GENERALIZED LIPSCHITZ ALGEBRAS

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Introduction. The purpose of this paper is to generalize the results of Sherbert on Lipschitz algebras and to study the relationship between homomorphisms of these algebras and continuous maps of the underlying metric spaces. In Sections 1, 2, and 3 we associate with each metric space a class of Lipschitz-type algebras and extend Sherbert's results in [7] to this class; in particular Sherbert's theorem 5.1 is extended to non-compact metric spaces (3.3, 3.4, 3.5). In Section 4 the relation between homomorphisms of these generalized Lipschitz algebras and continuous metric space maps is shown to have a natural expression in categorical terms, and in Section 5 this expression is applied to the theory of quasiconformal mappings.

I. Moduli of Continuity

We refer the reader to Bourbaki [1] for basic properties of convex and concave functions.

Definition. A real modulus of continuity α is a convex or concave homeomorphism of the real half line $[0, \infty]$ onto itself with $\alpha(0) = 0$ and $\lim_{x\to\infty} \frac{\alpha(x)}{x} = 1$.

Let CC be the set of all concave moduli of continuity; CV the set of all convex ones; $C = CC \cup CV$. Note that $CC \cap CV$ consists of the identity map, $\alpha(x) = x$.

1.1. (Glaeser, [2, p. 8]). For any family F of bounded uniformly equicontinuous functions on a metric space (X, d) with complex values there exists a non-decreasing concave real function α, continuous at 0 with $\alpha(0) = 0$, such that $|f(x) - f(y)| \leq \alpha(d(x, y))$ for all $f \in F$, $x, y \in X$.

1.2. We define a partial order on C by : $\alpha_1 \leq \alpha_2$ if and only if $\alpha_1(x) \leq \alpha_2(x)$ for all x in $[0, \infty]$. With this order, C is a lattice.

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Proof. We note that if h is the identity map on $[0, \infty]$, then $\alpha \leq h$ for α in CV and $\alpha \geq h$ for α in CC. If α_1, α_2 are not comparable in C, they must be both in CC or both in CV.

Let α_1, α_2 be non-comparable in CC. By concavity, the lower envelope $g(x) = \alpha_1(x) \wedge \alpha_2(x)$ is concave. Suppose $g(x) = g(y)$ for $x < y$. Since α_1, α_2 are strictly increasing, this can happen only if:

$$\alpha_1(x) \wedge \alpha_2(x) = \alpha_1(x) = \alpha_2(y) = \alpha_1(y) \wedge \alpha_2(y) .$$

But by the first equality and the monotonicity of α_2, $\alpha_1(x) \leq \alpha_2(x) < \alpha_2(y)$. So g is strictly increasing on $[0, \infty]$. Also:

$$\lim_{x \to \infty} \frac{\alpha_1(x) \wedge \alpha_2(x)}{x} = \lim_{x \to \infty} \left(\frac{\alpha_1(x)}{x}\right) \wedge \lim_{x \to \infty} \left(\frac{\alpha_2(x)}{x}\right) = 1 .$$

Thus $\alpha_1(x) \wedge \alpha_2(x) = \alpha_1 \wedge a_2(x) \in CC$.

By a similar argument, $\lim_{x \to \infty} \frac{\alpha_1(x) \vee \alpha_2(x)}{x} = 1$. However, $\alpha_1(x) \vee \alpha_2(x)$ may not be concave. By 1.1, α_1 and α_2 have a concave upper bound h which is continuous at 0. If $h(x) = h(y)$ for $x < y$, then there exists a point $z > y$ with $h(z) > h(y)$, so the line segment $[x, h(x))$, $(z, h(z)]$ lies above the graph of h at the point $(y, h(y))$, which contradicts the concavity of h. Consequently h is strictly increasing.

The graph of h consists by definition of points lying on line segments both ends of which are in $\{(x, y) : 0 \leq x \leq \infty, 0 \leq y \leq \alpha_1(x) \vee \alpha_2(x)\}$ including degenerate segments of one point only. Considering these point sets, it is apparent that $\lim_{x \to \infty} \frac{h(x)}{x} = 1$. Hence $h(x) = \alpha_1 \vee \alpha_2(x) \in CC$.

For α_1, α_2 non-comparable in CV, $\alpha_1(x) \vee \alpha_2(x) \in CV$ by convexity and by the same argument as above, $\lim_{x \to \infty} \frac{1}{x} \alpha_1(x) \vee \alpha_2(x) = 1$. To form $\alpha_1 \wedge \alpha_2$, let the set U be the complement in the first quadrant of the ordinate set of $\alpha_1(x) \wedge \alpha_2(x)$. Take $g(x)$ to be that function whose ordinate set is the complement in the first quadrant of the convex hull of U. As in 1.1, $g \in CV$ and by a similar argument to that employed in forming the join in the concave case, $g(x) = \alpha_1 \wedge \alpha_2(x)$ in CV.

An elementary calculation shows that the sublattices CC and CV are closed under composition. However, for $\alpha_1 \varepsilon$ CC, $\alpha_2 \varepsilon$ CV, the composition $\alpha_1 o \alpha_2$, while still monotone increasing, may be neither concave nor convex. It also follows trivially from the definitions that for $\alpha \varepsilon$ CC, $\alpha^{-1} \varepsilon$ CV and vice versa.

1.3. Every $\alpha \varepsilon$ C may be represented as:

$$\alpha(x) = \int_o^x p(t)dt ,$$

where p is a right continuous function, bounded a.e. on any finite interval, monotone increasing (decreasing) for α convex (concave).

Proof. By absolute continuity, every $\alpha \varepsilon$ C is differentiable a.e. and is equal to the integral of its derivative as stated; in particular to the integral of its right derivative. For α convex, a detailed proof is given in [6, Theorem 1.1]; the proof for the concave case is similar. Since α is finite for finite x, p must be bounded a.e. on finite intervals.

We now consider the class C_a consisting of all $\alpha \varepsilon$ C such that

$$0 < \lim_{x \to o} \inf \frac{\alpha(\lambda x)}{\alpha(x)} \quad \text{for all} \quad \lambda > 0 .$$

By concavity, $CC \subset C_a$. By definition, for $\alpha \varepsilon C_a$, we can say that there exists $K > 0$ with $\frac{\alpha(\lambda x)}{\alpha(x)} > K$ for all x, all $\lambda > 0$.

For example, $\alpha(0) = 0$, $\alpha(x) = e^{x^{-\frac{1}{2}}}$, in some interval $[0, x_1]$ shows that $C_a \cap CV$ is properly contained in CV. All functions in C of form $\alpha(x) = x^k$, $k \geq 1$ in a neighbourhood of 0 are in $CV \cap C_a$; the example $\alpha(0) = 0$, $\alpha(x) = -\frac{x}{\ln x}$ in an interval $[0, x_1]$ shows that such functions are not all of $CV \cap C_a$.

For $\alpha \varepsilon$ CC we have the following inequalities by concavity:

$$\lambda\, \alpha(x) \leq \alpha(\lambda x) \leq \alpha(x) \qquad \lambda \leq 1$$

$$\alpha(x) \leq \alpha(\lambda x) \leq \lambda\, \alpha(x) \qquad \lambda > 1, \text{ for all } x.$$

For $\alpha \varepsilon CV \cap C_a$, by convexity and the definition of C_a,

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$$\inf_{t \varepsilon [o,\infty]} \left[\frac{\alpha(\lambda t)}{\alpha(t)}\right] \alpha(x) \leq \alpha(\lambda x) \leq \lambda \alpha(x) \qquad \lambda \leq 1$$

$$\lambda \alpha(x) \leq \alpha(\lambda x) \leq \sup_{t \varepsilon [o,\infty]} \left[\frac{\alpha(\lambda t)}{\alpha(t)}\right] \alpha(x) \qquad \lambda > 1 .$$

By the definition of C, the supremum here is bounded. Consequently, for $\alpha \varepsilon C_a$, any $\lambda > 0$, there exist $K_1, K_2 > 0$ with

$$K_1 \alpha(x) \leq \alpha(\lambda x) \leq K_2 \alpha(x) \qquad \text{for all } x.$$

A simple calculation shows that $CV \cap C_a$ is closed under composition, while it has already been established that CC has this property. We note that since $CV = (CC)^{-1}$, C_a is not closed under inversion. Also, by the remarks following 1.2, C_a is not closed under composition.

In order to get a more tractable class of functions, we consider the equivalence classes of C_a with respect to the relation:

$$\alpha_1 \sim \alpha_2 \text{ if and only if } 0 < \lim_{x \to o} \inf \frac{\alpha_1(x)}{\alpha_2(x)} \leq \lim_{x \to o} \sup \frac{\alpha_1(x)}{\alpha_2(x)} < \infty ,$$

It is easy to see that this is in fact an equivalence relation which is preserved under composition and inversion in C_a.

We note that $\alpha \sim \alpha o \lambda$, where $\alpha o \lambda(x) = \alpha(\lambda x)$, λ real, if and only if $\alpha \varepsilon C_a$.

The set G of equivalence classes of C_a is closed under composition, as follows:

1.4. <u>For</u> $\alpha, \beta \varepsilon C_a$, <u>there always exists</u> $\gamma \varepsilon C_a$ <u>with</u> $\alpha o \beta \sim \gamma$.

<u>Proof</u>. Suppose $\alpha o \beta \varepsilon C_a$. By 1.3,

$$\alpha(x) = \int_o^x q(t)dt \qquad \beta(x) = \int_o^x q(t)dt ,$$

where p and q are monotone right continuous. Then

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$$\alpha \circ \beta(x) = \int_0^x p(\beta(t))\, q(t)dt$$

$$= \int_0^x g(t)dt ,$$

where g is right continuous and > 0 for any $t > 0$. If, for some x_o, $g(t)$ is monotone on $0 < x \le x_o$, we can construct γ by:

$$\gamma(x) = \int_0^x g(t)dt \qquad 0 < x \le x_1$$

$$\gamma(x) = x - x_1 + \gamma(x_1) \qquad x > x_1 ,$$

where x_1 is the relative extremum of $\alpha \circ \beta(x) - x$ for $0 < x < x_o$.

If there is no neighbourhood of 0 on which g is monotone, we note that $0 < g(0) < \infty$, since by right continuity:

(i) If $g(0) = 0$, g must be monotone increasing in some neighbourhood of 0.

(ii) If $g(0) = \infty$, g must be monotone decreasing in some neighbourhood of 0.

We then set $\gamma(x) = x$. Obviously, $\lim_{x \to o} \frac{\alpha \circ \beta(x)}{\gamma(x)} = g(0)$ and $\lim_{x \to \infty} \frac{\gamma(x)}{x} = g(1)$. In either case the inequalities (a) and (b) imply that $\gamma \varepsilon C_a$ if α and β are. Consequently, G is a semi-group under composition.

II. The Algebras L_α.

For (X, d) a metric space, we denote by $L_\alpha(X, d)$ the algebra of all complex-valued continuous functions on X which are finite in the norm:

$$\| f \| = \| f \|_\infty + \sup_{x \neq y} \left\{ \frac{|f(x) - f(y)|}{\alpha(d(x, y))} : x, y \varepsilon X, \ \alpha \varepsilon C_a \right\}$$

$$= \| f \|_\infty + \| f \|_{d,\alpha} .$$

2.1. With this norm, $L_\alpha(X, d)$ is a Banach algebra for each $\alpha \varepsilon C_a$.

Proof. L_α is trivially a linear space. The proof of completeness is exactly the same as that given by Mirkil [4, Theorem 4.5].

The fact that the given norm is a Banach algebra norm follows from the inequality

$$\frac{|f(x)g(x) - f(y)g(y)|}{\alpha(d(x, y))} \leq |f(x)| \frac{|g(x) - g(y)|}{\alpha(d(x, y))} + |g(y)| \frac{|f(x) - f(y)|}{\alpha(d(x, y))} .$$

Since L_α is a function algebra containing the constants, it is semi-simple with unit $f(x) = 1$, $x \varepsilon X$. For each $\alpha \varepsilon C_a$, L_α is closed under complex conjugation, inversion and truncation; consequently, by a result given by Sherbert [7, Proposition 4.2], if L_α separates (X, d) it is a regular algebra.

2.2. Let X be a locally compact metric space, $\alpha \varepsilon C_a$. A necessary and sufficient condition that L_α separate X is that for each $s \varepsilon X$, there exist $K_s > 0$ such that $f_s(x) = K_s \alpha(d(x, s)) \wedge 1$ be in L_α.

Proof. Sufficiency is obvious.

Necessity: For arbitrary $s \varepsilon X$, let K be a compact ball of radius $r > 0$ about s. For each pair $(x, y) \varepsilon K$, $x \neq y$, there exists $g \varepsilon L_\alpha$ and open neighbourhoods S_x and S_y of x and y respectively such that $|g(u) - g(v)| \geq a > 0$ for any $(u, v) \varepsilon S_x \times S_y$. The sets $\{S_x \times S_y : (x, y) \in K\}$ form an open cover of $K \times K$ in $X \times X$ with the metric product topology. Let $\{S_{x_i} \times S_{y_i}\}$ $i = 1, \ldots, n$ be a finite cover with associated functions $g_i \varepsilon L_\alpha$. For each pair $(x, y) \varepsilon K$, $x \neq y$, $(x, y) \varepsilon S_{x_i} \times S_{y_i}$ for some i and $|g_i(x) - g_i(y)| \geq a_i$. Let $a = \min\{a_i : i = 1, \ldots, n\}$ and set $h_i = (r/a) g_i$. Then for each $(x, y) \varepsilon X$ and some i:

$$\frac{|h_i(x) - h_i(y)|}{\alpha(d(x, y))} \geq \frac{r}{\alpha(d(x, y))} \geq \frac{|\alpha(d(x, s)) \wedge r - \alpha(d(y, s)) \wedge r|}{\alpha(d(x, y))} .$$

Thus $K_s \, \alpha(d(x, s)) \wedge 1 \varepsilon L_\alpha$, where $K_s = \frac{1}{r}$.

It follows from the triangle inequality that if $\alpha \circ d$ is a metric, then L_α contains f_s with $K_s = k$ for all $s \varepsilon X$. The example : $\alpha(x) = \tan \frac{\pi x}{4}$ in a suitable interval $[0, x_1]$ shows that it is not necessary that $\alpha \circ d$ be a metric for L_α to separate x.

From the general theory, the maximal ideal space of L_α coincides with its spectrum for each α. If L_α separates X, X corresponds homeomorphically with a subset of the maximal ideal space M_α of L_α which is dense in the Gelfand topology and the metric topology on X coincides with the topology induced on X by the Gelfand topology on M_α; if X is compact, S and M_α are homeomorphic.

2.3. <u>For any</u> (X, d), <u>if</u> $\alpha_1 \sim \alpha_2$ <u>in</u> C_a, <u>then</u> L_{α_1} <u>and</u> L_{α_2} <u>contain the same functions</u>.

<u>Proof</u>. By the definition of equivalence, there exist $K_1, K_2 > 0$ with

$$K_1 \alpha_1(x) \leq \alpha_2(x) \leq K_2 \alpha_1(x) \quad \text{for all } x \varepsilon [0, \infty].$$

The d, α norms for L_{α_1} and L_{α_2} are thus boundedly equivalent, and the sup norm is independent of α.

2.4. <u>For any</u> (X, d), <u>set</u> $r = \frac{d}{1+d}$. <u>Then for any</u> $\alpha \varepsilon C_a$, $L_\alpha(X, d)$ <u>and</u> $L_\alpha(X, r)$ <u>contain the same functions</u>.

<u>Proof</u>. For $d(x, y) \leq 1$

$$\begin{aligned} \alpha(d(x, y)) &\geq \alpha\left(\frac{d(x, y)}{1 + d(x, y))}\right) = \alpha(r(x, y)) \\ &\geq \alpha \frac{d(x, y)}{2} \\ &\geq K\alpha(d(x, y)) \quad \text{for some } K > 0. \end{aligned}$$

For $d(x, y) > 1$, any bounded f,

$$\frac{|f(x) - f(y)|}{\alpha(d(x, y))} \leq \frac{2\|f\|_\infty}{\alpha(1)} < \infty$$

$$\frac{|f(x) - f(y)|}{\alpha(r(x, y))} \leq \frac{2\|f\|_\infty}{\alpha\left(\frac{1}{2}\right)} < \infty .$$

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Consequently, a bounded function f on X is bounded in $\|\cdot\|_{d,\alpha}$ if and only if it is bounded in $\|\cdot\|_{r,\alpha}$. Since the uniform norm is independent of the metric, $L_\alpha(X, d)$ and $L_\alpha(X, r)$ contain the same functions.

2.5. Let (X, d) be any locally compact metric space; let $\alpha, \beta \in C_a$ such that $\lim_{x\to 0} \inf \frac{\alpha(x)}{\beta(x)} = 0$ and $\lim_{x\to 0} \sup \frac{\alpha(x)}{\beta(x)} < \infty$ and such that L_β separates X. Then L_α is properly contained in L_β.

Proof. By 2.2, $f_s \in L_\beta$ for all $s \in X$. For any s and all x such that $\beta(d(s, x)) < 1$, we have

$$\frac{|f_s(s) - f_s(x)|}{K_s \beta(d(s, x))} = 1 = \frac{|f_s(s) - f_s(x)|}{R_s \alpha(d(s, x))} \cdot \frac{\alpha(d(s, x))}{\beta(d(s, x))} .$$

Thus $\|f_s\|_{d,\alpha}$ is infinite for each s and so f_s is not in L_α. Let $f \in L_\alpha$. By hypothesis, there exists K, $0 < K < \infty$, such that $\alpha(x) \leq K\beta(x)$ for all $x \in [0, \infty]$. Consequently, $f \in L_\beta$. Note that $L_\alpha \subseteq L_\beta$ whether L_β separates X or not.

For each $\alpha \in C_a$ such that L_α separates S, (X, d) a fixed metric space, we introduce the metric σ_α on M_α as follows: For $\phi, \psi \in M_\alpha$:

$$\sigma_\alpha(\phi, \psi) = \sup \{ |\phi(f) - \psi(f)| : f \in L_\alpha, \|f\| \leq 1\} .$$

We use the same symbol for the induced metric on X,

$$\sigma_\alpha(x, y) = \sup \{ |f(x) - f(y)| : f \in L_\alpha, \|f\| \leq 1\} .$$

2.6. For any locally compact metric space (X, d) of finite diameter and any $\alpha \in C_a$ such that L_α separates (X, d) and the set $\{f_s : s \in X\}$ is bounded in L_α, the metric σ_α is boundedly equivalent on X to the distance given by $\alpha \circ d$.

Proof. By definition, $\sigma_\alpha(x, y) \leq \alpha(d(x, y))$.

By hypothesis, $\|f_s\| < P$ for some finite P, all $s \in X$. Then for $x, y \in X$, set $g(x) = {}^1/P \, f_x(s)$ for $s \in X$. Then:

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$\sigma_\alpha(x, y) \geq |g(x) - g(y)| \geq K\alpha(d(x, y))$.

COROLLARY. For any metric space (X, d) and any $\alpha \varepsilon C_a$ satisfying the above conditions, there exists a metric on X such that L_α is precisely the Lipschitz algebra with respect to that metric.

We note that if the triangle inequality for $\alpha o d$ holds at all points of X, or fails at all points of X, (for locally compact X) the set $\{f_s : s \varepsilon X\}$ is bounded in L_α. The case in which $\alpha o d$ satisfies the triangle inequality at some points of X and not at others remains open.

Also, the uniform structures U_α on X generated by the sets $\{(x, y) : \alpha(d(x, y)) \leq \varepsilon\}$ are all equivalent to the metric uniformity determined by d (i.e., U_1). Hence for α with $\{f_s : s \varepsilon X\}$ bounded in L_α, the σ_α are uniformly equivalent metrics on X.

III. Homomorphisms and Space Maps.

For every $\alpha \varepsilon C_a$ such that $L_\alpha(X, d)$ separates X, L_α is regular, closed under inversion and complex conjugation and contains the constant functions. For a homomorphism $T: L_\alpha(X, d) \to L_\beta(Y, r)$, $\alpha, \beta \varepsilon C_a$, we refer to the induced map $t: M_\beta \to M_\alpha$ as the adjoint of T and write T^* for t. From the general theory we know that t is continuous; to ensure that $t(Y) \subseteq X$ we have:

3.1. Let A, B be algebras of continuous bounded complex-valued functions on the topological spaces X, Y respectively, such that X is dense in M_A and every compact open neighbourhood of the unit of A contains a function of compact support in X. Then for any unitary homomorphism $T: A \to B$ which is continuous in the compact open topology, the adjoint $t: M_B \to M_A$ carries Y into X.

Proof. Let $y \varepsilon Y$. $M_y = \{f \varepsilon B : f(y) \neq 0\}$ is closed in the compact open topology on B. Suppose $ty \notin X$.

$$M_{ty} = \{ f \varepsilon A : f(ty) = Tf(y) = 0\}$$

is exactly $T^{-1}(M_y)$ and hence is compact open closed in A. Since $ty \notin X$, all functions f in A of compact support in X are in M_{ty}, for if K is the support of f, $X - K$ is dense in $M_A - X$ and thus

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$\hat{f}$ vanishes on $M_A - X$. But every compact open neighbourhood of the unit, 1, of A contains a function of compact support. Hence $1 \in M_{ty}$, which is a contradiction of the fact that T is unitary, so $ty \in X$ for all $y \in Y$.

In other words, the given conditions ensure that the mapping t carries fixed ideals of B to fixed ideals of A. An argument similar to this was given by Nakai [3, Lemma 3.2] in the special case of the Royden ring of functions on a Riemann surface, but as far as we know this is its first statement as a general proposition.

3.2. For X a locally compact metric space, let A be any regular sub-algebra of C(X) which is closed under truncation and complex conjugation. Then every compact open neighbourhood of the unit of A contains a function $h \in A$ with compact support in X.

Proof. Let K be any compact set in S, U an open neighbourhood of K with compact closure. For each $p \in M_A - U$, we have by regularity a function $f_p \in A$ with: $f_p(p) = 1$, $f_p(K) = 0$. Set $V_p = \{q : |f_p(q)| > \frac{1}{2}\}$.

Then $p \in V_p$, and the sets U and $\{V_p\}_{p \notin U}$ form an open cover of M_A. We select a finite cover, $U, V_{p_1} \ldots V_{p_n}$.

Now set $f = \sum_{i=1}^{n} f_{p_i} \overline{f}_{p_i} \in A$.

For $q \in X$, $q \notin U$ we have $q \in V_{p_j}$ for some j, so

$f(q) \geq |f_{p_j}(q)|^2 > \frac{1}{4}$. For $q \in K$, $f(q) = 0$ we take $h = 1 - [(4f) \wedge 1] \in A$.

h is then the required function.

It follows that we can refer to the adjoint $t : Y \to X$ of a compact open continuous algebra homomorphism $T : L_\alpha(X) \to L_\beta(Y)$ when $L_\alpha(X)$ separates X and X is locally compact.

Where $t : X \to Y$ is continuous, we will also refer to the induced algebra homomorphism $T : C(X) \to C(Y)$ defined by $Tf(x) = f(tx)$ as the adjoint of t and write t^* for T.

We classify the continuous maps of one metric space (X, d) into another (Y, r) as follows:

A continuous map $t: X \to Y$ is β-modally continuous if it satisfies $r(tx, ty) \leq K\beta(d(x, y))$ (*) for $\beta \in C_a$, some $K > 0$ and all $x, y \in X$. We

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will denote the fact that a continuous map t satisfies condition (*) for a particular $\beta \in C_a$ by writing $t: (X,d) \to (Y,r)$ (β). We note that if t is β-modally continuous and $\alpha \sim \beta$ in C_a, then t is α-modally continuous.

3.3. Let $t:(X,d) \to (Y,r)$ be β-modally continuous. For $\alpha \in C_a$, the restriction T_α of the adjoint T of t to L_α determines a compact open continuous homomorphism $T_\alpha : L_\alpha(Y) \to L_{\alpha o \beta}(X)$. (We consider $\alpha o \beta$ to be replaced by an equivalent element of C_a when necessary, as in 1.5.)

Proof. For $f \in L_\alpha(Y)$;

$$\begin{aligned} \| Tf \|_\infty &= \sup \{ |f(tx)| : x \in X\} \\ &\leq \sup \{ |f(y)| : y \in Y\} = \|f\|_\infty . \end{aligned}$$

$$\begin{aligned} \| Tf \|_{d, \alpha o \beta} &= \sup_{x \neq y} \left\{ \frac{|Tf(x) - Tf(y)|}{\alpha o \beta (d(x,y))} : x, y \in X \right\} . \\ &= \sup \left\{ \frac{|f(tx) - f(ty)|}{\alpha o \beta (d(x,y))} : x, y \in X \right\} . \\ &\leq \sup \left\{ \frac{|f(tx) - f(ty)|}{K' \alpha(r(tx,ty))} : x, y \in X \right\} . \\ &\leq \frac{1}{K'} \sup_{s \neq t} \left\{ \frac{|f(s) - f(t)|}{\alpha(r(s,t))} : s, t \in Y \right\} = \frac{1}{K'} \|f\|_{r,\alpha} . \end{aligned}$$

Thus $T(L_\alpha(Y)) \subseteq L_{\alpha o \beta}(X)$ and T is trivially a homomorphism. To show that T is compact open continuous, for $\varepsilon > o$, $g \in L_\alpha(Y)$, K compact in X, consider:

$$N(Tg, K, \varepsilon) = \{ f \in L_{\alpha o \beta}(X) : \|f - Tg|K\|_\infty < \varepsilon \} ,$$

and

$$N(g, tK, \varepsilon) = \{ f \in L_\alpha(Y) : \|f - g|tK\|_\infty < \varepsilon \} .$$

The continuity of t implies that tK is compact in Y, so $N(g, tK, \varepsilon)$ is an element of the subbase of the compact open neighbourhood system of g and $T(N(g, tK, \varepsilon)) \subseteq N(Tg, K, \varepsilon)$ since $\|f - g|tK\|_\infty = \|Tf - Tg|K\|_\infty$. This establishes the compact-open continuity of T. In the case where Y is compact, T is continuous in the uniform topology of $L_\alpha(Y)$, since the two topologies coincide.

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3.4. Let (X, d), (Y, r), where (X, d) is locally compact be metric spaces with $\alpha \in C_a$ such that $L_\alpha(X)$ separates S, $\alpha^{-1} \in C_a$ and the set $\{f_s : s \in X\}$ is bounded in $L_\alpha(X)$. Let $T : L_\alpha(X) \to L_\beta(Y)$ be a compact-open continuous homomorphism for some $\beta \in C_a$. Then its adjoint t is an open $\alpha^{-1} \circ \beta$-modally continuous map of (Y, r) into (X, d).

Proof. By the preceding discussion, t carries Y into $X \subseteq M_\alpha$. Since T is a Banach algebra homomorphism, the set $\{Tf_s : s \in X\}$ is bounded in norm in L_β, say by K. For all $x, y \in Y$,

$$K \geq \frac{|Tf_s(x) - Tf_s(y)|}{\beta(r(x, y))}$$

$$\geq K_s \frac{|\alpha(d(s, tx)) - \alpha(d(s, ty))|}{\beta(r(x, y))} .$$

Then for $s = ty$:

$$K_s \alpha(d(tx, ty)) \leq K \beta(r(x, y))$$

i.e.,

$$d(tx, ty) \leq K' \alpha^{-1} \circ \beta(r(x, y))$$

for suitable $K' > 0$ since $\alpha^{-1} \in C_a$ by hypothesis.

In the case where X is compact, we note that all three of the conditions of 3.3 are still necessary in order to reach the same conclusion. If $\alpha = \beta = 1$, the identity in C_a, this reduces to the case of the Lipschitz algebras of X and Y as considered by Sherbert. In this case, the separation of X by L_α, the inclusion of α^{-1} in C_a and the boundedness of $\{f_s : s \in X\}$ in L_α follow from the definition of L_α.

From here on, we will consider all metric spaces used to be locally compact.

3.5. A compact-open continuous homomorphism

$$T : L_1(X, d) \to L_\alpha(Y, r)$$

where $\alpha^{-1} \in C_a$ and the set $\{f_s : s \in Y\}$ is bounded in $L_\alpha(Y)$, is an isomorphism of $L_1(X)$ onto $L_\alpha(Y)$ if and only if the adjoint $t : Y \to X$ is a homeomorphism of Y onto X satisfying

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(3.5.0) $$K'\alpha(r(x,y)) \le d(tx,ty) \le K\alpha(r(x,y))$$

for some $K, K' > 0$, all $x, y \in Y$.

Proof. Let T be a compact open continuous isomorphism of $L_1(X)$ onto $L_\alpha(Y)$. Since T is onto, t is 1-1, so $t^{-1}: M_1 \to M_2$ is defined on $t(M_\alpha)$. Now suppose $t: M_\alpha \to M_1$ is not onto, so that there exists $\psi \in M_1$, $\psi \notin t(M_\alpha)$. Since M_α is compact, $t(M_\alpha)$ is compact in M_1. By the regularity of L_1, there exists $f \in L_1$ with $\hat{f}(y) = 1$, $\hat{f}(x) = 0$ for all $x \in t(M_\alpha)$.

So $$T\hat{f}(\psi) = \hat{f}(t\psi) = 0 \qquad \text{for all } \psi \in M_\alpha .$$

This contradicts the assumption that T is 1-1, so t must be onto, $M_\alpha \to M_1$. Likewise $t^{-1}: M_1 \to M_\alpha$ is onto and

$$t^{-1}(X) \subseteq Y \subseteq t^{-1}(M_1) = M_\alpha$$

i.e., $X \subseteq tY$ since t is 1-1. By the compact open continuity of T, $tY \subseteq X$, so $tY = X$ and t is onto.

The rest of the proof follows exactly as in Sherbert [7, Theorem 5.1].

COROLLARY. Every compact open continuous automorphism T of $L_\alpha(X, d)$ where $\alpha^{-1} \in C_a$ and the set $\{f_s : s \in X\}$ is bounded in L_α, is of the form: $Tf(x) = f(tx)$ $f \in L_\alpha$, $x \in X$ where $t: X \to X$ is a homeomorphism satisfying

$$K'd(x,y) \le d(tx,ty) \le K\, d(x,y) \qquad x, y \in X,\ K, K' > 0.$$

IV. Categorical Considerations

We can express some aspects of the relationship between modally continuous space maps and compact open continuous algebra homomorphisms in terms of suitable categories. The following proposition is obvious:

4.1. Let C_a be directed by the lattice order and assign to each $\alpha \in C_a$ the corresponding algebra L_α on a fixed metric space (X, d).

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For $\alpha \leq \beta$ in C_a let $h_\alpha^\beta : L_\alpha \to L_\beta$ be the mapping of L_α onto itself as a subset of L_β. $(L_\alpha, h_\alpha^\beta)$ then forms a direct system of Banach algebras and Banach algebra homomorphisms.

Since by 1.1 every uniformly continuous function on (X, d) satisfies some modulus of continuity in C_a, the union of this system is the algebra of all uniformly continuous complex-valued bounded functions on X. It is conjectured that the uniform closure of this algebra is the direct limit of this system in the category of commutative semi-simple Banach algebras.

Now let G stand for the set of equivalence classes of C_a under the relation defined above. G inherits the lattice order of C_a and is closed under composition by 1.4. We have then associated with each metric space (X, d) the direct system of algebras $(L_\phi(X), h_\phi^\eta)_{\eta, \phi \varepsilon G}$ where $L_\phi(X)$ is the algebra with compact-open topology given by $\alpha \varepsilon C_a$ and ϕ is the equivalence class of α in G. $L_\phi(X)$ then consists of the same set of functions as $L_\alpha(X)$ for all $\alpha \varepsilon \phi$, and it is easily seen that this is again a direct system of algebras over (X, d). Moreover, any β-modally continuous map $t: (X, d) \to (Y, r)$ with $\beta \varepsilon C_a$ determines by its adjoint the algebra homomorphisms $T_\phi : L_\phi(Y) \to L_{\phi o \eta}(X)$ for all $\phi \varepsilon G$, where η is the equivalence class of β in C_a.

This leads us to consider the category $\mathfrak{B}$ defined as follows :

The objects are the direct systems of topological algebras:

$$(B_\phi, h_\phi^\eta)\phi, \quad \eta \ \varepsilon \ G, \quad \phi \leq \eta$$

with the property that, for $\phi \leq \eta$, B_ϕ is a sub-algebra of B_η, and the algebra homomorphism $h_\phi^\eta : B_\phi \to B_\eta$ is in fact the natural injection. The morphisms are pairs of the forms:

$$\theta = ((T_\phi)\phi \ \varepsilon \ G, \ \eta) \ \eta \ \varepsilon \ G$$

where $T_\phi : B_\phi \to B_{\phi o \eta}$ is compact open continuous with

$$\theta \circ \theta' = ((T'_{\phi o \eta} \circ T_\phi), \ \eta \circ \eta')$$

where defined, and for any $\phi \leq \eta$, $T_\eta \circ h^\eta_\phi = T_\phi$, i.e. $T_\eta \mid B_\phi = T_\phi$. The composition defined is evidently associative and θ is the identity homomorphism of B_ϕ and η is the equivalence class in G containing the identity map in C_a.

On the other hand, we consider the category $\mathfrak{L}$ whose objects are the metric spaces (X, d) and whose morphisms are the η-modally continuous maps for $\eta \in G$; (t, η) such that $t: (X, d) \to (Y, r)\ (\beta)$ for any β in the class η of G.

Again, it is readily seen that

$$(t', \eta') \circ (t, \eta) = (t' \circ t, \eta \circ \eta')$$

where defined; this composition is associative and the identity for (X, d) is simply the pair (t, η) where η is the identity equivalence class in G and t is the identity mapping on X.

We can now define the category mapping $F : \mathfrak{L} \to \mathfrak{B}$ as follows

$$F(X, d) = (L(X, d, \phi), h^\eta_\phi)_{\phi,\ \eta \in G,\ \phi < \eta},$$

$$F(t, \eta) = ((T_\phi)_{\phi \in G}, \eta)$$

on the objects and morphisms respectively of $\mathfrak{L}$.

4.2. <u>This mapping</u> $F : \mathfrak{L} \to \mathfrak{B}(G)$ <u>is a contravariant functor which maps the set of modally continuous</u> $t : (X, d) \to (Y, r)\ (\eta)\ \eta \in G$ <u>one to one onto the set of all morphisms</u> $F(Y, r) \to F(X, d)$.

<u>Proof</u>. Let $\theta = ((T_\phi)_{\phi \in G}, \eta)$ for $\eta \in G$. In order to show that F is onto, we need to find (t, η) in $\mathfrak{L}$ such that $F(t, \eta) = \theta$. We take $t = T_\rho{}^*$, the adjoint of an algebra homomorphism $T_\rho : L_\rho(Y) \to L_{\rho \circ \eta}(X)$ in θ such that $L_\rho(X)$ satisfies the requirements of L_α in 3.4. It is sufficient to take ρ the equivalence class of the identity map in C_a.

By 3.4, $t : (X, d) \to (Y, r)\ (\eta)$. Moreover, we claim that $T^*_\rho(X) = T^*_\phi(X)$ for all $\phi \in G$, all $x \in X$ where T^*_ρ is defined. Suppose that $\phi \leq \rho$, so that $L_\phi(Y) \subseteq L_\rho(Y)$. Then $T_\phi = T_\rho \mid L_\phi$ i.e., $T^*_{\phi(x)} = T^*_\rho(x)$ where defined, so the above assertion holds.

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For $L_\rho(Y) \subseteq L(Y)$, we recall that the adjoint t of $T_\rho : L_\rho(Y) \to L_{\rho\circ\eta}(X)$ carries all points (i.e., fixed maximal ideals) of X to points of Y. Since $L_\rho(Y)$ separates Y, if two ideals are identified by t, they correspond to the same point of Y. Suppose $tx_1 = tx_2$; consider the adjoint h of $T_\phi : L_\phi(Y) \to L_{\phi\circ\eta}(X)$. Suppose $hx_1 \neq hx_2$. Then for all $s \in X$, $f_s(hx_1) \neq f_s(hx_2)$;

$$\text{i.e.,} \quad T_\phi f_s(x_1) = T_\phi f_s(x_2).$$

But since $f_s \in L_\rho(Y)$ for all $s \in Y$:

$$T_\phi f_s(x_1) = T_\rho f_s(x_1) = T_\rho f_s(x_2) = T_\phi f_s(x_2).$$

So we must have $hx_1 = hx_2 = ty_1$ for the adjoint of T_ρ. This takes care of all cases, since for ϕ, $\rho \in G$ we have that $\phi \le \rho$, $\rho \le \phi$ or ϕ and ρ are not comparable, we can find $\eta \in G$ with $\eta \ge \phi$ and $\eta \ge \rho$, and in this case, $L_\eta(Y)$ will have the same properties as $L_\rho(Y)$. This reduces the matter to the two cases we have considered. It follows that the adjoint of every $T_\phi \in \theta$ is determined by that of T_ρ, so we can take $t = T^*_\rho$ and then $F(t,\eta) = \theta$ as required.

Finally, let

$$\theta_1 = ((T_{1\phi})_{\phi \in G}, \eta_1)$$

$$\theta_2 = ((T_{2\phi})_{\phi \in G}, \eta_2).$$

Then if $\theta_1 = \theta_2$, we have $\eta_1 = \eta_2$, $T_{1\rho} = T_{2\phi}$, for all $\phi \in G$. In particular, $T_{1\rho} = T_{2\rho}$ for ρ as above, and since these homomorphisms are enough to determine t, θ_1 and θ_2 are the image under F of the same (t, η) in $\mathfrak{L}$; so F is one to one as stated.

Since F is a functor, it takes isomorphisms of $\mathfrak{L}$ to isomorphisms of $\mathfrak{B}$. An isomorphism in $\mathfrak{L}$ is a map $t : (X,d) \to (Y,r)$ (η) such that t^{-1} exists with $t^{-1} : (Y,r) \to (X,d)$ (η^{-1}) where $\eta^{-1} \in G$. Such a map must correspond under F with $\theta = ((T_\phi)_{\phi \in G}, \eta)$ where all T_ϕ are isomorphisms $T_\phi : L_\phi(Y) \to L_{\phi\circ\eta}(X)$. We note that the existence of t and t^{-1} as above implies that t satisfies

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$$r(tx, ty) \leq K \; \alpha(d(x, y)) \quad x, y \in X, \quad \alpha \in \eta$$

$$d(t^{-1}z, t^{-1}w) \leq C \; \beta(r(z, w)) \quad z, w \in Y, \; \beta \in \eta^{-1} .$$

Hence, for suitable $K_1, K_2 > 0$,

$$K_1 \; \alpha \; (d(x, y)) \leq r(tx, ty) \leq K_2 \; \alpha \; (d(x, y)) .$$

Thus the isomorphisms of $\mathfrak{B}$ come from maps (t, η) in $\mathfrak{L}$ such that t^{-1} is defined, $\eta^{-1} \in G$ and t satisfies the above double condition with respect to α, for any α in the equivalence class η.

V. Quasiconformality.

We take as definition of quasiconformality the following, shown by Gehring [5, p. 97] to be equivalent to the classical definition in terms of the moduli of rings:

For t a homeomorphism of a plane domain D, we define, for $x \in D$:

$$L(x, r) = \sup_{|x-y|=r} |tx - ty|$$

$$\ell(x, r) = \inf_{|x-y|=r} |tx - ty|$$

$$H(x) = \limsup_{r \to o} \frac{L(x, r)}{\ell(x, r)} .$$

We say that a topological mapping of a domain D is quasiconformal if and only if $H(x)$ is bounded on D.

We have the following relation between quasiconformal mappings and the morphisms discussed in the last section:

5.1. Let X and Y be plane domains. The direct systems $S(X)$ and $S(Y)$ of generalized Lipschitz algebras over X and Y are objects of the category $\mathfrak{B}$. Let θ be a $\mathfrak{B}$-isomorphism, $\theta : S(X) \to S(Y)$. Then the adjoint t of θ is a quasiconformal map of Y onto X.

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Proof. As shown above, t is a homeomorphism satisfying

$$K_1\, \alpha(|x - y|) \leq |tx - ty| \leq K_2\, \alpha(|x - y|) \qquad x, y \in Y.$$

where α is in the equivalence class of y and $|x - y|$ is the usual metric in the plane. Then for every $y \in Y$:

$$H(y) = \lim_{r \to o} \sup \frac{\sup |tx - ty|}{\inf |tx - ty|} \qquad |x - y| = r$$

$$\leq \lim_{r \to o} \frac{K_2\, \alpha(|x - t|)}{K_1\, \alpha(|x - y|)} = \frac{K_2}{K_1} .$$

In general, a quasiconformal map of one domain onto another cannot be shown to give rise to a $\mathfrak{B}$-isomorphism of the associated direct systems of algebras, since such a map does not necessarily satisfy the double inequality

$$K_1\, \alpha(|x - y|) \leq |tx - ty| \leq K_2\, \alpha(|x - y|)$$

for any $\alpha \in C_a$. However, using a well known result from the theory of quasiconformal mappings, we have:

5.2. Let t be a K-quasiconformal map of the open unit disc U in the plane onto itself, with $t(0) = 0$. The adjoint T of t determines a $\mathfrak{B}$-morphism

$$\epsilon = ((T_\phi)_{\phi \in G}, \eta), \qquad \text{i.e.} \quad \epsilon : S(U) \to S(U) \qquad (\eta)$$

of the direct system S(U) into istelf, where η is the equivalence class of $\alpha(x) = x^{\frac{1}{K}}$. Moreover, for each $\phi \in G$, we have :

$$L_{\phi \circ \beta}(U) \subseteq T_\phi(L_\phi(U)) \subseteq L_{\phi \circ \alpha}(U)$$

where $\beta(x) = x^k$.

Proof. This follows from the results in Section 3 and 4 and from the fact that for the map t and each pair of points z_1, z_2 in U:

$$C^{-K} |z_1 - z_2|^K \leq |t(z_1) - t(z_2)| \leq C|z_1 - z_2|^{\frac{1}{K}}$$

where C is an absolute constant whose smallest possible value is 16. [5, p. 102].

These results may be extended without difficulty to the case of domains in real n-space.

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