

Discrete groups and duality

On a discrete space-time lattice the notion of continuity is lost. Remarkably, this gives us more freedom in formulating a gauge theory. Whereas classical continuum gauge fields require a continuous gauge group for non-triviality, this is no longer so in the Wilson theory. Indeed, it is straightforward to consider the lattice link variables to be elements of some finite discrete group. The simplest such model considers elements of the group $Z_2 = \{1, -1\}$ and represents a gauge-invariant interaction of a set of Ising spins. Wegner (1971) first introduced this system as an example of a case with non-trivial phase structure but without a local order parameter.

An amusing point is that although the classical theory based on a discrete group has no continuum limit, this does not necessarily carry over to the quantum theory. If the system has a second-order phase transition at an appropriate zero of its renormalization group function, one should be able to define a continuum quantum field theory.

One reason to study discrete variables is that the resulting models are often inherently more amenable to analysis. For example, the two-dimensional Ising model is exactly solvable for the thermodynamic functions, and yet it has a non-trivial ferromagnetic phase transition. A hope with gauge theories is to gain some insight into the nature of their phase structures. Furthermore, these models provide a useful testing ground for new techniques.

In this chapter we concentrate on the cyclic groups Z_P , where the elements are the P 'th roots of unity. These models are all Abelian and as $P \rightarrow \infty$ we approach the $U(1)$ model. Obtaining the phase structure of the latter is essential because this is the gauge group of electrodynamics. Any attempt to understand quark confinement must also explain why QED , the prototype gauge theory, does not confine.

We shall use the Z_P gauge models in four dimensions as a framework for the discussion of duality transformations. This technique, which has also been extensively developed for spin systems (Savit, 1980; Cardy, 1980) relates the strong and weak coupling domains and in some cases determines

phase transition temperatures exactly. Under duality, the thermodynamic functions of a model map onto a related system but with different couplings. Singularities either occur in dual pairs or are restricted to special self-dual points. Historically, this was first applied to the Ising model and gave an exact determination of the critical point (Kramers and Wannier, 1941). The extension to gauge theories is direct (Balian, Drouffe and Itzykson, 1975*a*; Korthals-Altes, 1978; Yoneya, 1978).

As in the usual formulation of lattice gauge theory, our variables are elements of the gauge group, which we take to be Z_P :

$$U_{ij} \in Z_P = \{e^{2\pi i k/P} \mid k = 0, \dots, P-1\}. \tag{16.1}$$

Absorbing a factor of -1 in the action, the path integral is

$$Z = \sum_{U \in Z_P} e^{S(U)}. \tag{16.2}$$

Again as in the usual theory, the action is a sum over the plaquettes of the lattice

$$S(U) = \sum_{\square} S_{\square}(U_{\square}). \tag{16.3}$$

In the normal case $S_{\square}(U_{\square})$ is the real part of the trace of U_{\square} . We deviate slightly at this point because the discussion of duality is simplified if we consider a more general action per plaquette. To interpret e^S as a Boltzmann weight, we require S_{\square} to be a real function. So that the orientation of the plaquettes is irrelevant, we also require

$$S_{\square}(U_{\square}) = S_{\square}(U_{\square}^{-1}). \tag{16.4}$$

Beyond these constraints, the action is arbitrary. For a general gauge group one usually requires that S_{\square} is a class function over the group. For an Abelian group such as considered here all functions are class functions.

With any gauge group the general plaquette action has a character expansion

$$S_{\square}(U) = \sum_n \beta_n \chi_n(U), \tag{16.5}$$

where χ_n represents the trace in the n 'th irreducible representation of the group. For Z_P there are precisely P such representations, all one-dimensional and given by

$$R_n(U) = U^n, \quad n = 0, \dots, P-1. \tag{16.6}$$

The representation property

$$R_n(U) R_n(U') = R_n(UU') \tag{16.7}$$

is a trivial consequence of the Abelian nature of the group. To combine representations we have the rule

$$R_m(U) R_n(U) = R_{m+n}(U), \tag{16.8}$$

where the index $m+n$ is understood modulo P . The orthogonality of the

characters is

$$P^{-1} \sum_U R_m(U) R_n(U) = P^{-1} \sum_U U^{m+n} = \delta_{m, P-n}. \tag{16.9}$$

The crucial point which results in a simple duality structure for the Z_P theories is that the combination rule for characters has precisely the same structure as the original group. The dual theory will again be a Z_P theory. Duality has been much less useful in other theories, such as with non-Abelian groups, where the representation structure is more complicated.

With the above representations, eq. (16.5) becomes

$$S_{\square}(U) = \sum_n \beta_n U^n. \tag{16.10}$$

In terms of these variables the constraint of eq. (16.4) becomes

$$\beta_n = \beta_{P-n}. \tag{16.11}$$

The parameter β_0 is simply an overall normalization, irrelevant to thermodynamics but convenient to keep. Character orthogonality inverts eq. (16.5) with the result

$$\beta_n = P^{-1} \sum_U U^{-n} S_{\square}(U). \tag{16.12}$$

For the discussion of duality it is convenient to expand the Boltzmann weight

$$e^{S_{\square}(U)} = \sum_n b_n U^n, \tag{16.13}$$

where

$$b_n = P^{-1} \sum_U U^{-n} e^{S_{\square}(U)} \tag{16.14}$$

and eq. (16.11) becomes

$$b_n = b_{P-n}. \tag{16.15}$$

Up to a factor of b_0 , these are the parameters which proved so useful in the strong coupling expansion. The energy shift represented by β_0 becomes an overall scale factor in the b_n . Thermodynamics depends on the latter parameters only in a projective sense.

To proceed, we take the path integral and insert the character expansion of eq. (16.14) for each plaquette. This gives a sum over an integer n_{\square} associated with every elementary square of the lattice. Pulling this sum to the outside, the partition function is

$$Z = \sum_{\{n_{\square}\}} \left(\prod_{\square} b_{n_{\square}} \right) \prod_{\{ij\}} \left(\sum_{U_{ij}} \left(\prod_{\square \supset ij} U_{ij}^{n_{\square}} \right) \right). \tag{16.16}$$

Here the innermost product is over the six plaquettes containing the link ij . The sum over the U_{ij} is immediate from the orthogonality relations and gives

$$Z = \sum_{\{n_{\square}\}} \left(\prod_{\square} b_{n_{\square}} \right) \prod_{\{ij\}} (P \delta_{\sum_{\square \supset ij} n_{\square}, 0}), \tag{16.17}$$

where the Kronecker delta is understood modulo P in its indices. The factor of P multiplying the delta functions occurs because we have not normalized

our sums over the group. We now wish to make an appropriate change of variables which will enable us to do some of the sums with the Kronecker functions.

The factor
$$\delta_{\sum_{\square \supset ij} n_{\square}, 0} \quad (16.18)$$

involves the six plaquettes containing the link ij . In figure 16.1 we illustrate four of these, the remaining two utilize the unvisualized fourth dimension. The key to simplify this construct is to go to the dual lattice. We associate a new site with the center of each of the hypercubes of the original lattice.

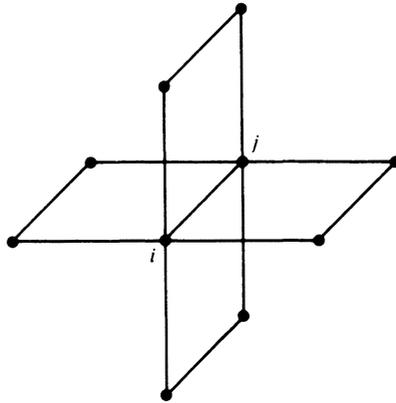


Fig. 16.1. Four of the six plaquettes containing the link ij . The remaining two utilize the fourth dimension.

For each site, link, plaquette, cube or hypercube on the original lattice there is a one-to-one correspondence with a hypercube, cube, plaquette, link or site, respectively, on the dual lattice. For example, dual to a link is the cube representing the common boundary of the two hypercubes which are dual to the ends of the link in question. Furthermore, the duality mapping can carry a sense of orientation if we invent a four-dimensional 'left hand' rule. For example, for the dual of a link in the positive t direction, we have a three-dimensional cube in xyz space. We can then adopt the convention that all plaquettes on this cube are oriented to the left when viewed from its center. For all other directions, we make even permutations on the indices x, y, z , and t . The dual of a plaquette is another plaquette, common to the four cubes which are dual to the links of the original plaquette. The orientation of the dual plaquette is specified by the above convention on any one of the original plaquette's links.

The utility of the dual mapping for the Z_p problem begins to appear with the observation that the six plaquettes needed in eq. (16.18) are dual to the set of six plaquettes which forms the three-dimensional cube dual

to the link ij . Regarded on the dual lattice, the partition function is a sum over integers associated with each plaquette but subject to the constraint that the sum of these variables over any three-dimensional cube is zero modulo P , where the plaquettes are oriented with the appropriate handed rule. The next step is to solve this constraint.

If each of these dual plaquette variables were a modulo P sum of integers associated with the links of the dual lattice, then the constraint of eq. (16.18) would be automatically satisfied. This is because each link variable would occur twice, once in each orientation, in forming the cube variable and would thus cancel out. Remarkably, this solution of the constraint equation is unique up to gauge transformations. To see this, consider a completely fixed gauge in the sense discussed in chapter 9. Thus we set to zero all link variables on a maximal tree containing no closed loops. Any unfixed link must then form a unique closed loop with a set of fixed links. To solve the constraint, we set this link to the sum, modulo P , of the plaquette variables on any two-dimensional surface with this loop as its boundary. The constraint condition on cubes permits deformation of this surface and thus assures the uniqueness of the selection procedure. If we now undo the gauge fixing, we obtain P^{N^4} gauge equivalent configurations giving the dual plaquette variables as sums over the corresponding links.

This process eliminates the delta functions in eq. (16.17) and replaces the sum over plaquette variables with one over the dual link quantities

$$Z = P^{3N^4} \sum_{\{n_{ij}\}} \prod_{\square} b_{n_{\square}}. \tag{16.19}$$

Here n_{\square} is the modulo P sum of the n_{ij} around the given plaquette. We now identify n_{\square} with an element of Z_P in the natural way

$$U_{\square} = e^{2\pi i n_{\square}/P}, \tag{16.20}$$

and do a character expansion for b_n

$$b_{n_{\square}} = P^{-\frac{1}{2}} e^{\tilde{S}(U_{\square})} = P^{-\frac{1}{2}} \sum_n \tilde{b}_n U_{\square}^n, \tag{16.21}$$

where this equation defines the dual action $\tilde{S}(U_{\square})$. In terms of these new variables we reproduce the original partition function but with a new set of parameters \tilde{b}_n

$$Z(b) = Z(\tilde{b}). \tag{16.22}$$

The relation between b and \tilde{b} is simply a linear transformation

$$\left. \begin{aligned} \tilde{b}_n &= A_{nm} b_m, \\ b_n &= A_{nm}^* \tilde{b}_m, \end{aligned} \right\} \tag{16.23}$$

where A is the unitary matrix which generates discrete Fourier transforms

$$A_{nm} = P^{-\frac{1}{2}} e^{2\pi i mn/P}. \tag{16.24}$$

This matrix has the properties

$$A^{-1} = A^* \quad (\text{unitarity}), \tag{16.25}$$

$$A = A^T \quad (\text{symmetric}), \tag{16.26}$$

$$(A^2)_{mn} = \delta_{m, P-n}, \tag{16.27}$$

$$A^4 = I. \tag{16.28}$$

Equation (16.22) is the key consequence of duality for the Z_P models. We note that the criterion of eq. (16.15) for orientation invariance automatically carries over to the dual variables because

$$A_{P-n, m} = A_{n, P-m}. \tag{16.29}$$

However, duality does not always result in a physically sensible model. If any of the \tilde{b}_n are negative, then one cannot interpret them as new Boltzmann weights as in eq. (16.21). For those domains of the parameter space which are dual to another physical model, we have an interesting constraint on the singularities which can occur in the partition function. These must either occur in pairs, dual to each other, or must occur at self-dual points where $\tilde{b} = b$. There are examples of each of these possibilities.

To illustrate these ideas in a more specific case, we now turn to the Z_2 theory. Here the variables are from the set $\{1, -1\}$ and the action is

$$S_{\square}(U) = \beta_0 + \beta_1 U. \tag{16.30}$$

The parameters b_0 and b_1 follow from the expansion

$$\exp(\beta_1 U) = \cosh(\beta_1) + U \sinh(\beta_1). \tag{16.31}$$

This immediately gives

$$\left. \begin{aligned} b_0 &= \exp(\beta_0) \cosh(\beta_1), \\ b_1 &= \exp(\beta_0) \sinh(\beta_1). \end{aligned} \right\} \tag{16.32}$$

Inverting these equations gives

$$\left. \begin{aligned} \beta_0 &= \frac{1}{2} \log(b_0^2 - b_1^2), \\ \beta_1 &= \frac{1}{2} \log((b_0 - b_1)/(b_0 + b_1)). \end{aligned} \right\} \tag{16.33}$$

The Fourier matrix A is

$$A = 2^{-\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{16.34}$$

This gives the dual variables

$$\left. \begin{aligned} \tilde{b}_0 &= 2^{-\frac{1}{2}}(b_0 + b_1), \\ \tilde{b}_1 &= 2^{-\frac{1}{2}}(b_0 - b_1). \end{aligned} \right\} \tag{16.35}$$

In terms of the variables β_0 and β_1 we have

$$\left. \begin{aligned} \tilde{\beta}_0 &= \beta_0 + \frac{1}{2} \log (\sinh (2\beta_1)), \\ \tilde{\beta}_1 &= \frac{1}{2} \log (\tanh (\beta_1)). \end{aligned} \right\} \quad (16.36)$$

The change in β_0 merely represents an overall normalization. The shift in β_1 , however, represents a non-trivial change in the model. Small β_1 maps onto large $\tilde{\beta}_1$ and vice versa. Knowledge of the thermodynamic functions of the model in, say, the weak coupling regime determines, via eq. (16.22) and its derivatives, the corresponding functions in strong coupling. One point maps onto itself; this self duality occurs at

$$\beta_1 = \frac{1}{2} \log (1+2^{\frac{1}{2}}) = 0.4406867 \dots \quad (16.37)$$

At exactly this coupling, numerical work has demonstrated that the model has a strong first-order phase transition, exhibited in chapter 9, figure 9.1 (Creutz, Jacobs and Rebbi, 1979a).

Returning now to general P , various contours in the multiparameter space reduce to standard models. The simple Wilson Z_P theory considers only β_0 and $\beta_{\pm 1}$. In this system for $P = 2, 3$, and 4 , the model maps onto itself under duality. Monte Carlo analysis (Creutz, Jacobs, and Rebbi, 1979b) indicates strong first-order phase transitions at the self-dual points. At $P = 5$ or more, the model ceases to be exactly self-dual, the dual model requiring more than just β_0 and β_1 . Numerical work on these models indicates two second-order phase transitions, one moving to larger β_1 as P increases, and the other remaining in the $U(1)$ limit. These features can be understood in terms of duality with a slightly modified one-parameter action which is self-dual.

The Villain (1975) variation of the Wilson theory considers the action

$$e^{S_{\square}(U)} = \sum_{l=-\infty}^{\infty} e^{-\frac{1}{2}\beta(\theta - 2\pi l)^2}, \quad (16.38)$$

where the angle θ is defined

$$U = e^{i\theta}, \quad -\pi < \theta \leq \pi. \quad (16.39)$$

For this action the parameters b_n are given by the double sum

$$b_n = P^{-1} \sum_{m=1}^P \sum_{l=-\infty}^{\infty} \exp (-\pi\beta(m/P-l)^2 - 2\pi imn/P). \quad (16.40)$$

To simplify this, we first complicate it by replacing the sum over l with an integral over a continuous angle and inserting a sum of delta functions from the formula

$$\sum_{l=-\infty}^{\infty} \delta(l - \theta/2\pi) = \sum_{k=-\infty}^{\infty} e^{ik\theta}. \quad (16.41)$$

This gives

$$b_n = P^{-1} \sum_{m=1}^P \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} (d\theta/2\pi) \exp(-\frac{1}{2}\beta(\theta - 2\pi m/P)^2 - 2\pi imn/P + ik\theta). \tag{16.42}$$

The theta integral is now Gaussian and yields

$$b_n = P^{-1}(2\pi\beta)^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} \sum_{m=1}^P \exp(-\frac{1}{2}k^2/\beta - 2\pi im(n+k)/P). \tag{16.43}$$

The sum over m constrains $n+k$ to be zero modulo P . Thus we do this sum and replace the sum over k by a sum over multiples of P . This gives the final result

$$b_n = (2\pi\beta)^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} \exp(-\frac{1}{2}(P^2/\beta)(k-n/P)^2). \tag{16.44}$$

If we now return all the way back to eq. (16.21) and interpret this as the dual Boltzmann weight, we see that it has the same form as in eq. (16.38) but with β mapped onto

$$\tilde{\beta} = P^2/(2\pi\beta). \tag{16.45}$$

We have a self-dual model with the self-dual point at

$$\beta = (2\pi)^{-\frac{1}{2}}P. \tag{16.46}$$

As the parameter P goes to infinity the model goes over into $U(1)$. To avoid confinement in electrodynamics formulated with this lattice prescription, this model should exhibit a deconfining phase transition to a photon phase at weak coupling. Guth (1980) has rigorously proven the existence of such a transition. If the transition persists in the finite P models, then the latter must, by duality, have another transition at the dual point. This second transition, a consequence of the discreteness of the group, should move towards zero temperature with β growing as P^2 as P becomes large (Elitzur, Pearson, and Shigemitsu, 1979; Horn, Weinstein, and Yankielowicz, 1979; Ukawa, Windey and Guth, 1980). This is the empirically observed behavior for the Wilson theory, of which the Villain form is an approximation more amenable to analytic treatment.

Problems

1. Show that the two-dimensional Ising model is self-dual.
2. Show that the three-dimensional Z_2 gauge theory is dual to the three-dimensional Ising model.
3. Consider the P state gauge Potts model where all b_n except b_0 are equal to each other. Show that this model is self-dual. This system has a

single first-order phase transition at the self-dual point (Kogut, Pearson, Shigemitsu and Sinclair, 1980).

4. Show the self-duality of the Wilson Z_4 model with only the couplings β_0 and $\beta_1 = \beta_3$.