} \sigma^{(i),n}(D^{(i),n}(nt), Y^{(i)}(n)) + \sigma_c^{(0),n}(\Sigma^{(0),n}(nt), n^{(0),n})$$

for any t.

Taking $t = M$, we have $\limsup_{n\to\infty} X_c^n/n \leq M = T \vee cn^{(0)}/\lambda = \pi_{i_0}^{\boldsymbol{\alpha}}/\mu^{(i_0)} \vee cn^{(0)}/\lambda$, and the result follows.

Proof of Theorem 4.1. It is easy to see that the assumptions of Proposition 3.3 hold for the Jackson networks $\mathbf{JN}_c^n = \{\sigma_c, \nu, N^n\}$, with $n^{(0)} = 1$ and $n^{(i)} = 0$, $i \neq 0$.

4.3. Stability of generalized Jackson networks

We now give the connection between this fluid limit and the stability region of generalized Jackson networks under stationary ergodicity assumptions, following [1].

Assume that we have a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, endowed with an ergodic measure-preserving shift θ. Consider a sequence of simple Euler networks $\{E(n)\}_{n=-\infty}^{\infty}$, say, where $E(n) = \{\sigma(n), \nu(n), (1, 0, \dots, 0)\}$. Let $\xi(n) = \{\{\sigma(n)\}, \{\nu(n)\}\}$. The stochastic assumptions of [1, Section 4.1] are as follows.

- The variables $\{\sigma(n)\}$ and $\{\nu(n)\}$ are random variables defined on $(\Omega, \mathcal{F}, \mathrm{P})$.
- The random variable $\xi(n)$ satisfies the relation $\xi(n) = \xi(0) \circ \theta^n$ for all n, which implies that $\{\xi(n)\}_n$ is stationary and ergodic.
- All the expectations $\mathrm{E}[\phi^{(k)}(0)]$ and $\mathrm{E}[\sum_{i=1}^{\phi^{(k)}(0)} \sigma_i^{(k)}(0)]$ are finite ($\phi^{(j)}(n)$ is obtained by Procedure 1 on $E(n)$).

In such a setting, we can find Ω_0 such that, on Ω_0, (4.1), (4.2), and Condition 3.1 (noncapture) hold and $\mathrm{P}(\Omega_0) = 1$. By the strong law of large numbers, we have, almost surely,

$$\frac{1}{n}(\phi^{(j)}(1) + \cdots + \phi^{(j)}(n)) \to \mathrm{E}[\phi^{(j)}(0)] < \infty,$$

$$\frac{1}{n}\Bigg(\sum_{i=1}^{\phi^{(j)}(1)} \sigma_i^{(j)}(1) + \cdots + \sum_{i=1}^{\phi^{(j)}(n)} \sigma_i^{(j)}(n)\Bigg) \to \mathrm{E}\Bigg[\sum_{i=1}^{\phi^{(j)}(0)} \sigma_i^{(j)}(0)\Bigg] < \infty.$$

From these equations, we derive condition (4.1):

$$\lim_{n\to\infty}\frac{\sigma^{(j)}(1,n)}{n}=\frac{\mathrm{E}[\sum_{i=1}^{\phi^{(j)}(0)}\sigma_i^{(j)}(0)]}{\mathrm{E}[\phi^{(j)}(0)]}=:\frac{1}{\mu^{(j)}}\quad\text{a.s.}$$

By the same kind of argument, we can show that limit (4.2) holds almost surely. To show that $\boldsymbol{P}$ satisfies the NC condition, we write $V^{(j)}=\mathrm{E}[\phi^{(j)}(0)]$ and $V^{(j)}(n)=\phi^{(j)}(1)+\cdots+\phi^{(j)}(n)$. Owing to the Euler property of the graphs, we have $V^{(i)}(n)=P_{0,i}(n)+\sum_{j=1}^{K}P_{j,i}(V^{(j)}(n))$, whence $V^{(i)}=p_{0,i}+\sum_{j=1}^{K}p_{j,i}V^{(j)}$. Equation (4.3) has a finite solution, so $\boldsymbol{P}$ satisfies the NC condition and $V^{(i)}=\pi_i$ (see Lemma 3.3). Now we can define Ω_0 as follows:

$$\Omega_0=\left\{\frac{\sigma^{(k)}(1,n)}{n}\to\frac{1}{\mu^{(k)}},\ \frac{P_{i,j}(n)}{n}\to p_{i,j},\ \frac{V^{(j)}(n)}{n}\to\pi_j\right\}.$$

We will use the conventional notation $\mu^{(0)}=\lambda$ for the intensity of the external arrival.

The limit calculated in Theorem 4.1 is exactly the constant $\delta(c)$ defined in [1, Equation (85)]. In the event Ω_0, Theorem 4.1 applies and gives a new proof of Theorem 15 of [1], which says that $\delta(0)=\gamma(0)=\max_i\pi_i/\mu^{(i)}$. Moreover, the lower bound of Lemma 6 (in [1]) is in fact shown to be the exact value of $\delta(c)$. Theorems 13 and 14 of [1] give the stability condition of a Jackson-type queueing network in an ergodic setting. To be more precise, for $m\le n\le 0$ we define $\sigma_{[m,n]}$ and $\nu_{[m,n]}$ to be the concatenations of $\{\sigma(k)\}_{m\le k\le n}$ and $\{\nu(k)\}_{m\le k\le n}$, and then define the corresponding generalized Jackson networks as

$$\mathbf{JN}_{[m,n]}=\{\sigma_{[m,n]},\nu_{[m,n]},N_{[m,n]}\},\qquad\text{with } N_{[m,n]}=(n-m+1,0,\dots,0).$$

We define $X_{[m,n]}$ to be the time taken to empty the generalized Jackson network $\mathbf{JN}_{[m,n]}$ and denote by $Z_{[m,n]}=X_{[m,n]}-\sum_{i=1}^{n-m+1}\sigma_{[m,n],i}^{(0)}$ the associated maximal dater. (Note that our notation is consistent with [1].) The sequence $Z_{[-n,0]}$ is increasing, so there exists a limit $Z=\lim_{n\to\infty}Z_{[-n,0]}$ (which may be either finite or infinite). We call this limit the maximal dater of the generalized Jackson network $\mathbf{JN}=\{\sigma,\nu,N\}$ where σ and ν are the infinite concatenations of $\{\sigma(k)\}_{k\le0}$ and $\{\nu(k)\}_{k\le0}$, respectively, and $N=(\infty,0,\dots,0)$. Let A be the event

$$A=\Big\{Z=\lim_{n\to\infty}Z_{[-n,0]}=\infty\Big\}.$$

This event is of crucial interest, since a finite, stationary construction of the state of the network can only be made on the complementary part of A. In other words, $Z<\infty$ if and only if the network is stable. The following theorem follows from Theorems 13 and 14 of [1].

Theorem 4.2. *Let* $\rho=\lambda\max_{1\le i\le K}\pi_i/\mu^{(i)}$. *If* $\rho<1$ *then* $\mathrm{P}(A)=0$ *while, if* $\rho>1$, *then* $\mathrm{P}(A)=1$.

Remark 4.1. There exists a parallel stream of work that uses sample path methods – quite different from those described in this paper – to prove a weaker form of stability called pathwise stability or rate stability. Rate stability means that, with probability 1, the long-run average departures must equal the long-run average arrivals at each station. In Chen [5], it was proved that, for a multiclass queueing network under work-conserving service disciplines, the weak stability of the fluid model implies the rate stability of the stochastic network. We refer the reader to [5] for a detailed definition of the fluid model and weak stability; the main result of [5] is that, under the usual traffic conditions, a generalized Jackson network is rate stable.

Furthermore, in Dai [8] it was proved, under weak strong-law-of-large-numbers assumptions, that if $\rho > 1$ (with our notation) then the number of customers in the network diverges to infinity with probability 1 as time $t \to \infty$ (see Proposition 5.1 of [8]). This result corresponds to the second part of our Theorem 4.2. To prove that $\rho < 1$ ensures stability of the generalized Jackson network, Dai [7] needed i.i.d. assumptions and additional conditions on the interarrival times, namely that they be unbounded and spread out. In this paper, we use fluid limits to derive the same result under stationarity and ergodicity conditions only.

5. Rare events in generalized Jackson networks

The aim of this section is to give a picture of a particular kind of rare event in which the maximal dater of a generalized Jackson network becomes very big. Under some stochastic assumptions, we can prove that large maximal daters occur when a single large service time has 'taken place' at one of the stations, while all other service times are close to their mean (see [3]). We now give the corresponding fluid picture.

5.1. The single-big-event framework

We consider a sequence of simple Euler networks $\{E(n)\}_{n=-\infty}^{\infty}$, say, where $E(n) = \{\sigma(n), \nu(n), 1\}$. Considering the corresponding $\mathbf{JN}_{[-n,\infty]}$ network, we assume that

$$\hat{\Sigma}^{(0),n}(t) \to t/a \quad \text{for all } t, \tag{5.1}$$

$$\hat{\Sigma}^{(k),n}(t) \to \mu^{(k)} t \quad \text{for all } t \text{ and all } k \geq 1, \tag{5.2}$$

$$\hat{P}^{n}_{i,j}(t) \to p_{i,j} t \quad \text{for all } t, i, \text{ and } j. \tag{5.3}$$

We further assume that $\boldsymbol{P} = (p_{i,j})_{1 \leq i,j \leq K}$ satisfies the NC condition, and we use the following notation:

$$\pi_i = p_{0,i} + \sum_{k=1}^{K} p_{k,i}\pi_k, \qquad i = 1, \dots, K, \tag{5.4}$$

$$x_i = p_{0,i} + \sum_{k \neq j} p_{k,i} x_k \Rightarrow x_j = p_j, \qquad i = 1, \dots, K, \tag{5.5}$$

$$\pi_{j,i} = \delta_{j,i} + \sum_{k=1}^{K} p_{k,i}\pi_{j,k}, \qquad i = 1, \dots, K. \tag{5.6}$$

Equation (5.4) is the traditional traffic equation of the network in terms of numbers of customers. In (5.5), p_j corresponds to the amount of traffic coming in queue j if queue i is blocked (its departure process is null). Note that $x_i \leq \pi_i$ in this case. Equation (5.6) corresponds to the traffic equation in the network when there is no input from the outside world and only queue j is active. We introduce the corresponding loads

$$b_j = \frac{\pi_j}{\mu^{(j)}}, \qquad b = \max_j b_j, \qquad b_{j,i} = \frac{\pi_{j,i}}{\mu^{(i)}}, \quad \text{and} \quad B_j = \max_i b_{j,i},$$

and assume that the stability condition $b < a$ holds. We suppose that the big event (i.e. the large service time) occurs in the simple Euler network $-n$ and, so, we replace $E(-n)$ by an extra E. The replacement network E is atypical in the sense that a large service time σ takes place at station j and within the set of service times of E.

Let us look at the corresponding maximal dater $Z_{[-n,0]}(E)$ in the fluid scale suggested by the limit of Proposition 4.1. If $\sigma > na$ then the number of customers blocked at station j at time σ is of order np_j, whereas the numbers of customers at the other stations are small. So, according to Proposition 4.1, the time taken to empty the network from time σ on should be of order np_jB_j; hence, in this case, the maximal dater in question should be of order $\sigma - na + np_jB_j$. On the other hand, if $\sigma < na$ then, at time σ, the number of customers blocked at station j is of order $p_j\sigma/a$, and the other stations have few customers; from time σ to the time of the last arrival (which is of order na), station k has to serve approximately the load $(p_j\sigma/a)b_{j,k}$ generated by these blocked customers plus the load $(na-\sigma)b_k/a$ generated by the external arrivals in the time interval from σ to the last arrival. In this time interval, the service capacity is of order $na-\sigma$. Hence, the maximal dater should be of order $\max_k\{p_jb_{j,k}\sigma/a + (na-\sigma)b_k/a - (na-\sigma)\}^+$.

It is now natural to introduce the following function:

$$f^j(\sigma, n) = \mathbf{1}_{\{\sigma > na\}}(\sigma - na + np_jB_j) + \mathbf{1}_{\{\sigma \le na\}} \max_k \left\{ p_jb_{j,k}\frac{\sigma}{a} + \left(\frac{b_k}{a} - 1\right)(na-\sigma)\right\}^+.$$

To make our discussion more rigorous, consider the network $\mathbf{JN}^n(E)$ with input $\{\tilde{E}(k)\}_{k=-n}^{\infty}$, where $\tilde{E}(k) = E(k)$ for all $k > -n$ and $\tilde{E}(-n) = E := \{\sigma(E), \nu(E), (1, 0, \dots, 0)\}$. That is, if we denote by $\sigma^{(k),n}$ and $\nu^{(k),n}$ the concatenations $(\{\sigma^{(k)}(E)\}, \{\sigma^{(k)}(-n+1)\}, \dots, \{\sigma^{(k)}(0)\}, \dots)$ and $(\{\nu^{(k)}(E)\}, \{\nu^{(k)}(-n+1)\}, \dots, \{\nu^{(k)}(0)\}, \dots)$, respectively, then

$$\mathbf{JN}^n(E) = \{\sigma^n(E), \nu^n(E), N^n\}, \quad \text{with } N^n = (n, 0, \dots, 0).$$

The maximal dater of order $[-n, 0]$ in this network will be denoted by $\tilde{Z}^n(E)$; of course, $\tilde{Z}^n(E(n)) = Z_{[-n,0]}$. For all simple Euler networks $E = \{\sigma, \nu, (1, 0, \dots, 0)\}$, let $Y^{(j)}(E) = \sum_{u=1}^{\phi^{(j)}} \sigma_u^{(j)}$. For some sequence of positive real numbers z_n, we define

$$\mathbf{U}^j(n) = \{E \text{ is a simple Euler network such that } Y^{(k)}(E) \le z_n \text{ for all } k \ne j\},$$
$$\mathbf{V}^j(n) = \{E \in \mathbf{U}^j(n),\ Y^{(j)}(E) \ge n(a-b),\ \phi^{(j)} \le L\}.$$

Proposition 5.1. *Under the previous assumptions, there exists a sequence $z_n \to \infty$, with $z_n/n \to 0$, such that*

$$\sup_{E \in \mathbf{V}^j(n)} \left| \frac{1}{n}(\tilde{Z}^n(E) - f^j(Y^{(j)}(E), n)) \right| \xrightarrow{n\to\infty} 0. \tag{5.7}$$

5.2. Computation of the fluid limit (Proof of Proposition 5.1)

We take a sequence of simple Euler networks $F_n \in \mathbf{V}^j(n)$ and write $\mathbf{JN}^n = \mathbf{JN}^n(F_n)$. Since z_n/n tends to 0, we have

$$\hat{\Sigma}^{(0),n}(t) \to t/a \quad \text{for all } t, \text{ a.s.},$$
$$\hat{\Sigma}^{(k),n}(t) \to \mu^{(k)}t \quad \text{for all } k \ne j \ge 1 \text{ and all } t, \text{ a.s.},$$
$$\hat{P}^n_{i,j}(t) \to p_{i,j}t \quad \text{for all } i \text{ and } j \text{ and all } t, \text{ a.s.}$$

We write $\zeta_n = Y^{(j)}(F_n) \in [n(a-b), \infty)$ and denote by T_n the time taken by station j to complete its first $\phi^{(j)}(F_n)$ services in the network $\mathbf{JN}^n$. From monotonicity, we obtain $\zeta_n \le T_n \le \zeta_n + \sum_{k\ne j} Y^{(k)}(F_n)$. Hence, we have $\lim_{n\to\infty} T_n/\zeta_n = 1$, since $z_n/\zeta_n \le z_n/n(a-b) \to 0$.

If we assume that $\zeta_n/n \to \zeta < \infty$, then $\mathbf{JN}^n$ is such that $\Sigma^{(j),n}(t) \le L$ for $t \le T_n$. Hence, $\Sigma^{(j),n}(nt)/n \le L/n$ for $nt \le T_n$, so that $\hat{\Sigma}^{(j),n}(t) \to 0$ for all $t \le \zeta_n$. We see that this last fluid limit does not hold on the whole positive real line. Nevertheless, consider the Jackson networks with the same driving sequences as $\mathbf{JN}^n$ except for at station j, where we take the concatenation of $(\{\sigma^{(j)}(F_n)\}, \infty, \dots)$. For this new network, the fluid limit for station j holds on $\mathbb{R}_+$ and we can directly apply Proposition 3.3. However, it is easy to see that, for $t \le T_n$, this network and the original Jackson network $\mathbf{JN}^n$ have exactly the same dynamics. Hence, Proposition 3.3 applies for $t \le \zeta$, so that for each k the sequence $\{\hat{A}^{(k),n}\}$ converges u.o.c. to a limit $\hat{A}^{(k)}$ when n tends to ∞, with a similar result and notation for the departure process. We have, with $\lambda = a^{-1}$,

$$\begin{aligned}
\hat{A}^{(i)}(t) &= p_{0,i}\lambda(t \wedge a) + \sum_{k=1}^{K} p_{k,i}\hat{D}^{(k)}(t),\\
\hat{D}^{(i)}(t) &= \hat{A}^{(i)}(t) \wedge \tilde{\mu}^{(i)}t \quad \text{with } \tilde{\mu}^{(i)} = \mu^{(i)} \text{ for } i \ne j \text{ and } \tilde{\mu}^{(j)} = 0.
\end{aligned}$$

We can rewrite the first expression as

$$\hat{A}^{(i)}(t) = \lambda p_{0,i}(t \wedge a) + \sum_{k \ne j} p_{k,i}\hat{D}^{(k)}(t).$$

Hence, with the notation introduced in Section 5.1, we have

$$\begin{aligned}
\hat{A}^{(i)}(t) &= \hat{D}^{(i)}(t) = \lambda x_i(t \wedge a) \le \lambda \pi_i(t \wedge a) \quad \text{for } t \le \zeta \text{ and } i \ne j,\\
\hat{A}^{(j)}(t) &= \lambda p_j(t \wedge a) \quad \text{for } t \le \zeta.
\end{aligned}$$

In what follows, we will consider the (new) Jackson network obtained by taking the state of the initial network at time T_n as initial condition and, as routeing and service sequences, the routeing decisions and (residual) service still unused at this time. This network will be denoted by $\bar{\mathbf{JN}}^n = \{\bar{\sigma}^n, \bar{\nu}^n, \bar{N}^n\}$, with

$$\begin{aligned}
\bar{\sigma}^{(0),n} &= \{\Sigma^{(0),n\leftarrow}(\Sigma^{(0),n}(T_n)+1) - T_n, \sigma^{(0),n}_{\Sigma^{(0),n}(T_n)+2}, \dots\},\\
\bar{\nu}^{(0),n} &= \{\nu^{(0),n}_{\Sigma^{(0),n}(T_n)+1}, \nu^{(0),n}_{\Sigma^{(0),n}(T_n)+2}, \dots\},\\
\bar{N}^{(0),n} &= n - \Sigma^{(0),n}(T_n),
\end{aligned}$$

and, for $i \ne 0$,

$$\begin{aligned}
\bar{\sigma}^{(i),n} &= \{r^{(i),n}, \sigma^{(i),n}_{D^{(i),n}(T_n)+2}, \dots\},\\
\bar{\nu}^{(i),n} &= \{\nu^{(i),n}_{D^{(i),n}(T_n)+1}, \nu^{(i),n}_{D^{(i),n}(T_n)+2}, \dots\},\\
\bar{N}^{(i),n} &= A^{(i),n}(T_n) - D^{(i),n}(T_n), \quad \text{where}\\
r^{(i),n} &= \begin{cases} \sigma^{(i),n}_{D^{(i),n}(T_n)+1} & \text{if } A^{(i),n}(T_n) = D^{(i),n}(T_n),\\ D^{(i),n\leftarrow}(D^{(i),n}(T_n)+1) - T_n & \text{otherwise.}\end{cases}
\end{aligned}$$

We have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} A^{(i),n}(T_n) = \lim_{n\to\infty} \frac{1}{n} D^{(i),n}(T_n) = \hat{A}^{(i)}(\zeta) = \hat{D}^{(i)}(\zeta) \quad \text{for } i \ne j,\\
&\lim_{n\to\infty} \frac{1}{n} A^{(j),n}(T_n) = \lambda p_j(\zeta \wedge a), \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} D^{(j),n}(T_n) = 0.
\end{aligned}$$

Hence,

$$\begin{aligned} \frac{\bar{N}^n}{n} &\to (\lambda(a-\zeta)^+, 0, \dots, \lambda p_j(\zeta \wedge a), \dots, 0) \\ \hat{\hat{\Sigma}}^{(0),n}(t) &\to \lambda t \quad \text{for all } t, \\ \hat{\hat{\Sigma}}^{(i),n}(t) &\to \mu^{(i)} t \quad \text{for all } i \geq 1 \text{ and all } t, \\ \hat{\hat{P}}^n_{i,j}(t) &\to p_{i,j} t \quad \text{for all } i \text{ and } j \text{ and all } t. \end{aligned}$$

We can apply Proposition 4.1 with a parameter $\boldsymbol{\alpha}$ that depends on the quantity $a - \zeta$. If $\zeta \geq a$ we then have $\boldsymbol{\alpha} = p_j \boldsymbol{e}_j$, where $\boldsymbol{e}_j = (0, \dots, 1, \dots, 0)$ with the 1 in the jth position, and

$$\pi_i^{\boldsymbol{\alpha}} = p_j \pi_{j,i}.$$

Proposition 4.1 gives

$$\frac{\tilde{Z}_n(F_n) + na - T_n}{n} \to p_j \max_i \frac{\pi_{j,i}}{\mu^{(i)}} \tag{5.8}$$

and, hence, we have

$$\tilde{Z}_n(F_n) = (T_n - na + np_j B_j)(1 + o(n)) = f^j(T_n, n)(1 + o(n)). \tag{5.9}$$

On the other hand, if $\zeta < a$ we then have $\boldsymbol{\alpha} = \lambda(a-\zeta)\boldsymbol{P}_0 + \lambda p_j \zeta \boldsymbol{e}_j$, where $\boldsymbol{P}_0 = (p_{0,1}, \dots, p_{0,K})$, and

$$\pi_i^{\boldsymbol{\alpha}} = \lambda[(a-\zeta)\pi_i + p_j \pi_{j,i}\zeta].$$

In this case, Proposition 4.1 gives

$$\frac{\tilde{Z}_n(F_n) + na - T_n}{n} \to (a-\zeta) \vee \lambda \max_i \left[\frac{(a-\zeta)\pi_i + p_j \pi_{j,i}\zeta}{\mu^{(i)}}\right]$$

and, hence, we have

$$\begin{aligned} \tilde{Z}_n(F_n) &= (1 + o(n)) \max_i \left[p_j b_{j,i} \frac{T_n}{a} + (na - T_n)\left(\frac{b_i}{a} - 1\right)\right]^+ \\ &= f^j(T_n, n)(1 + o(n)). \end{aligned}$$

In the case $\zeta_n / n \to \infty$, which corresponds to $\zeta = \infty$, our results up until (5.8) still hold and, hence, (5.9) holds as well.

This proves that

$$\left|\frac{\tilde{Z}_n(F_n) - f^j(\zeta_n, n)}{n}\right| \xrightarrow{n\to\infty} 0 \tag{5.10}$$

for any sequence $F_n \in \mathbf{V}^j(n)$ with $Y^{(j)}(F_n) = \zeta_n \in [n(a-b), \infty)$ such that $\zeta_n/n \to \zeta \leq \infty$. However, this result holds for any sequence $F_n \in \mathbf{V}^j(n)$. Consider any $F_n \in \mathbf{V}^j(n)$ and suppose that

$$\limsup_{n\to\infty} \left|\frac{\tilde{Z}_n(F_n) - f^j(Y^{(j)}(F_n), n)}{n}\right| = l > 0.$$

By extracting a subsequence of $\{F_n\}$, we can replace 'lim sup' by 'lim'. Moreover, by making a further extraction, we may suppose that $Y^{(j)}(F_n)/n \to \zeta \le \infty$ and, for this subsequence, limit (5.10) is violated. Hence, for any sequence F_n, we have

$$\left|\frac{\tilde{Z}_n(F_n) - f^j(Y^{(j)}(F_n), n)}{n}\right| \xrightarrow{n\to\infty} 0. \tag{5.11}$$

Now consider a sequence $F_n \in \mathbf{V}^j(n)$ such that

$$\left|\frac{\tilde{Z}_n(F_n) - f^j(Y^{(j)}(F_n), n)}{n}\right| \ge \sup_{E\in\mathbf{V}^j(n)}\left|\frac{\tilde{Z}_n(E) - f^j(Y^{(j)}(E), n)}{n}\right| - \varepsilon_n,$$

with $\varepsilon_n \to 0$. We then see that (5.7) follows from (5.11).

Remark 5.1. In the stochastic framework of Section 4.3, we see that our assumptions on the limits (5.1), (5.2), and (5.3) are justified. In particular, if the sequence of simple Euler networks $\{E(n)\}_{n=-\infty}^{\infty}$ is i.i.d., then we deduce from the previous proposition that

$$\sup_{E\in\mathbf{V}^j(n)}\left|\frac{\tilde{Z}^n(E) - f^j(Y^{(j)}(E), n)}{n}\right| \xrightarrow{n\to\infty} 0 \quad \text{a.s.}$$

Appendix A. Proof of Lemma 2.1

For $1 \le j$, we define the point process Γ_j as follows:

$$\begin{aligned} \tau_n^{\Gamma_j} &= 0 && \text{for } 1 \le n \le j-1, \\ \tau_n^{\Gamma_j} &= \tau_j^A + \sigma(j, n) && \text{for } n \ge j. \end{aligned}$$

The construction of Γ_j is depicted in Figure 1. For $j \ge 1$, we have

$$\Gamma_j(t) = \begin{cases} j-1 & \text{for } t < \tau_j^A, \\ \Sigma(t - \tau_j^A + \sigma(1, j-1)) & \text{for } t \ge \tau_j^A, \end{cases}$$

with the convention that $\sigma(1, 0) = 0$.

From (2.1), we have $t \ge \tau_n^D \Leftrightarrow t \ge \tau_n^{\Gamma_j}$ for all $j \le n$ and, hence,

$$D(t) \ge n \Leftrightarrow \inf_{j\le n} \Gamma_j(t) \ge n.$$

However, for all $j \ge n+1$, we have $\Gamma_j(t) \ge n$ for all t so, in fact, $D(t) = \inf_{j\ge 1}\Gamma_j(t)$. We also have

$$\inf_{j\ge 1}\Gamma_j(t) = \inf_{j\ge 1, \tau_j^A \le t} \Sigma[t - \tau_j^A + \sigma(1, j-1)] \wedge A(t),$$

and we now show that $\inf_{j\ge 1}\Gamma_j(t) = A(t) \wedge \inf_{0\le s\le t}\Sigma[t - s + \Sigma^{\leftarrow}(A(s))]$. Since $\tau_j^A \in \mathbb{A}_2^*$, on each interval $[\tau_{j-1}^A, \tau_j^A)$ (we use the convention $\tau_0^A = 0$), we have $A(s) = j-1$ and the function $s \mapsto \Sigma[t - s + \Sigma^{\leftarrow}(j-1)]$ is nonincreasing. Hence,

$$\inf_{s\in[\tau_{j-1}^A, \tau_j^A)} \Sigma[t - s + \Sigma^{\leftarrow}(A(s))] = \Sigma[t - \tau_j^A + \sigma(1, j-1)].$$

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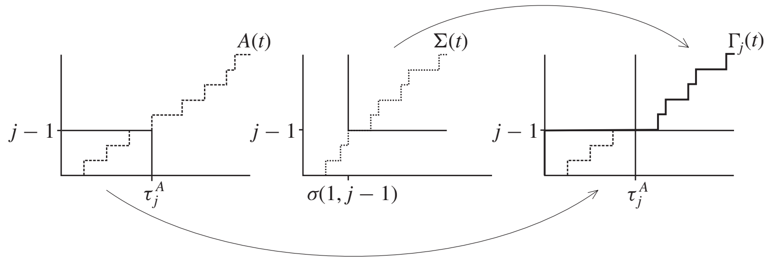

FIGURE 1: Construction of Γ_j.

Moreover, for $\tau_k^A \le t < \tau_{k+1}^A$ we have

$$\inf_{s\in[\tau_k^A, t)} \Sigma[t-s+\Sigma^{\leftarrow}(A(s))] = \Sigma(\Sigma^{\leftarrow}(k)) \ge k = A(t).$$

Finally, we have

$$\begin{aligned} A(t) \wedge \inf_{0\le s\le t} \Sigma[t-s+\Sigma^{\leftarrow}(A(s))] &= A(t) \wedge \inf_{j\ge 1, \tau_j^A \le t} \Sigma[t-\tau_j^A + \sigma(1, j-1)] \\ &= \inf_{j\ge 1} \Gamma_j(t). \end{aligned}$$

Appendix B. Construction of arrival and departure processes

Here we give a procedure that constructs the processes **A** and **D**.

Procedure 2.

–1– $t := 0$;
for $i \ge 0$ **do**
$R^{(i)}(t) := \sigma_1^{(i)}$; $A^{(i)}(t) := n^{(i)}$; $D^{(i)}(t) := 0$;
od

–2– $V := \min_{\{i\colon A^{(i)}(t)-D^{(i)}(t)\ge 1\}} R^{(i)}(t)$; $\gamma := \arg\min_{\{i\colon A^{(i)}(t)-D^{(i)}(t)\ge 1\}} R^{(i)}(t)$;

–3– **if** $V = \infty$ **then** *END*;
fi

–4– $D^{(\gamma)}(t+V) := D^{(\gamma)}(t) + 1$; $A^{(\gamma)}(t+V) := A^{(\gamma)}(t)$;
if $A^{(\gamma)}(t+V) - D^{(\gamma)}(t+V) \ge 1$ **then** $R^{(\gamma)}(t+V) := \sigma^{(\gamma)}_{D^{(\gamma)}(t+V)+1}$; **fi**
$j := \nu^{(\gamma)}_{D^{(\gamma)}(t+V)}$;
if $j \ne K+1$ **then** $A^{(j)}(t+V) := A^{(j)}(t) + 1$; $D^{(j)}(t+V) := D^{(j)}(t)$;
if $A^{(j)}(t) - D^{(j)}(t) = 0$ **then** $R^{(j)}(t+V) := \sigma^{(j)}_{A^{(j)}(t+V)}$; **fi**
fi
for $i \notin \{\gamma, j\}$ **do**
$R^{(i)}(t+V) := R^{(i)}(t) - V$; $A^{(i)}(t+V) := A^{(i)}(t)$; $D^{(i)}(t+V) := D^{(i)}(t)$;
od
$t := t + V$;

–5– **goto** 2;

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Remark B.1. Since the sequences $\{\sigma_j^{(k)}\}_{j\geq 1}$ and $\{\nu_j^{(k)}\}_{j\geq 1}$ are infinite, the variables $\nu^{(\gamma)}_{D^{(\gamma)}(t+V)}$, $\sigma^{(\gamma)}_{D^{(\gamma)}(t+V)}$, and $\sigma^{(j)}_{A^{(j)}(t+V)}$ in step –4– are always available. Also note that the procedure ends in step –3– if $\sum_{i=0}^{K} n^{(i)} < \infty$. If $\sum_{i=0}^{K} n^{(i)} = \infty$, on the other hand, the procedure never ends. The latter situation corresponds to a network with an infinite number of customers. In this case, there exists $T \leq \infty$ such that $\lim_{t\to T} A(t) = \lim_{t\to T} D(t) = \infty$.

Proof of Proposition 2.1. If we define $J^{(k)} = \sup\{j\colon \sum_{i=1}^{j} \sigma_i^{(k)} = 0\}$, the generalized Jackson network is equivalent to

$$\left\{\{\sigma_j^{(k)}\}_{j\geq J^{(k)}+1}, \{\nu_j^{(k)}\}_{j\geq J^{(k)}+1}, n^{(k)} + \sum_{i=0}^{K} P_{i,k}(J^{(i)})\right\}.$$

Hence, we can assume that $J^{(k)} = 0$ for all k and we have $A^{(i)}(0) = n_i$ and $D^{(i)}(0) = 0$ for time $t = 0$. For $t \geq 0$, let

$$\tilde{D}^{(i)}(t) = A^{(i)}(0) \wedge \inf_{0\leq s\leq t} \Sigma^{(i)}[t - s + \Sigma^{(i)\leftarrow}(A^{(i)}(0))],$$

$$\tilde{A}^{(i)}(t) = n^{(i)} + P_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} P_{j,i}(\tilde{D}^{(j)}(t)).$$

Denote by

$$t_1 = \inf\{t \geq 0\colon \Sigma^{(0)}(t-) \neq \Sigma^{(0)}(t) \text{ or there exists an } i \text{ such that } \tilde{D}^{(i)}(t-) \neq \tilde{D}^{(i)}(t)\}$$

the first time of jump for processes $\tilde{D}$ and $\tilde{A}$. Then,

$$A^{(i)}(t) = \tilde{A}^{(i)}(t) \quad \text{for } 0 \leq t \leq t_1,$$
$$D^{(i)}(t) = \tilde{D}^{(i)}(t) \quad \text{for } 0 \leq t \leq t_1,$$

provide a solution pair to (2.4) over $t \in [0, t_1]$. In fact, this solution is exactly the one constructed by Procedure 2. Now suppose that a solution pair $(\mathbf{A}, \mathbf{D})$ has been constructed on $[0, t_n]$, where t_n is a jump point for either $A^{(i)}$ or $D^{(i)}$. As above, let $\mathbf{X}(s) = \mathbf{A}(s)$ for $s \leq t_n$ and $\mathbf{X}(s) = \mathbf{A}(t_n)$ for $s > t_n$, and for $t \geq t_n$ define

$$\tilde{D}^{(i)}(t) = X^{(i)}(t) \wedge \inf_{0\leq s\leq t} \Sigma^{(i)}[t - s + \Sigma^{(i)\leftarrow}(X^{(i)}(s))],$$

$$\tilde{A}^{(i)}(t) = n^{(i)} + P_{0,i}(\Sigma^{(0)}(t) \wedge n^{(0)}) + \sum_{j=1}^{K} P_{j,i}(\tilde{D}^{(j)}(t)).$$

Letting

$$t_{n+1} = \inf\{t \geq t_n\colon \Sigma^{(0)}(t-) \neq \Sigma^{(0)}(t) \text{ or there exists an } i \text{ such that } \tilde{D}^{(i)}(t-) \neq \tilde{D}^{(i)}(t)\}$$

leads us to the same conclusion as above. The uniqueness of $(\mathbf{A}, \mathbf{D})$ is a consequence of this construction procedure.

Remark B.2. This construction is very similar to the construction of the reflection mapping made in the proof of Theorem 2.1 of [6].

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Appendix C. Proof of Lemma 3.3

By Corollaries 1 and 2 of Seneta [15], part 3 of our Lemma 3.3 follows from part 2, which in turn is a consequence of part 1. To see that part 3 implies part 1, just write the equations for the expected number of visits of the Markov chain (X_n) with transition matrix $\boldsymbol{R}$ to state $i \neq K+1$, i.e.

$$V_i = \mathrm{E}\Bigg[\sum_n \mathbf{1}_{\{X_n=i\}}\Bigg] = \mathrm{P}(X_0 = i) + \sum_{j=1}^{K} p_{j,i} V_j \quad \text{for all } i \in [1, K]. \tag{C.1}$$

Since $(\boldsymbol{I} - \boldsymbol{P}^\top)$ is invertible, (C.1) has a finite solution. Hence, the only absorbing state of (X_n) is $K+1$.

References

[1] Baccelli, F. and Foss, S. (1994). Ergodicity of Jackson-type queueing networks. *Queuing Systems Theory Appl.* **17,** 5–72.

[2] Baccelli, F. and Foss, S. (1995). On the saturation rule for the stability of queues. *J. Appl. Prob.* **32,** 494–507.

[3] Baccelli, F., Foss, S. and Lelarge, M. (2005). Tails in generalized Jackson networks with subexponential service-time distributions. *J. Appl. Prob.* **42,** 513–530.

[4] Billingsley, P. (1979). *Probability and Measure*. John Wiley, New York.

[5] Chen, H. (1995). Fluid approximations and stability of multiclass queueing networks: work-conserving disciplines. *Ann. Appl. Prob.* **5,** 637–665.

[6] Chen, H. and Mandelbaum, A. (1991). Discrete flow networks: bottleneck analysis and fluid approximations. *Math. Operat. Res.* **16,** 408–446.

[7] Dai, J. G. (1995). On positive Harris recurrence of multiclass queueing networks: a unified approach via fluid limit models. *Ann. Appl. Prob.* **5,** 49–77.

[8] Dai, J. G. (1996). A fluid limit model criterion for the instability of multiclass queueing networks. *Ann. Appl. Prob.* **6,** 751–757.

[9] Foss, S. (1991). Ergodicity of queueing networks. *Siberian Math. J.* **32,** 184–203.

[10] Gordon, W. J. and Newell, G. F. (1967). Closed queueing systems with exponential servers. *Operat. Res.* **15,** 254–265.

[11] Harrison, J. M. and Reiman, M. I. (1981). Reflected Brownian motion on an orthant. *Ann. Prob.* **9,** 302–308.

[12] Jackson, J. R. (1963). Jobshop-like queueing systems. *Manag. Sci.* **10,** 518–527.

[13] Majewski, K. (2000). Single class queueing networks with discrete and fluid customers on the time interval $\mathbb{R}$. *Queueing Systems* **36,** 405–435.

[14] Massey, W. A. (1981). Non-stationary queues. Doctoral Thesis, Department of Mathematics, Stanford University.

[15] Seneta, E. (1981). *Nonnegative Matrices and Markov Chains*, 2nd edn. Springer, New York.

[16] Skorokhod, A. V. (1961). Stochastic equations for diffusions in a bounded region. *Theory Prob. Appl.* **6,** 264–274.

[17] Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific J. Math.* **5,** 285–309.