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A SUFFICIENT CONDITION FOR THE SECOND DERIVED FACTOR GROUP TO BE FINITE

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1. Introduction

This paper concerns an application of an algorithm for the second derived factor group as described by Howse and Johnson in [3]. This algorithm has as its basis the Fox derivative (see [1]), a mapping from the free group F to the group-ring $\mathbb{Z}F$, defined as follows: let X be a set of generators of a group G, and let $w = y_1 \dots y_k$ with each $y_i \in X^{\pm 1}$. Then the Fox derivative of the word w with respect to any generator $x \in X$ is defined to be

$$\frac{\partial w}{\partial x} = \sum_{i=1}^{k} a_i, \quad \text{where} \quad a_i = \begin{cases} y_1 \dots y_{i-1}, & \text{when} \quad y_i = x, \\ -y_1 \dots y_i, & \text{when} \quad y_i = x^{-1}, \\ 0, & \text{when} \quad y_i \neq x^{\pm 1}. \end{cases}$$

Let $\phi: F \to G$ (and also $\phi: \mathbb{Z}F \to \mathbb{Z}G$, etc.) and $\psi: G \to G/G'$ (and also $\psi: \mathbb{Z}G \to \mathbb{Z}(G/G')$, etc.). The Jacobian $J = \partial R/\partial X$ of the presentation $G = \langle X \mid R \rangle$ is the $|R| \times |X|$ matrix whose (i,j) entry is $\partial r_i/\partial x_j$. Let $G/G' = \{z_1, \dots, z_n\}$ and A be a matrix over $\mathbb{Z}(G/G')$. Any entry $\gamma \in \mathbb{Z}(G/G')$ of A is of the form $\gamma = \sum_{i=1}^{n} \alpha_i z_i$ and thus defines an n-tuple $(\alpha_1, \dots, \alpha_n)$. The n-tuple corresponding to $z_j \gamma (1 \leqq j \leqq n)$ is a rearrangement of this, and we let $m(\gamma)$ denote the $n \times n$ matrix having this as its jth row. Let $m(A)$ denote the matrix of integers obtained by applying m to each entry of A. Then the integer matrix $M = m(\psi\phi(J))$ is a relation matrix for the group $G'/G'' \oplus Z^{\oplus(n-1)}$. The invariant factors of G'/G'' can be computed from M by diagonalisation.

The proof of this algorithm together with examples illustrating it and applications of it can be found in [2] and [3].

This paper applies the algorithm to 2-generator groups with finite derived factor groups. The main result obtained is that the second derived factor group is finite if the determinant of the matrix, with the above notation, $A_{ij} = m(\psi\phi(\partial r_i/\partial x_j))$ for some $r_i \in R$ and $x_j \in X$ of the group presentation $G = \langle X \mid R \rangle$ ($|X| = 2$), is non-zero. This result is then applied to groups with cyclic derived factor group and which have a presentation which

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contains at least one relator having a small number of syllables; in which case much more explicit conditions for the second derived factor group to be finite are determined.

In the application of the algorithm the integer matrix $m(\psi\phi(\partial r/\partial x))$ can be represented by the "polynomial" $\psi\phi(\partial r/\partial x)$, e.g. if $r=x^3$, then $\psi\phi(\partial r/\partial x)=1+x+x^2$ and this can represent the integer matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(assuming that $G/G'\cong\mathbb{Z}_3$). Moreover the integer relation matrix $m(\psi\phi(J))$ can be represented by the "polynomial" matrix $\psi\phi(J)$. Indeed row and column operations can be performed on this "polynomial" matrix.

2. The main theorem

Consider the 2-generator group $G=\langle x,y|r_1,\ldots,r_q\rangle$, where $2\leqq q<\infty$, with finite derived factor group and $|G:G'|=n$. For $i=1,\ldots,q$, let $r_i=x^{a_{i1}}y^{b_{i1}}\ldots x^{a_{ik_i}}y^{b_{ik_i}}$ where $a_{ih}\neq 0$ and $b_{ih}\neq 0$ for $h=1,\ldots,k_i$, if $k_i>1$.

Let $\sum_{h=1}^{k_i} a_{ih}=a_i$ and $\sum_{h=1}^{k_i} b_{ih}=b_i$. Let $G/G'=\{z_1,\ldots,z_n\}$. Then the Fox derivatives of the relator r_i with respect to the generators x and y, modulo G', are of the form

$$\frac{\partial r_i}{\partial x}\equiv\alpha_{i1}z_1+\cdots+\alpha_{in}z_n \pmod{G'},$$

$$\frac{\partial r_i}{\partial y}\equiv\beta_{i1}z_1+\cdots+\beta_{in}z_n \pmod{G'}.$$

Lemma 1. *$\alpha_{i1}+\cdots+\alpha_{in}=a_i$ and $b_{i1}+\cdots+b_{in}=b_i$.*

The proof is obvious from the definition of Fox derivatives.

The matrix (given in polynomial form)

$$J=\begin{pmatrix} \alpha_{11}z_1+\cdots+\alpha_{1n}z_n & \beta_{11}z_1+\cdots+\beta_{1n}z_n \\ \vdots & \vdots \\ \alpha_{q1}z_1+\cdots+\alpha_{qn}z_n & \beta_{q1}z_1+\cdots+\beta_{qn}z_n \end{pmatrix}$$

is a relation matrix for $G'/G''\oplus\mathbb{Z}^{\oplus(n-1)}$. $m(J)$ is a $qn\times 2n$ matrix, thus $|G':G''|$ is equal to the h.c.f. of the determinants of all $(n+1)$-rowed minors of $m(J)$ when finite, and is infinite when all these are zero.

Consider the $2n\times 2n$ "submatrix" of J

$$\begin{pmatrix} \alpha_{i1}z_1+\cdots+\alpha_{in}z_n & \beta_{i1}z_1+\cdots+\beta_{in}z_n \\ \alpha_{j1}z_1+\cdots+\alpha_{jn}z_n & \beta_{j1}z_1+\cdots+\beta_{jn}z_n \end{pmatrix}=K_{ij}\text{ (say)},$$

where $i \neq j$. In integer form we will write this matrix as

$$m(K_{ij}) = \begin{pmatrix} m(\alpha_{i1}, \ldots, \alpha_{in} & m(\beta_{i1}, \ldots, \beta_{in} \\ m(\alpha_{j1}, \ldots, \alpha_{jn} & m(\beta_{j1}, \ldots, \beta_{jn} \end{pmatrix}.$$

Replace row $n+1$ by the sum of the last n rows, and column $n+1$ by the sum of the last n columns and then consider the first $n+1$ rows and $n+1$ columns of the resulting matrix to get (using Lemma 1)

$$\begin{pmatrix} m(\alpha_{i1}, \ldots, \alpha_{in}) & & & b_i \\ & & & \vdots \\ & & & b_i \\ a_j & \ldots & a_j & nb_j \end{pmatrix} = M_{ij} \text{ (say)}.$$

Now M_{ij} is an $(n+1) \times (n+1)$ matrix (while not an $(n+1)$-rowed minor of J, M_{ij} was produced from J by matrix operation), thus we have

$$|G':G''| \,|\, |\det M_{ij}|. \tag{1}$$

Lemma 2. *If* $a_i \neq 0$, *then*

$$|\det M_{ij}| = n|a_ib_j - a_jb_i|\,|\det A_i|/|a_i|,$$

where $A_i = m(\alpha_{i1}, \ldots, \alpha_{in})$.

Proof.

$$|\det M_{ij}| = \left|\det \begin{pmatrix} A_i & & & b_i \\ & & & \vdots \\ & & & b_i \\ a_j & \ldots & a_j & nb_j \end{pmatrix}\right|$$

$$\xrightarrow[c_{n+1} - \frac{b_i}{a_i}\sum_{k=1}^{n} c_k]{} = \left|\det \begin{pmatrix} A_i & & & 0 \\ & & & \vdots \\ & & & 0 \\ a_j & \ldots & a_j & nb_j - \dfrac{na_jb_i}{a_i} \end{pmatrix}\right|$$

$$= n\left|\frac{a_ib_j - a_jb_i}{a_i}\right| |\det A_i|, \text{ as required.} \qquad \square$$

Lemma 3. *If* $a_i \neq 0$, *then* $a_ib_j - a_jb_i \neq 0$ *for some* j.

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Proof. From the original definition of G, we have

$$|G:G'| = \text{h.c.f.}(a_i b_j - a_j b_i; i=1,\dots,q, j=1,\dots,q). \tag{2}$$

We will assume throughout this proof that $a_i \neq 0$. If $b_i = 0$, then there exists j such that $b_j \neq 0$ from (2), because G/G' is finite. So if $b_i = 0$, then $a_i b_j - a_j b_i \neq 0$ for some j. Now consider the case $b_i \neq 0$. Assume, for a contradiction, that $a_i b_j - a_j b_i = 0$ for all j. Thus $a_j = a_i b_j / b_i$ for all j. So for all j, k we have

$$a_k b_j - a_j b_k = \frac{a_i b_k}{b_i} b_j - \frac{a_i b_j}{b_i} b_k = 0$$

contradicting (2), because G/G' is finite. Thus $a_i b_j - a_j b_i \neq 0$ for some j. □

We can now state a sufficient condition for $|G':G''|$ to be finite.

Theorem 1. *Let G be a 2-generator group with G/G' finite. Let $A_i = m(\psi\phi(\partial r_i/\partial x_j))$ for a given generator x_j. If* $\det A_i \neq 0$ *for some i, then G'/G'' is finite.*

Proof. If $a_i = 0$, then $\det A_i = 0$, because each row-sum $= a_i = 0$, by Lemma 1; however, there exists i so that $a_i \neq 0$, by the hypothesis that $\det A_i \neq 0$. Assume that $a_i \neq 0$. By Lemma 3, $a_i b_j - a_j b_i \neq 0$ for some j; so if $\det A_i \neq 0$, then $\det M_{ij} \neq 0$ by Lemma 2. Thus by (1), G'/G'' is finite. □

3. Groups with cyclic derived factor group

When the derived factor group is cyclic, the matrix A_i of Theorem 1 is circulant. A formula for the determinant of a circulant matrix is given in Lemma 4 below. From this formula conditions can be found such that the determinant is not zero, giving further conditions for the second derived factor group to be finite.

Lemma 4. *Let $C = C(\alpha_1, \alpha_2, \dots, \alpha_n)$ be a circulant matrix, and ω be a primitive* n*th root of unity. Then*

$$\det C = \prod_{i=1}^{n} \sum_{j=1}^{n} \alpha_j \omega^{i(j-1)}.$$

The proof of this lemma can be found in [4].

The following theorems are concerned with groups with cyclic derived factor group. We will consider groups having a presentation which contains at least one relator having a small number of syllables, i.e. a relator of the form $r = x^a y^b$, or of the form $r = x^{a_1} y^{b_1} x^{a_2} y^{b_2}$.

Let $G = \langle x, y | R \rangle$, where R is a finite set of relators, with $G/G' = \langle z | z^n \rangle$, where $x = z^{m_1} (\text{mod } G')$ and $y \equiv z^{m_2} (\text{mod } G')$, where $0 < m_1 \leqq n-1$ and $0 < m_2 \leqq n-1$. (It should be noted that $0<m_1, m_2$ is an extra assumption, but the case $x \equiv 1 \pmod{G'}$, i.e. $m_1=0$, is considered in [2]. Moreover, either m_1 or m_2 is not zero, unless $G=G'=G''$.)

Let $M_j=\{s; s|m_j\}$ $(j=1,2)$ the set of all divisors of m_j.

Theorem 2. *Let G be as just stated. Let $r=x^a y^b$, where $a \neq 0$, $r \in R$. If $(am_1, n) \in M_1$, then G'/G'' is finite.*

Proof. Without loss of generality we can assume that $a>0$ (if $a<0$, then the relator $r=x^a y^b$ can be rewritten as $y^{-b}x^{-a}$ and then as $x^{-a}y^{-b}$). Now

$$\frac{\partial r}{\partial x}=\frac{\partial x^a}{\partial x}=1+x+\cdots+x^{a-1}$$

$$\equiv 1+z^{m_1}+z^{2m_1}+\cdots+z^{(a-1)m_1} \pmod{G'}.$$

Let $f(z)=1+z^{m_1}+z^{2m_1}+\cdots+z^{(a-1)m_1}$. (Recall that $z^n=1$). Then, by Lemma 4, $\det A=\prod_{z^n=1} f(z)$ (where A represents, in this case, the matrix A_i of Theorem 1).

If $\det A=0$, then $f(w)=0$ for some nth root w of unity, and if $\det A \neq 0$, then G'/G'' is finite by Theorem 1. Let $(am_1, n) \in M_1$, and, for a contradiction, assume that $f(w)=0$ where w is an nth root of unity. Then

$$(1-w^{m_1})f(w)=0 \Rightarrow 1-z^{am_1}=0 \Rightarrow z^{am_1}=1.$$

Thus $w^{(am_1,n)}=1$ and hence $w^{m_1}=1$, because $(am_1, n) \in M_1$ and so (am_1, n) divides m_1. So

$$f(w)=1+w^{m_1}+\cdots+w^{(a-1)m_1}=a \neq 0$$

the required contradiction. So $\det A \neq 0$ and G'/G'' is finite by Theorem 1. □

Theorem 3. *Let G be as defined above. Let $r=x^{a_1}y^{b_1}x^{a_2}y^{b_2}$, where $r \in R$ and $a_1+a_2 \neq 0$.*

(i) *Let n be odd. If $(a_1m_1, a_2m_1, n) \in M_1$ and $(b_1m_2, b_2m_2, n) \in M_1$, then G'/G'' is finite.*

(ii) *Let n be even. If $(a_1m_1, a_2m_1, n) \in M_1$, $(b_1m_2, b_2m_2, n) \in M_1$, and $((a_1-a_2)m_1, n) \in M_1$, then G'/G'' is finite.*

Proof. The proof proceeds along similar lines to that of Theorem 2. We have

$$\frac{\partial r}{\partial x}=\frac{\partial x^{a_1}}{\partial x}+x^{a_1}y^{b_1}\frac{\partial x^{a_2}}{\partial x}$$

$$\equiv \alpha_1+\alpha_2 z+\cdots+\alpha_n z^{n-1} \pmod{G'}.$$

Let $f(z)=\alpha_1+\alpha_2 z+\cdots+\alpha_n z^{n-1}$.

There are four cases to consider depending on the signs of a_1 and a_2. However in each case we obtain

$$(1-z^{m_1})f(z)=1-z^{a_1m_1}+z^{a_1m_1+b_1m_2}-z^{a_1m_1+b_1m_2+a_2m_1}. \tag{3}$$

Now, by Lemma 4, $\det A=\prod_{z^n=1} f(z)$ (where A is the matrix equivalent to the matrix A_i of Theorem 1). If $\det A=0$, then $f(w)=0$ for some nth root w of unity, and if $\det A\neq 0$, then G'/G'' is finite by Theorem 1. Assume, for a contradiction, that $\det A=0$, so there exists w such that $f(w)=0$, where $w^n=1$. Then $(1-w^{m_1})f(w)=0$ and so, by (3),

$$1-w^{a_1m_1}+w^{a_1m_1+b_1m_2}-w^{a_1m_1+b_1m_2+a_2m_1}=0. \tag{4}$$

Let $w^{a_1m_1}=p_1+iq_1$, $w^{a_1m_1+b_1m_2}=p_2+iq_2$, and $w^{a_1m_1+b_1m_2+a_2m_1}=p_3+iq_3$, where $p_j^2+q_j^2=1$ $(j=1,2,3)$, (where $i^2=-1$).

From (4) we have

$$1-p_1+p_2-p_3=0\Rightarrow p_1=1+p_2-p_3 \tag{5}$$

and $-q_1+q_2-q_3=0\Rightarrow q_1=q_2-q_3$.

Now $p_1^2+q_1^2=1$, so $(1+p_2-p_3)^2+(q_2-q_3)^2=1$. Multiplying out, factorising, and squaring, we have

$$(1+p_2)^2(1-p_3)^2=(1-p_2^2)(1-p_3^2).$$

There are three cases to consider

(a) $p_3=1$, so $w^{a_1m_1+b_1m_2+a_2m_1}=1$, hence, by (4), $w^{a_1m_1}=w^{a_1m_1+b_1m_2}\Rightarrow w^{b_1m_2}=1$. Hence $w^{(a_1+a_2)m_1}=1$. Also $w^{(a_1+a_2)m_1+(b_1+b_2)m_2}=1$ (because $r=z^{(a_1+a_2)m_1+(b_1+b_2)m_2} \pmod{G'}$), so $w^{b_2m_2}=1$.

(b) $p_2=-1$, so $w^{a_1m_1+b_1m_2}=-1$.

(c) $(1+p_2)(1-p_3)=(1-p_2)(1+p_3)$ i.e. $1+p_2-p_3-p_2p_3=1-p_2+p_3-p_2p_3$, hence $p_2=p_3$. Thus, from (5), $p_1=1$, so $w^{a_1m_1}=1$. By (4), $w^{b_1m_2}=w^{b_1m_2+a_2m_1}$, so $w^{a_2m_1}=1$.

Assume that n is odd. In case (b) $w^{a_1m_1+b_1m_2}=-1$, so this case has no solution (because n is odd), so we need only consider cases (a) and (c). In case (a) $w^{b_1m_2}=1$ and $w^{b_2m_2}=1$, while in case (c) $w^{a_1m_1}=1$ and $w^{a_2m_1}=1$. So if $(a_1m_1,a_2m_1,n)\in M_1$ and $(b_1m_2,b_2m_2,n)\in M_1$, then $w^{m_1}=1$. Hence $f(w)=a_1+a_2w^{a_1m_1+b_1m_2}$. Now $w^n=1$, where n is odd, and $a_1+a_2\neq 0$, so $f(w)\neq 0$ contradicting $\det A=0$. So $\det A\neq 0$ and G'/G'' is finite by Theorem 1, proving (i).

Assume that n is even. In case (b) $w^{a_1m_1+b_1m_2}=-1$, so, by (4), $w^{b_1m_2+a_2m_1}=-1$ and thus $w^{(a_1-a_2)m_1}=1$. In case (a) $w^{b_1m_2}=1$ and $w^{b_2m_2}=1$. In case (c) $w^{a_1m_1}=1$ and $w^{a_2m_1}=1$. So if $(a_1m_1,a_2m_1,n)\in M_1$, $(b_1m_2,b_2m_2,n)\in M_1$, and $((a_1-a_2)m_1,n)\in M_1$, then $w^{m_1}=1$. Thus $f(w)=a_1+a_2w^{a_1m_1+b_1m_2}$. $f(w)\neq 0$, because $a_1+a_2\neq 0$ and $a_1\neq a_2$ (because $((a_1-a_2)m_1,n)\in M_1$) contradicting $\det A=0$. So $\det A\neq 0$ and G'/G'' is finite by Theorem 1, proving (ii). □

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