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The Classification of 7- and 8-dimensional Naturally Reductive Spaces

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Abstract. A new method for classifying naturally reductive spaces is presented. This method relies on a new construction and the structure theory of naturally reductive spaces recently developed by the author. This method is applied to obtain the classification of all naturally reductive spaces in dimension 7 and 8.

1 Introduction

A locally naturally reductive space is a Riemannian manifold together with a metric connection that has parallel skew-torsion and parallel curvature. A locally symmetric space can thus be seen as a naturally reductive space with zero torsion. In the seminal paper [5], Cartan classified all symmetric spaces.

The story for naturally reductive spaces is quite different, of not available. A good place to start is to classify them in small dimensions. This has been done in dimensions 3, 4, and 5 in [13, 14, 19] and more recently in dimension 6 in [1]. These classifications essentially rely on being able to parametrize the possible torsion and curvature tensors of the naturally reductive connections and then solving the first Bianchi identity. Such a parametrization breaks down in higher dimensions. The recent developments in [16, 17] tell us however that naturally reductive spaces are still very rigid. This gives us the ability to present a completely new way to classify naturally reductive spaces.

1.1 Results

The new approach presented here can be applied in any dimension, but becomes increasingly more elaborate as the dimension increases. Therefore, it becomes important to find ways to limit the possible cases. This will be carried out explicitly for naturally reductive spaces of dimension 7 and 8. An important point is the division of naturally reductive spaces into two types, as in [17]:

Type I The transvection algebra is semi-simple.
Type II The transvection algebra is not semi-simple.

Another simplification of the classification comes from the partial duality of naturally reductive spaces defined in [17]; see also Definition 2.7. Every naturally reductive space of Type I can be related to a compact one. Therefore, we will only list

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the compact spaces of Type I and in case a non-compact partial dual space exists we will mention this. For the spaces of Type II, we will only list the ones for which the semi-simple factor of the canonical base space is compact, and we will mention if a partial dual spaces exist. It is not difficult to explicitly obtain all partial dual spaces. This makes the classification much more transparent. The classification of all 7- and 8-dimensional naturally reductive spaces is summarized in Theorem 3.6 for Type I and in Theorem 4.5 for Type II.

2 Preliminaries

The essential structure of a locally homogeneous space is encoded in the infinitesimal model. We now briefly discuss this below.

Theorem 2.1 (Ambrose–Singer [3]) *A complete simply connected Riemannian manifold (M, g) is a homogeneous Riemannian manifold if and only if there exists a metric connection ∇ with torsion T and curvature R such that*

$$\nabla T = 0 \quad \textit{and} \quad \nabla R = 0. \tag{2.1}$$

A metric connection satisfying (2.1) is called an *Ambrose–Singer connection*. The torsion T and curvature R of an Ambrose–Singer connection evaluated at a point $p \in M$ are linear maps

$$T_p : \Lambda^2 T_pM \longrightarrow T_pM \quad \text{and} \quad R_p : \Lambda^2 T_pM \longrightarrow \mathfrak{so}(T_pM) \tag{2.2}$$

that satisfy, for all $x, y, z \in T_pM$,

$$R_p(x, y) \cdot T_p = R_p(x, y) \cdot R_p = 0, \tag{2.3}$$

$$\mathfrak{S}^{x,y,z} R_p(x, y)z - T_p\big(T_p(x, y), z\big) = 0, \tag{2.4}$$

$$\mathfrak{S}^{x,y,z} R_p\big(T_p(x, y), z\big) = 0, \tag{2.5}$$

where $\mathfrak{S}^{x,y,z}$ denotes the cyclic sum over x, y, and z, and $\cdot$ denotes the natural action of $\mathfrak{so}(T_pM)$ on tensors. The first equation encodes that T and R are parallel objects for ∇, and under this condition the first and second Bianchi identity become equations (2.4) and (2.5), respectively. A pair of tensors (T, R), as in (2.2), on a vector space $\mathfrak{m}$ with a metric g satisfying (2.3), (2.4) and (2.5) is called an *infinitesimal model* on $(\mathfrak{m}, g)$.

Conversely, given an infinitesimal model (T, R) of a homogeneous space, one can construct a homogeneous space with that infinitesimal model. This construction is known as the *Nomizu construction*; see [15]. This construction goes as follows. Let

$$\mathfrak{h} := \{h \in \mathfrak{so}(\mathfrak{m}) : h \cdot T = 0,\ h \cdot R = 0\}.$$

and set $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{m}$. On $\mathfrak{g}$, the following Lie bracket is defined for all $h, k \in \mathfrak{h}$ and $x, y \in \mathfrak{m}$:

$$[h + x, k + y] := [h, k]_{\mathfrak{so}(\mathfrak{m})} - R(x, y) + h(y) - k(x) - T(x, y), \tag{2.6}$$

where $[\cdot, \cdot]_{\mathfrak{so}(\mathfrak{m})}$ denotes the Lie bracket in $\mathfrak{so}(\mathfrak{m})$. The bracket from (2.6) satisfies the Jacobi identity if and only if R and T satisfy equations (2.3), (2.4), and (2.5). We will call $\mathfrak{g}$ the *symmetry algebra* of the infinitesimal model (T, R). Let G be the simply

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connected Lie group with Lie algebra $\mathfrak{g}$ and let H be the connected subgroup with Lie algebra $\mathfrak{h}$. The infinitesimal model is *regular* if H is a closed subgroup of G. If this is the case, then clearly the canonical connection on G/H has the infinitesimal model (T, R) we started with. In [18, Thm. 5.2], it was proved that an infinitesimal model coming from an invariant connection on a globally homogeneous Riemannian manifold, as in (2.2), is regular.

Definition 2.2 Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$ be a Lie algebra together with a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a complement $\mathfrak{m}$ of $\mathfrak{h}$, and a metric g on $\mathfrak{m}$. Suppose $\mathrm{ad}(\mathfrak{h})\mathfrak{m} \subset \mathfrak{m}$ and, for all $x, y, z \in \mathfrak{m}$, that

$$g([x,y]_{\mathfrak{m}}, z) = -g(y, [x,z]_{\mathfrak{m}}).$$

Then we call $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$ a *naturally reductive decomposition* with $\mathfrak{h}$ the *isotropy algebra*. We will mostly refer to just $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as a naturally reductive decomposition and let the metric be implicit. The infinitesimal model of the naturally reductive decomposition is defined by

$$\begin{aligned} &(2.7) \qquad T(x,y) := -[x,y]_{\mathfrak{m}}, && \forall x, y \in \mathfrak{m}, \\ &\qquad\qquad R(x,y) := -\mathrm{ad}([x,y]_{\mathfrak{h}}) \in \mathfrak{so}(\mathfrak{m}), && \forall x, y \in \mathfrak{m}, \end{aligned}$$

where $[x,y]_{\mathfrak{m}}$ and $[x,y]_{\mathfrak{h}}$ are the $\mathfrak{m}$- and $\mathfrak{h}$-component of $[x,y]$, respectively. We call the decomposition an *effective* naturally reductive decomposition if the restricted adjoint map $\mathrm{ad}\colon \mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ is injective. We will say that $\mathfrak{g}$ is the *transvection algebra* of the naturally reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ if the decomposition is effective and $\mathrm{im}(R) = \mathrm{ad}(\mathfrak{h}) \subset \mathfrak{so}(\mathfrak{m})$. Note that (2.3) implies that $\mathrm{im}(R) \subset \mathfrak{so}(\mathfrak{m})$ is a subalgebra and that the transvection algebra is always a Lie subalgebra of the symmetry algebra. When we simply refer to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as a *naturally reductive transvection algebra*, we mean that this is a naturally reductive decomposition for which $\mathfrak{g}$ is also the transvection algebra.

The following theorem is a classical result by Kostant (see also [6]).

Theorem 2.3 (Kostant, [11]) *Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, g)$ be an effective naturally reductive decomposition. Then $\mathfrak{k} := [\mathfrak{m},\mathfrak{m}]_{\mathfrak{h}} \oplus \mathfrak{m}$ is an ideal in $\mathfrak{g}$ and there exists a unique $\mathrm{ad}(\mathfrak{k})$-invariant non-degenerate symmetric bilinear form $\overline{g}$ on $\mathfrak{k}$ such that $\overline{g}|_{\mathfrak{m}\times\mathfrak{m}} = g$ and $[\mathfrak{m},\mathfrak{m}]_{\mathfrak{h}} \perp \mathfrak{m}$. Conversely, any $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form $\overline{g}$ on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{m} = \mathfrak{h}^{\perp}$ and $\overline{g}|_{\mathfrak{m}\times\mathfrak{m}}$ positive definite gives a naturally reductive decomposition.*

In [16] a new construction of naturally reductive spaces is defined. This starts from a naturally reductive transvection algebra and a certain subalgebra $\mathfrak{k}$ of derivations of the transvection algebra together with an $\mathrm{ad}(\mathfrak{k})$-invariant metric B on $\mathfrak{k}$ and constructs a new naturally reductive decomposition. This new decomposition we call the $(\mathfrak{k}, B)$*-extension*, and generally this new space is a homogeneous fiber bundle over the original space. More explicitly, for $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ a non-zero transvection algebra the algebra, $\mathfrak{k}$ of derivations has to be a subalgebra of

$$\mathfrak{s}(\mathfrak{g}) := \{f \in \mathrm{Der}(\mathfrak{g}) : f(\mathfrak{h}) = \{0\},\ f(\mathfrak{m}) \subset \mathfrak{m},\ f|_{\mathfrak{m}} \in \mathfrak{so}(\mathfrak{m})\}.$$

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Let $\mathfrak{so}(\infty)$ be the Lie algebra of all skew-symmetric matrices of finite rank. If $\mathfrak{g} = \{0\}$, then we define $\mathfrak{s}(\{0\}) = \mathfrak{so}(\infty)$. For every finite dimensional subalgebra $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$ with an $\mathrm{ad}(\mathfrak{k})$-invariant metric B on $\mathfrak{k}$ a new naturally reductive decomposition is obtained, which is called a $(\mathfrak{k}, B)$-extension of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$; see [16].

One can imagine that simply connected spaces of Type I are relatively easy to classify in low dimensions. This is indeed the case, and similar results with different purposes can be found in the literature, *e.g.*, [10]. The theory developed in [17] together with the construction of [16] allow us to classify all remaining naturally reductive spaces. The main reason for this is the following slightly rephrased result from [17].

Theorem 2.4 ([17, Thm. 4]) *For every naturally reductive decomposition of Type II, there exists a unique naturally reductive transvection algebra of the form*

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n, \tag{2.8}$$

where $\mathfrak{h} \oplus \mathfrak{m}$ is a semi-simple algebra and $\oplus_{L.a.}$ denotes the direct sum of Lie algebras, such that the original Type II decomposition is a $(\mathfrak{k}, B)$-extension of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$.

Definition 2.5 The unique naturally reductive decomposition from Theorem 2.4 in (2.8) is called the *canonical base space* of the Type II space. The ideal $\mathfrak{h} \oplus \mathfrak{m}$ is called the *semi-simple* factor, and the ideal $\mathbb{R}^n$ is called the *Euclidean factor*.

The above theorem is the fundamental result on which the here-presented classification method is based. It also suggests some further simplifications for the classification that are discussed below.

Definition 2.6 A Lie algebra $\mathfrak{g}$ together with a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ define a *naturally reductive pair* $(\mathfrak{g}, \mathfrak{h})$ if there exists an $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form $\overline{g}$ for which $\overline{g}|_{\mathfrak{m}\times\mathfrak{m}}$ is positive definite, where $\mathfrak{m} = \mathfrak{h}^\perp$ and such that $\mathfrak{g}$ is the transvection algebra of the corresponding naturally reductive decomposition.

Definition 2.7 A naturally reductive pair $(\mathfrak{g}^*, \mathfrak{h}^*)$ is a *partial dual* of a naturally reductive pair $(\mathfrak{g}, \mathfrak{h})$ when $\mathfrak{g}^*$ is a real form of $\mathfrak{g} \otimes \mathbb{C}$ different from $\mathfrak{g}$ and the complexified Lie algebra pairs are isomorphic: $(\mathfrak{g} \otimes \mathbb{C}, \mathfrak{h} \otimes \mathbb{C}) \cong (\mathfrak{g}^* \otimes \mathbb{C}, \mathfrak{h}^* \otimes \mathbb{C})$.

Remark 2.8 In [17] it is shown that every naturally reductive pair of Type I admits exactly one compact partial dual pair of Type I and for a space of Type II there exists exactly one partial dual pair for which the semi-simple factor of the canonical base space is compact. Moreover, a partial dual of a naturally reductive pair $(\mathfrak{g}, \mathfrak{h})$ of Type I exists if and only if $(\mathfrak{g}_i, \mathrm{proj}_{\mathfrak{g}_i}(\mathfrak{h}))$ is a symmetric pair for some i, where $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n$ is the decomposition of $\mathfrak{g}$ into simple ideals. We will use this partial duality to deflate our classification list and make it more comprehensive.

In a classification, it is common to only list the irreducible objects. This is also the case for naturally reductive spaces.

Definition 2.9 An infinitesimal model (T, R) on $\mathfrak{m}$ is *reducible* if there exists a non-trivial orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $T = T_1 \oplus T_2$ and $R = R_1 + R_2$,

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where $T_i \in \Lambda^3\mathfrak{m}_i$ and $R_i \in L(\Lambda^2\mathfrak{m}_i, \mathfrak{so}(\mathfrak{m}_i))$. A naturally reductive decomposition is *reducible* if and only if its infinitesimal model its reducible.

Remark 2.10 Given a simply connected Riemannian manifold with an Ambrose–Singer connection (M, g, ∇), its infinitesimal model is reducible in the sense of Definition 2.9 if and only if $(M, g, \nabla) \cong (M_1, g_1, \nabla_1) \times (M_2, g_2, \nabla_2)$; *i.e.*, the Riemannian manifold is a product and the Ambrose–Singer connection is a product connection.

The local reducibility of a naturally reductive space or equivalently the reducibility of its infinitesimal model only depends on the torsion.

Theorem 2.11 ([17,20]) *A naturally reductive decomposition is reducible in the sense of Definition 2.9 if and only if there exists a non-trivial orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ such that $T = T_1 \oplus T_2$ with $T_i \in \Lambda^3\mathfrak{m}_i$.*

The following propositions will simplify the classification procedure significantly; see [17] for the proofs. The first proposition gives a necessary and sufficient condition for a $(\mathfrak{k}, B)$-extension to be irreducible.

Proposition 2.12 *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$ be a naturally reductive transvection algebra with $\mathfrak{h} \oplus \mathfrak{m}$ semi-simple. Furthermore, let $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$ and let B be some* $\mathrm{ad}(\mathfrak{k})$*-invariant inner product on $\mathfrak{k}$. Consider the following decomposition*

$$\mathfrak{g} = (\mathfrak{h}_1 \oplus \mathfrak{m}_1) \oplus_{L.a.} \cdots \oplus_{L.a.} (\mathfrak{h}_p \oplus \mathfrak{m}_p) \oplus_{L.a.} \mathfrak{m}_{p+1} \oplus_{L.a.} \cdots \oplus_{L.a.} \mathfrak{m}_{p+q}, \tag{2.9}$$

where $\mathfrak{h}_i \oplus \mathfrak{m}_i$ is an irreducible naturally reductive decomposition with $\mathfrak{h}_i \subset \mathfrak{h}$ and $\mathfrak{m}_i \subset \mathfrak{m}$ for $i = 1, \dots, p$ and $\mathfrak{m}_{p+j} \subset \mathbb{R}^n$ is an irreducible $\mathfrak{k}$-module for $j = 1, \dots, q$. We choose the $\mathfrak{m}_1, \dots, \mathfrak{m}_{p+q}$ mutually orthogonal. Suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$ is the canonical base space of the $(\mathfrak{k}, B)$-extension. The $(\mathfrak{k}, B)$-extension is reducible if and only if there exists a non-trivial partition

$$\{\mathfrak{m}_1, \dots, \mathfrak{m}_p, \mathfrak{m}_{p+1}, \dots, \mathfrak{m}_{p+q}\} = W' \cup W'', \quad W' \cap W'' = \varnothing,$$

and an orthogonal decomposition of ideals $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{k}''$ with respect to B such that $\mathfrak{k}'$ acts trivially on all elements of W'' and $\mathfrak{k}''$ acts trivially on all elements of W'.

We also need to recall some definitions from [16].

Definition 2.13 Let $(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n, g)$ be a as in (2.8). Let $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$ be a Lie subalgebra and let B be an $\mathrm{ad}(\mathfrak{k})$-invariant inner product on $\mathfrak{k}$. Let $\varphi\colon \mathfrak{k} \to \mathfrak{so}(\mathfrak{m} \oplus \mathbb{R}^n)$ be the natural Lie algebra representation and let $\psi\colon \mathfrak{k} \to \mathfrak{so}(\mathfrak{k} \oplus \mathfrak{m} \oplus \mathbb{R}^n)$ be the Lie algebra representation $\psi := \mathrm{ad} \oplus \varphi$.

Then we define $\varphi_1\colon \mathfrak{k} \to \mathfrak{so}(\mathfrak{m})$ and $\varphi_2\colon \mathfrak{k} \to \mathfrak{so}(\mathbb{R}^n)$ to be the restricted representations of $\varphi\colon \mathfrak{k} \to \mathfrak{so}(\mathfrak{m} \oplus \mathbb{R}^n)$. Next we put

$$\mathfrak{k}_1 := \ker(\varphi_2), \quad \mathfrak{k}_3 := \ker(\varphi_1), \quad \mathfrak{k}_2 := (\mathfrak{k}_1 \oplus \mathfrak{k}_3)^\perp \subset \mathfrak{k},$$

where the orthogonal complement is taken with respect to B. Furthermore, we have $\mathfrak{s}(\mathfrak{g}) \cong \mathcal{Z}(\mathfrak{h}) \oplus \mathfrak{p} \oplus \mathfrak{so}(n)$, where $\mathfrak{p} := \{m \in \mathfrak{m} : [h, m] = 0,\ \forall h \in \mathfrak{h}\}$ and $\mathcal{Z}(\mathfrak{h})$ denotes the center of $\mathfrak{h}$. In this way, we identify $\mathfrak{k}_1 \oplus \mathfrak{k}_2 \subset \mathrm{Aut}(\mathfrak{h} \oplus \mathfrak{m})$ with inner derivations: $\mathfrak{d}_1 \oplus \mathfrak{d}_2 \subset \mathcal{Z}(\mathfrak{h}) \oplus \mathfrak{p} \subset \mathfrak{h} \oplus \mathfrak{m}$.

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All the spaces of Type II are constructed as $(\mathfrak{k}, B)$-extensions. Generically, a $(\mathfrak{k}, B)$-extension results in a space of Type II. However, in general it does not. The following proposition guaranties that all of the $(\mathfrak{k}, B)$-extensions we list are of Type II. This is to assure that none of the spaces we list are isomorphic.

Proposition 2.14 *Let $\mathfrak{f}$ be the transvection algebra of a $(\mathfrak{k}, B)$-extension of a naturally reductive transvection algebra of the form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$. The canonical base space of $\mathfrak{f}$ is isomorphic to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$ if and only if the following hold:*

(i) $\pi_{\mathfrak{m}}(\mathcal{Z}(\mathfrak{b}_1)) = \{0\}$;

(ii) $\ker(R|_{\mathrm{ad}(\mathfrak{h})+\psi(\mathfrak{k})}) = \{0\}$;

where $\pi_{\mathfrak{m}}$ denotes the projection onto $\mathfrak{m}$ and R is the curvature tensor associated with $\mathfrak{f}$.

For Lemma 2.14, we need to be able to compute $\ker(R)$ the following lemma simplifies this.

Lemma 2.15 *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n$ be a naturally reductive transvection algebra. Let $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$ and let B be an $\mathrm{ad}(\mathfrak{k})$-invariant inner product on $\mathfrak{k}$. Let (T, R) be the infinitesimal model of the $(\mathfrak{k}, B)$-extension. Then*

$$\mathrm{ad}(\mathfrak{h}^{ss}) \oplus \mathrm{ad}(\mathfrak{k}^{ss}) = \mathrm{ad}(\mathfrak{h}^{ss} \oplus \mathfrak{k}^{ss}) \subset \mathrm{im}(R) \quad \text{and} \quad \ker(R) \subset \mathrm{ad}(\mathcal{Z}(\mathfrak{h} \oplus \mathfrak{k})),$$

where $\mathfrak{g}^{ss}$ denotes the semi-simple commutator ideal of a Lie algebra $\mathfrak{g}$ and $\mathcal{Z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$. Moreover, if $\mathfrak{k}_1 = \{0\}$, then $\ker(R) = \{0\}$.

Finally, we need to be able to decide whether two spaces of Type II are isomorphic. The following proposition does exactly this.

Proposition 2.16 *Let $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i = \mathfrak{h}_i \oplus \mathfrak{m}_{0,i} \oplus_{L.a.} \mathbb{R}^{n_i}$ be naturally reductive transvection algebras with $\mathfrak{h}_i \oplus \mathfrak{m}_{0,i}$ semi-simple or 0-dimensional for $i = 1, 2$. Suppose $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{m}_i$ is the canonical base space of some $(\mathfrak{k}_i, B_i)$-extension for $i = 1, 2$ and that the $(\mathfrak{k}_1, B_1)$-extension and $(\mathfrak{k}_2, B_2)$-extension are isomorphic. Then there is a Lie algebra isomorphism $\tau\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Furthermore, $\tau(\mathfrak{h}_1) = \mathfrak{h}_2$, $\tau|_{\mathfrak{m}_1}\colon \mathfrak{m}_1 \to \mathfrak{m}_2$ is an isometry and $\tau_*\colon \mathfrak{k}_1 \to \mathfrak{k}_2$ is an isometry, where $\tau_*\colon \mathrm{Der}(\mathfrak{g}_1) \to \mathrm{Der}(\mathfrak{g}_2)$ is the induced map on derivations.*

The above proposition also covers Type I spaces by considering a Type I space as a trivial $(\mathfrak{k}, B)$-extension over itself. Also note that the isomorphism τ from the above proposition is necessarily an isometry with respect to the unique invariant non-degenerate symmetric bilinear form of Theorem 2.3.

3 Classification of Type I

Now we describe how to classify all naturally reductive decompositions of Type I in some dimension k. First, we list all semi-simple Lie algebras $\mathfrak{g}$ of dimension less or equal to $\frac{1}{2}k(k+1)$. For all of these, we look for subalgebras $\mathfrak{h} \subset \mathfrak{g}$ together with all $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear forms $\overline{g}$ on $\mathfrak{g}$ such that the following hold:

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k	3	4	5	6	7	8
$\mathfrak{h}$	$\mathfrak{so}(3)$	n/a	$\mathfrak{u}(2)$	$\mathfrak{su}(3)$	$\mathfrak{g}_2$	$\mathfrak{su}(3)$
$D_k = \dim(\mathfrak{h})$	3	n/a	4	8	14	8

Table 1: Stabilizers of some irreducible 3-form of the largest dimension possible.

(1) $\dim(\mathfrak{g}/\mathfrak{h}) = k$;
(2) $\overline{g}|_{\mathfrak{m}\times\mathfrak{m}}$ is positive definite, where $\mathfrak{m} = \mathfrak{h}^{\perp}$,
(3) the torsion T from (2.7) is irreducible;
(4) $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} = \mathfrak{h}$.

We will refer to these as conditions (1) to (4), as we will use them regularly. Condition (3) implies that the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ will be irreducible; see Theorem 2.11, Condition (4) implies that $\mathfrak{g}$ is the transvection algebra; see [17, Lem. 8]. This produces all irreducible naturally reductive transvection algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of Type I, and thus, after finding all isomorphic ones, we obtain a classification of all naturally reductive transvection algebras of Type I in dimension k.

Remark 3.1 The naturally reductive structures on globally homogeneous spaces are the ones that are regular. To obtain all regular structures, we only have to investigate when H is closed in G, where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and H is the connected subgroup with Lie subalgebra $\mathfrak{h}$; see [12, 18]. For all the cases we discuss, we mention when the naturally reductive structure at hand is regular.

The above approach is very crude and becomes quite a lot of work even in dimensions 7 and 8. We can make our method more efficient by first looking for an upper bound for the dimension of $\mathfrak{h}$. We used above that $\mathfrak{h}$ is always a subalgebra of $\mathfrak{so}(k)$, and thus $\dim(\mathfrak{h}) \leq \frac{1}{2}k(k-1)$. However, since $\mathfrak{h}$ is the stabilizer of an irreducible 3-form $T \in \Lambda^3\mathfrak{m}$, we can improve this estimate. In Table 1 we list stabilizers of irreducible 3-forms in $\mathbb{R}^k$ for $3 \leq k \leq 8$ that are of the largest dimension possible. The dimension of this stabilizer is denoted by D_k. Additionally, we can also look for the stabilizer with the second largest dimension d_k. In dimension 7, we find this is $\mathfrak{u}(3)$, which has dimension 9. In dimension 8, we find it has at most dimension 5. Now we can follow the same approach as before, but only the semi-simple Lie algebras up to dimension $k + D_k$ have to be listed, and we also do not have to list semi-simple Lie algebras $\mathfrak{g}$ with $k + d_k \leq \dim(\mathfrak{g}) \leq k + D_k$. This is already a big improvement compared to the initial approach.

The next step is to find all subalgebras of these semi-simple Lie algebras, such that the conditions (1)–(4) are satisfied. We do this for every semi-simple Lie algebra by listing all reductive algebras $\mathfrak{h}$ that satisfy $\dim(\mathfrak{h}) = \dim(\mathfrak{g}) - k$ and $\operatorname{rank}(\mathfrak{h}) \leq \min\{\operatorname{rank}(\mathfrak{g}), \operatorname{rank}(\mathfrak{so}(k))\}$. Once we have listed all such pairs $(\mathfrak{g}, \mathfrak{h})$, we have to find all possible injective Lie algebra homomorphisms $\mathfrak{h} \to \mathfrak{g}$ up to conjugation by an automorphism of $\mathfrak{g}$, such that the conditions (3) and (4) are satisfied for some non-degenerate symmetric bilinear form $\overline{g}$. For condition (3), it is often easier to check the condition in the following lemma; see [17, Lem. 5] for a proof.

Lemma 3.2 *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a naturally reductive transvection algebra. Let $\overline{g}$ be the unique $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form from Kostant's theorem; see Theorem 2.3. The reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is reducible if and only if there exist two non-trivial orthogonal ideals $\mathfrak{g}_1 \subset \mathfrak{g}$ and $\mathfrak{g}_2 \subset \mathfrak{g}$ with respect to $\overline{g}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_i \subset \mathfrak{g}_i$ for $i = 1, 2$.*

The following lemma is useful when listing all conjugacy classes of subalgebras of $\mathfrak{so}(n)$ and $\mathfrak{su}(n)$ in small dimensions.

Lemma 3.3 *Let $\mathfrak{g} = \mathfrak{so}(n)$ or $\mathfrak{g} = \mathfrak{su}(n)$. Let $\pi\colon \mathfrak{g} \to \mathrm{End}(K^n)$ be the vector representation, with $K = \mathbb{R}^n$, $\mathbb{C}^n$. Let $f_i\colon \mathfrak{h} \to \mathfrak{g}$ be an injective Lie algebra homomorphism for $i = 1, 2$. We denote the image of f_i by $\mathfrak{h}_i := f_i(\mathfrak{h})$. If the representations $\pi \circ f_1$ and $\pi \circ f_2$ are equivalent, then the subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are conjugate by an automorphism of $\mathfrak{g}$.*

Note that the above lemma implies the naturally reductive pairs defined by $(\mathfrak{g}, \mathfrak{h}_1)$ and $(\mathfrak{g}, \mathfrak{h}_2)$ are isomorphic if and only if the representations representations of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are equivalent. The last step is to find all $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear forms on $\mathfrak{g}$ such that condition 2 is satisfied.

Let us briefly illustrate how one can obtain Table 1 by explaining it in dimension 8. The largest dimensional stabilizer will be a proper subalgebra of $\mathfrak{so}(8)$ of dimension greater than or equal to 8, since the adjoint representation of $\mathfrak{su}(3)$ stabilizes the irreducible 3-form defined by $T(x, y, z) := B_{\mathfrak{su}(3)}([x, y], z)$, where $B_{\mathfrak{su}(3)}$ is the Killing form of $\mathfrak{su}(3)$. Any stabilizer is a reductive Lie algebra and its commutator ideal is equal to one of the following semi-simple Lie subalgebras of $\mathfrak{so}(8)$:

$$\mathfrak{su}(2), \quad \mathfrak{su}(2)^2, \quad \mathfrak{su}(3), \quad \mathfrak{su}(2)^3, \quad \mathfrak{sp}(2),$$
$$\mathfrak{so}(4) \oplus \mathfrak{so}(4), \quad \mathfrak{sp}(2) \oplus \mathfrak{sp}(1), \quad \mathfrak{g}_2, \quad \mathfrak{su}(4), \quad \mathfrak{so}(7).$$

The only Lie algebras $\mathfrak{h}$ with semi-simple part $\mathfrak{su}(2)$ and $\mathrm{rank}(\mathfrak{h}) \leq \mathrm{rank}(\mathfrak{so}(8)) = 4$ are $\mathfrak{h} = \mathfrak{su}(2) \oplus \mathbb{R}^i$ for $i = 1, 2, 3$. These are of dimension less than or equal to 6. Hence to find the stabilizer with the largest dimension, we can forget about these cases. For the other Lie algebras we can list all faithful 8-dimensional real representations and check if there exists an irreducible invariant 3-form. Then one needs to check if the action can be extended to a larger Lie algebra and see if the 3-form is still stabilized by this larger Lie algebra.

The following lemma will exclude many Lie subalgebras $\mathfrak{h} \subset \mathfrak{so}(k)$ from having an invariant irreducible 3-form.

Lemma 3.4 *Suppose that $\mathfrak{so}(l) \subset \mathfrak{h} \subset \mathfrak{so}(k)$, where the inclusion $\mathfrak{so}(l) \subset \mathfrak{so}(k)$ is the standard block embedding and $l \geq 3$. Then there is no $\mathfrak{h}$-invariant irreducible 3-form $T \in \Lambda^3\mathbb{R}^k$.*

Proof We show that there is no irreducible 3-form invariant under $\mathfrak{so}(l)$, and this implies that there is no invariant irreducible 3-form under the $\mathfrak{h}$-action. As an $\mathfrak{so}(\mathfrak{l})$-module $\mathbb{R}^k$ splits into two orthogonal submodules, $\mathbb{R}^k = \mathbb{R}^l \oplus \mathbb{R}^{k-l}$. This implies

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that

$$T \in \Lambda^3\mathbb{R}^l \oplus \Lambda^2\mathbb{R}^l \otimes \mathbb{R}^{k-l} \oplus \mathbb{R}^l \otimes \Lambda^2\mathbb{R}^{k-l} \oplus \Lambda^3\mathbb{R}^{k-l},$$

and all direct sums are preserved by $\mathfrak{so}(l)$. Let T_2 denote the component of T in $\Lambda^2\mathbb{R}^l \otimes \mathbb{R}^{k-l}$. We can identify T_2 with an $\mathfrak{so}(l)$-equivariant map $T_2\colon \Lambda^2\mathbb{R}^l \to \mathbb{R}^{k-l}$. Since $\mathfrak{so}(l)$ acts trivially on $\mathbb{R}^{k-l}$ and has no fixed 2-forms, because $\Lambda^2\mathbb{R}^l \cong \mathfrak{so}(l)$ is the adjoint representation. We conclude by Schur's lemma that $T_2 = 0$. By a similar argument the component of T in $\mathbb{R}^l \otimes \Lambda^2\mathbb{R}^{k-l}$ vanishes. We conclude $T \in \Lambda^3\mathbb{R}^l \oplus \Lambda^3\mathbb{R}^{k-l}$, and thus T is reducible. ∎

Note that $\mathfrak{su}(2)^2$ is a subalgebra of each of the following Lie algebras

$$\mathfrak{su}(2)^3, \quad \mathfrak{sp}(2), \quad \mathfrak{so}(4) \oplus \mathfrak{so}(4), \quad \mathfrak{sp}(2) \oplus \mathfrak{sp}(1), \quad \mathfrak{g}_2, \quad \mathfrak{su}(4), \quad \mathfrak{so}(7).$$

Therefore, if there is no representation of $\mathfrak{su}(2)^2$ that stabilizes an irreducible 3-form, then there is also no representation of any of these Lie algebras that stabilizes and irreducible 3-form. In the following, we will denote a highest weight representations of a semi-simple Lie algebra $\mathfrak{g}$ with highest weight $n_1\lambda_1 + \cdots + n_p\lambda_p$ as $R(n_1, \ldots, n_p)$, where $\lambda_1, \ldots, \lambda_p$ are the fundamental weights of $\mathfrak{g}$ in the Bourbaki labeling. The real irreducible representations correspond to a subset of the complex irreducible representations; see, for example, [4]. All complexifications of 8-dimension faithful real representations of $\mathfrak{su}(2)^2$ are

$$\begin{aligned} &2R(1,0) \oplus 2R(0,1), && 2R(1,0) \oplus R(0,2) \oplus R(0,0), && R(1,1) \oplus 4R(0,0),\\ &R(1,1) \oplus 2R(0,1), && R(1,1) \oplus R(0,2) \oplus R(0,0), && R(1,1) \oplus R(1,1),\\ &R(4,0) \oplus R(0,2), && R(2,0) \oplus R(0,2) \oplus 2R(0,0). \end{aligned}$$

For example, the real representation underlying $2R(1,0) \oplus 2R(0,1)$ is $\mathbb{R}^4 \oplus \mathbb{R}^4$ such that the first $\mathfrak{su}(2)$-summand only acts on the first $\mathbb{R}^4$-summand, where $\mathbb{R}^4$ denotes the representation $\mathbb{C}^2$ by restricting scalars to $\mathbb{R}$ and similarly for the second summand. For the representations $2R(1,0) \oplus R(0,2) \oplus R(0,0)$, $R(1,1) \oplus R(0,2) \oplus R(0,0)$, $R(2,0) \oplus R(0,2) \oplus 2R(0,0)$, $R(1,1) \oplus 4R(0,0)$, and $R(4,0) \oplus R(0,2)$, we can apply Lemma 3.4 to see that there is no invariant irreducible 3-form. For the other three representations it follows that there are no irreducible invariant 3-forms by a similar argument as that in Lemma 3.4.

We conclude that the stabilizer of some irreducible 3-form of the largest dimension possible has $\mathfrak{su}(3)$ as its commutator ideal. The representation $R(1,1)$ is the complexified adjoint representation of $\mathfrak{su}(3)$, and it is of real type. Hence, the endomorphism ring is trivial, and $\mathfrak{su}(3)$ is the stabilizer of an irreducible 3-form with the largest dimension. We also see from the table that the stabilizer of an irreducible 3-form of the second largest dimension has $\mathfrak{su}(2)$ as its semi-simple part. Let us consider the algebra $\mathfrak{su}(2) \oplus \mathbb{R}^3 \cong \mathfrak{u}(2) \oplus \mathbb{R}^2$. There is only one faithful Lie algebra representation of this algebra on $\mathbb{R}^8$, namely: $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$, where $\mathbb{R}^4 \cong \mathbb{C}^2$ is the vector representation of $\mathfrak{u}(2)$ and each $\mathbb{R}^2$-summand is an irreducible $\mathbb{R}$-representation. Again by a similar argument as in Lemma 3.4, we conclude there is no irreducible invariant 3-form for this representation. Thus, the biggest dimension of a stabilizer of an irreducible 3-form of dimension less than 8 has dimension less than or equal to 5. So for

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$\mathfrak{h}$	$R_{\mathbb{C}}$	inv. irred. 3-form
$\mathfrak{su}(3)$	$2R(1,0) \oplus R(0,0)$	✓
$\mathfrak{su}(2)^3$	$R(1,1,0) \oplus R(0,0,2)$	✗
$\mathfrak{so}(5)$	$R(1,0) \oplus 2R(0,0)$	✗
$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$	$\varnothing$	n/a
$\mathfrak{su}(2)^4$	$\varnothing$	n/a
$\mathfrak{so}(5) \oplus \mathfrak{su}(2)$	$\varnothing$	n/a
$\mathfrak{g}_2$	$R(1,0)$	✓
$\mathfrak{su}(3) \oplus \mathfrak{su}(2)^2$	$\varnothing$	n/a

Table 2: 7-dimensional representations with irreducible 3-forms.

the case $k = 8$, we only have to list all semi-simple Lie algebras $\mathfrak{g}$ with $\dim(\mathfrak{g}) \leq 13$ and add those of dimension 16.

For $k = 7$ there is a stabilizer of an irreducible 3-form with a relatively large dimension, namely G_2. There is only one naturally reductive decomposition that has $\mathfrak{g}_2$ as isotropy algebra, the decomposition of $\mathrm{Spin}(7)/G_2$. In Table 2 we list all semi-simple Lie algebras with their dimension between 8 and 14 together with all of their 7-dimensional faithful representations. In the third column we indicated if the representation admits an invariant irreducible 3-form. Lemma 3.4 implies that there does not exists an invariant irreducible 3-form for the representations of $\mathfrak{su}(2)^3$ and $\mathfrak{so}(5)$. The endomorphism ring of the $\mathfrak{su}(3)$-representation $2R(1,0) \oplus R(0,0)$ is 1-dimensional. Consequently, the stabilizer of an irreducible 3-form in dimension 7 with the second largest dimension is $\mathfrak{u}(3)$. For a particular choice of basis in $\mathbb{R}^7$ the $\mathfrak{u}(3)$-invariant torsion forms are spanned by $e_7 \wedge (e_{12} + e_{34} + 2e_{56})$, where e_{ij} denotes $e_i \wedge e_j$. Thus, for $k = 7$ we only have to list all semi-simple Lie algebras $\mathfrak{g}$ with $\dim(\mathfrak{g}) \leq 16$ and add to this the pair $(\mathfrak{so}(7), \mathfrak{g}_2)$.

3.1 Classification of Type I in Dimension 7

Now we follow the classification approach described above in dimension 7. In the second column of Table 3, all compact semi-simple Lie algebras $\mathfrak{g}$ of dimension $7 \leq k \leq 16$ are listed and to this the case $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(7), \mathfrak{g}_2)$ is added. In the third column, all Lie algebras $\mathfrak{h}$ of dimension $\dim(\mathfrak{g}) - 7$ and with $\mathrm{rank}(\mathfrak{h}) \leq \min\{\mathrm{rank}(\mathfrak{g}), \mathrm{rank}(\mathfrak{so}(7))\}$ are listed. The following result will exclude many cases from satisfying condition 3.

Lemma 3.5 *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$, with $\mathfrak{g}_i$ simple for $i = 1, \dots, k$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra with a naturally reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to some $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is irreducible, then*

$$\mathrm{rank}\,\mathfrak{g} \geq \mathrm{rank}\,\mathfrak{h} + k - 1.$$

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$\dim(\mathfrak{g})$	$\mathfrak{g}$	$\mathfrak{h}$
8	$\mathfrak{su}(3)$	$\mathbb{R}$
9	$\mathfrak{su}(2)^3$	$\mathbb{R}^2$
10	$\mathfrak{so}(5)$	$\mathfrak{su}(2)$
11	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathbb{R}$
12	$\mathfrak{su}(2)^4$	$\mathfrak{su}(2) \oplus \mathbb{R}^2$
13	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
14	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)^2$	$\mathfrak{su}(2)^2 \oplus \mathbb{R}$
14	$\mathfrak{g}_2$	$\varnothing$
15	$\mathfrak{su}(2)^5$	$\mathfrak{su}(3)$
15	$\mathfrak{su}(4)$	$\mathfrak{su}(3)$
16	$\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2$	$\mathfrak{su}(3) \oplus \mathbb{R}$, $\mathfrak{su}(2)^3$
16	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathbb{R}$, $\mathfrak{su}(2)^3$
21	$\mathfrak{so}(7)$	$\mathfrak{g}_2$

Table 3: Candidates for 7-dimensional Type I naturally reductive pairs.

Proof For $k = 1$ the statement is true. Suppose that it is true for a certain $k \in \mathbb{N}$. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \mathfrak{g}_{k+1}$ and let us denote $\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Let $\pi_1 \colon \mathfrak{g} \to \mathfrak{g}'$ and $\pi_2 \colon \mathfrak{g} \to \mathfrak{g}_{k+1}$ be the projections. Let $\mathfrak{h}_1 := \ker(\pi_2)$, $\mathfrak{h}_3 := \ker(\pi_1)$ and $\mathfrak{h}_2 \subset \mathfrak{h}$ a complementary ideal of $\mathfrak{h}_1 \oplus \mathfrak{h}_3$, which exists because $\mathfrak{h}$ is a reductive Lie algebra. Note that $\operatorname{rank} \mathfrak{h}_2 \geq 1$, because otherwise the decomposition is reducible by Lemma 3.2. By our induction hypothesis, we have

$$\operatorname{rank} \mathfrak{g}' \geq \operatorname{rank} \mathfrak{h}_1 \oplus \mathfrak{h}_2 + k - 1.$$

Furthermore, we have $\operatorname{rank} \mathfrak{g}_{k+1} \geq \operatorname{rank} \mathfrak{h}_2 + \operatorname{rank} \mathfrak{h}_3$. Combining these yields

$$\operatorname{rank} \mathfrak{g} \geq \operatorname{rank} \mathfrak{h}_1 + \operatorname{rank} \mathfrak{h}_2 + k - 1 + \operatorname{rank} \mathfrak{h}_2 + \operatorname{rank} \mathfrak{h}_3 \geq \operatorname{rank} \mathfrak{h} + k. \qquad \blacksquare$$

Now that we have all candidates for the pairs $(\mathfrak{g}, \mathfrak{h})$, it remains to find all possible conjugacy classes of injective Lie algebra homomorphisms $\mathfrak{h} \to \mathfrak{g}$ such that conditions (3) and (4) from the beginning of this section are satisfied. The pairs $(\mathfrak{g}, \mathfrak{h})$, which are excluded by Lemma 3.5, are

$$(\mathfrak{su}(2)^4, \mathfrak{su}(2) \oplus \mathbb{R}^2), \quad (\mathfrak{su}(3) \oplus \mathfrak{su}(2)^2, \mathfrak{su}(2)^2 \oplus \mathbb{R}), \quad (\mathfrak{su}(2)^5, \mathfrak{su}(3))$$
$$(\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2, \mathfrak{su}(3) \oplus \mathbb{R}), \quad (\mathfrak{so}(5) \oplus \mathfrak{su}(2)^2, \mathfrak{su}(2)^3).$$

For the pair $(\mathfrak{su}(3) \oplus \mathfrak{su}(3), \mathfrak{su}(2)^3)$ there does not exist an injective Lie algebra homomorphism from $\mathfrak{h}$ to $\mathfrak{g}$. It is easily seen that no injective Lie algebra homomorphism $\mathfrak{su}(3) \oplus \mathbb{R} \to \mathfrak{su}(3) \oplus \mathfrak{su}(3)$ satisfies condition (3) or (4). The remaining pairs are

$$(\mathfrak{su}(3),\mathbb{R}),\quad (\mathfrak{su}(2)^3,\mathbb{R}^2),\quad (\mathfrak{so}(5),\mathfrak{su}(2)),\quad (\mathfrak{su}(3)\oplus\mathfrak{su}(2),\mathfrak{su}(2)\oplus\mathbb{R}),$$
$$(\mathfrak{so}(5)\oplus\mathfrak{su}(2),\mathfrak{su}(2)\oplus\mathfrak{su}(2)),\quad (\mathfrak{su}(4),\mathfrak{su}(3)),\quad (\mathfrak{so}(7),\mathfrak{g}_2).$$

For these remaining cases, we now describe explicitly all subalgebras $\mathfrak{h}$ together with all the possible non-degenerate symmetric bilinear forms.

Case $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{su}(3),\mathbb{R})$. Every subalgebra $\mathbb{R}\subset\mathfrak{su}(3)$ is conjugate to one spanned by

$$r(a,b) := \begin{pmatrix} ia & 0 & 0\\ 0 & ib & 0\\ 0 & 0 & -i(a+b)\end{pmatrix},$$

with $a,b\in\mathbb{R}$ and not both equal to zero. By Lemma 3.3, two pairs (a,b) and (c,d) will give an isomorphic infinitesimal model exactly when their subalgebras are conjugate by an element $A\in\operatorname{Aut}(\mathfrak{su}(3))$. If A is an inner automorphism, then $A(r(a,b))$ has the same eigenvalues as $r(a,b)$. Therefore, $A(r(a,b)) = \hat{A}(r(a,b))$ for some signed permutation matrix in $\hat{A}\in SU(3)$. An outer automorphism $\tau\colon\mathfrak{su}(3)\to\mathfrak{su}(3)$ is given by taking the negative transpose in $\mathfrak{su}(3)$. We have $\tau(r(a,b)) = r(-a,-b)$. The outer automorphism group of $\mathfrak{su}(3)$ is $\mathbb{Z}_2$. We can now see that all pairs (x,y) for which $\operatorname{span}\{r(x,y)\}$ is conjugate to $\operatorname{span}\{r(a,b)\}$ by an automorphism of $\mathfrak{su}(3)$ are

$$\pm(a,b),\ \pm(a,-a-b),\ \pm(b,a),\ \pm(b,-a-b),\ \pm(-a-b,a),\ \pm(-a-b,b).$$

By using these automorphisms, we can always arrange that $a\geq b>0$. Thus the isomorphism classes are precisely described by $\frac{a}{b}\geq 1$. The connected subgroup with this Lie algebra is closed if and only if $\frac{a}{b}=q\in\mathbb{Q}$. The homogeneous spaces are $SU(3)/S^1_q$, where S^1_q is the image of

$$S^1\longrightarrow SU(3);\quad \theta\longmapsto\begin{pmatrix} e^{iq\theta} & 0 & 0\\ 0 & e^{i\theta} & 0\\ 0 & 0 & e^{-i\theta(1+q)}\end{pmatrix}.$$

The $\operatorname{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form $\overline{g}$ on $\mathfrak{g}$ is induced from the Killing form of $\mathfrak{su}(3)$; hence, for every case there is a 1-parameter family of naturally reductive metrics.

Case $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{su}(2)^3,\mathbb{R}^2)$. Let x_1,x_2,x_3 be the following basis of $\mathfrak{su}(2)$:

$$x_1 := \begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad x_2 := \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\quad x_3 := \begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}. \tag{3.1}$$

The $\operatorname{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form on $\mathfrak{su}(2)^3$ is necessarily positive definite and given by $\overline{g} = \frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)}\oplus\frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}\oplus\frac{-1}{8\lambda_3^2}B_{\mathfrak{su}(2)}$. Without loss of generality, we assume that $0<\lambda_1\leq\lambda_2\leq\lambda_3$. If the naturally reductive decomposition is irreducible, then $\mathfrak{h}$ is conjugate by an automorphism of $\mathfrak{su}(2)^3$ to a subalgebra spanned by

$$h_1 := (a_1x_1, a_2x_1, 0)\quad\text{and}\quad h_2 := (0, b_1x_1, b_2x_1),$$

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with $a_1, a_2, b_1, b_2 > 0$. If $\lambda_1 = \lambda_2 < \lambda_3$, then $\mathfrak{h}$ is conjugate to one with $a_1 \le a_2$. Similarly, if $\lambda_1 < \lambda_2 = \lambda_3$, then we can arrange that $b_1 \le b_2$. Lastly, if $\lambda_1 = \lambda_2 = \lambda_3$, then we can arrange that $a_1 \le a_2$ and $b_1 \le b_2$. Under these conditions, every irreducible naturally reductive space is represented exactly once. The connected subgroup H of $SU(2)^3$ with $\mathrm{Lie}(H) = \mathfrak{h}$ is a closed subgroup precisely when $\frac{a_2}{a_1} = q_1 \in \mathbb{Q}$ and $\frac{b_2}{b_1} = q_2 \in \mathbb{Q}$. If H is closed, then it is isomorphic to $S^1 \times S^1$. We obtain a 3-parameter family of naturally reductive structures on $SU(2)^3/(S^1_{q_1} \times S^1_{q_2})$, where the parameters are $\lambda_1, \lambda_2, \lambda_3 > 0$ and $\mathrm{Lie}(S^1_{q_1} \times S^1_{q_2}) = \mathfrak{h}$. Note that $(\mathfrak{su}(2), \mathbb{R})$ is a symmetric pair with $(\mathfrak{sl}(2,\mathbb{R}), \mathbb{R})$ its dual symmetric pair. We obtain the partial dual spaces by replacing one or two of the $\mathfrak{su}(2)$-factors by $\mathfrak{sl}(2,\mathbb{R})$. If we replace the first factor, then $\overline{g}|_{\mathfrak{m}\times\mathfrak{m}}$ is positive definite if and only if $\frac{-a_1^2}{\lambda_1^2} + \frac{a_2^2}{\lambda_2^2} < 0$ and when we replace the last factor then $\overline{g}|_{\mathfrak{m}\times\mathfrak{m}}$ is positive definite if and only if $\frac{b_1^2}{\lambda_2^2} - \frac{b_3^2}{\lambda_3^2} < 0$. If we replace the middle factor, then the condition becomes

$$-\frac{\lambda_2^1}{\lambda_1^2} a_1^2 b_1^2 - \frac{\lambda_2^2}{\lambda_3^2} b_2^2 a_2^2 + \frac{\lambda_2^4}{\lambda_1^2 \lambda_3^2} a_1^2 b_2^2 < 0.$$

We get similar conditions if two out of the three factors are non-compact.

Case $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(5), \mathfrak{su}(2))$. For this pair, there are three nonequivalent faithful 5-dimensional real representations of $\mathfrak{su}(2)$. They correspond to the representations $R(2) \oplus 2R(0)$, $2R(1) \oplus R(0)$, $R(4)$, where each summand corresponds to a real irreducible representation. This gives us the following simply connected spaces:

$$SO(5)/SO(3)_{\mathrm{ir}}, \quad SO(5)/SO(3)_{\mathrm{st}}, \quad Sp(2)/Sp(1)_{\mathrm{st}},$$

where $SO(3)_{\mathrm{ir}}$ denotes the subgroup given by the 5-dimensional irreducible representation of $SO(3)$; $SO(3)_{\mathrm{st}}$ is the standard $SO(3)$ subgroup of $SO(5)$, and $Sp(1)_{\mathrm{st}} \subset Sp(2)$ is the standard $Sp(1)$ subgroup. The first space corresponds to the representation $R(4)$, the second space to $R(2) \oplus 2R(0)$ and the last space to $2R(1) \oplus R(0)$. In particular all the possible infinitesimal models for the pair $(\mathfrak{so}(5), \mathfrak{su}(2))$ are regular. The metric is induced from the Killing form on $\mathfrak{so}(5)$, and thus for each case, we get a 1-parameter family of naturally reductive metrics. We can easily see that these three naturally reductive spaces are not isomorphic, because they have pairwise different isotropy representations, and the isotropy representations are the same as the holonomy representations of the canonical connections.

Case $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{su}(3) \oplus \mathfrak{su}(2), \mathfrak{su}(2) \oplus \mathbb{R})$. Let $f\colon \mathfrak{h} \to \mathfrak{g}$ be an injective Lie algebra homomorphism. If $f(\mathfrak{su}(2)) \subset \mathfrak{su}(2)$, then $f(\mathbb{R}) \subset \mathfrak{su}(3)$, since $f(\mathfrak{su}(2))$ and $f(\mathbb{R})$ commute. Now conditions (3) and (4) from the beginning of this section are not satisfied. There are, up to conjugation, only two injective Lie algebra homomorphism from $\mathfrak{su}(2)$ to $\mathfrak{su}(3)$ associated with the irreducible representations on $\mathbb{C}^2$ and $\mathbb{C}^3$. The irreducible representation on $\mathbb{C}^3$ defines the irreducible symmetric pair $(\mathfrak{su}(3), \mathfrak{so}(3))$. This implies that $f(\mathbb{R}) \subset \mathfrak{su}(2)$ and thus results in a reducible space; see Lemma 3.2. In other words, condition (3) is not satisfied. Hence, the inclusion of $\mathfrak{su}(2)$ in $\mathfrak{su}(3)$ can only be the standard inclusion. We obtain the following subalgebras:

$$\mathfrak{su}(2)_{\mathrm{st}} \oplus \mathbb{R}_{a,b} \subset \mathfrak{su}(3) \oplus \mathfrak{su}(2) \quad \text{and} \quad \mathfrak{su}(2)_{\Delta} \oplus \mathbb{R} \subset \mathfrak{su}(3) \oplus \mathfrak{su}(2).$$

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In the first inclusion, $\mathfrak{su}(2)_{\mathrm{st}} = i_{\mathrm{st}}(\mathfrak{su}(2))$ with $i_{\mathrm{st}}\colon \mathfrak{su}(2) \to \mathfrak{su}(3)$ the standard inclusion, and $\mathbb{R}_{a,b}$ is the subalgebra spanned by

$$\left(\begin{pmatrix} ia & 0 & 0 \\ 0 & ia & 0 \\ 0 & 0 & -2ia \end{pmatrix}, \begin{pmatrix} ib & 0 \\ 0 & -ib \end{pmatrix} \right).$$

By Lemma 3.2, this naturally reductive decomposition is irreducible if and only if a and b are non-zero. In this case, the connected subgroup of $SU(3) \times SU(2)$ with Lie algebra $\mathfrak{su}(2)_{\mathrm{st}} \oplus \mathbb{R}_{a,b}$ is closed exactly when $\frac{a}{b} = q \in \mathbb{Q}$. Hence, the infinitesimal model is regular if and only if $\frac{a}{b} \in \mathbb{Q}$. This subalgebra is conjugate by an automorphism of $\mathfrak{su}(3) \oplus \mathfrak{su}(2)$ to one with $a, b > 0$. The $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form is given by $\overline{g} = \frac{-\lambda_1}{12} B_{\mathfrak{su}(3)} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}$. For this case, $\overline{g}$ has to be positive definite, *i.e.*, $\lambda_1, \lambda_2 > 0$. We obtain a 2-parameter family of naturally reductive structures on $(SU(3) \times SU(2))/(SU(2)_{\mathrm{st}} \times S^1_q)$, where $\mathrm{Lie}(S^1_q) = \mathbb{R}_{a,b}$.

The subalgebra $\mathfrak{su}(2)_\Delta \oplus \mathbb{R}$ is defined by $\mathfrak{su}(2)_\Delta := (i_{\mathrm{st}} \oplus \mathrm{id})(\mathfrak{su}(2))$ and $\mathbb{R}$ is spanned by

$$\left(\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

The corresponding naturally reductive decomposition is irreducible and regular. The $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form is the same as in the previous case. In this case, the space can be normal homogeneous or not. The normal homogeneous metrics correspond to $\lambda_1, \lambda_2 > 0$. For the non-normal homogeneous case, we have $\lambda_1 > 0$, $\lambda_2 < 0$ and $\lambda_1 + \lambda_2 < 0$. We obtain a 2-parameter family of naturally reductive structures on $(SU(3) \times SU(2))/(SU(2)_\Delta \times S^1)$. This space, together with one of the normal homogeneous metrics, has positive sectional curvature and is known as a Wilking's space, see [21].

Note that for both cases $(\mathfrak{su}(3), f(\mathfrak{su}(2) \oplus \mathbb{R}))$ is a symmetric pair. Therefore, by Remark 2.8, we see that both spaces have non-compact partial duals. For a non-compact partial dual the $\mathrm{ad}(\mathfrak{g}^*)$-invariant non-degenerate symmetric bilinear form is given by $\overline{g}^* = \frac{\lambda_1}{12} B_{\mathfrak{su}(2,1)} \oplus \frac{-\lambda_2}{8} B_{\mathfrak{su}(2)}$. For the first space, $\overline{g}^*|_{\mathfrak{h}\times\mathfrak{h}}$ is negative definite precisely when $-3a^2\lambda_1 + b^2\lambda_2 < 0$. For the second space, $\overline{g}^*|_{\mathfrak{h}\times\mathfrak{h}}$ is negative definite if and only if $\lambda_1, \lambda_2 > 0$ and $-\lambda_1 + \lambda_2 < 0$. For the first space, $(\mathfrak{su}(2), \mathrm{proj}_{\mathfrak{su}(2)}(\mathfrak{h}))$ is also a symmetric pair. If we replace this pair with its symmetric dual, we obtain a naturally reductive structure on

$$\big(SU(3) \times SL(2, \mathbb{R})\big)/(SU(2) \times S^1_q).$$

The $\mathrm{ad}(\mathfrak{g}^*)$-invariant non-degenerate symmetric bilinear form is $\overline{g}^* = \frac{-\lambda_1}{12} B_{\mathfrak{su}(3)} \oplus \frac{\lambda_2}{8} B_{\mathfrak{sl}(2,\mathbb{R})}$. We have that $\overline{g}^*|_{\mathfrak{m}\times\mathfrak{m}}$ is positive definite if and only if $\lambda_1, \lambda_2 > 0$ and $3a^2\lambda_1 - b^2\lambda_2 < 0$. Suppose we replace both factors by their non-compact dual. The invariant symmetric bilinear form is $\frac{\lambda_1}{12} B_{\mathfrak{su}(2,1)} \oplus \frac{\lambda_2}{8} B_{\mathfrak{sl}(2,\mathbb{R})}$ with $\lambda_1, \lambda_2 > 0$. This has signature $(6, 5)$, and thus $\overline{g}|_{\mathfrak{m}\times\mathfrak{m}}$ is never positive definite, which is not allowed.

Case $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(5) \oplus \mathfrak{su}(2), \mathfrak{su}(2) \oplus \mathfrak{su}(2))$. In order for condition (3) to be satisfied, we see that both $\mathfrak{su}(2)$ factors of $\mathfrak{h}$ need to have a non-zero image in $\mathfrak{so}(5)$.

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There is only one 5-dimensional orthogonal faithful representation of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\cong\mathfrak{so}(4)$, and this corresponds to the standard inclusion of $\mathfrak{so}(4)$ in $\mathfrak{so}(5)$. We will denote the image of the $\mathfrak{su}(2)$-summand, which has non-zero image in both $\mathfrak{so}(5)$ and $\mathfrak{su}(2)$ by $\mathfrak{su}(2)_\Delta$. The associated infinitesimal model is always regular, and this gives us the following naturally reductive space:

$$\big(\mathrm{Spin}(5)\times SU(2)\big)/\big(SU(2)_\Delta\times SU(2)\big).$$

On this homogeneous space, we have a 2-parameter family of $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear forms: $\overline{g}:=\frac{-\lambda_1}{6}B_{\mathfrak{so}(5)}\oplus\frac{-\lambda_2}{8}B_{\mathfrak{su}(2)}$. The normal homogeneous spaces correspond to the parameter $\lambda_1,\lambda_2>0$. The non-normal homogeneous spaces correspond to $\lambda_1>0$, $\lambda_2<0$, and $2\lambda_1+\lambda_2<0$. The inequality ensures that $\overline{g}|_{\mathfrak{su}(2)_\Delta}$ is negative definite, and thus $\overline{g}|_{\mathfrak{m}\times\mathfrak{m}}$ is positive definite, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{su}(2)_\Delta\oplus\mathfrak{su}(2)$ in $\mathfrak{spin}(5)\oplus\mathfrak{su}(2)$ with respect to $\overline{g}$. This space is known as the squashed 7-sphere. This is one of the homogeneous spaces for which there exists a proper nearly parallel G_2-structure; see [7].

Note that $(\mathfrak{so}(5),f(\mathfrak{su}(2)\oplus\mathfrak{su}(2))$ is a symmetric pair. From Remark 2.8, we see that there exists a non-compact partial dual. For the non-compact partial dual the $\mathrm{ad}(\mathfrak{g}^*)$-invariant non-degenerate symmetric bilinear form is given by $\overline{g}^*=\frac{\lambda_1}{6}B_{\mathfrak{so}(4,1)}\oplus\frac{-\lambda_2}{8}B_{\mathfrak{su}(2)}$. The parameters λ_1 and λ_2 have to satisfy $\lambda_1,\lambda_2>0$ and $-2\lambda_1+\lambda_2<0$ for the metric $g|^*_{\mathfrak{m}\times\mathfrak{m}}$ to be positive definite.

Case $(\mathfrak{g},\mathfrak{h})=(\mathfrak{su}(4),\mathfrak{su}(3))$. There are two non-equivalent faithful representations of $\mathfrak{su}(3)$ on $\mathbb{C}^4$. They correspond to the reducible representations $\mathbb{C}^3\oplus\mathbb{C}=R(1,0)\oplus R(0,0)$ and $\overline{\mathbb{C}^3}\oplus\mathbb{C}=R(0,1)\oplus R(0,0)$. The two subalgebras defined by these representations are conjugate by an outer automorphism of $\mathfrak{su}(4)$. Therefore, there is only one injective Lie algebra homomorphism $\mathfrak{su}(3)\to\mathfrak{su}(4)$ up to conjugation, and this is the standard inclusion. This yields the 7-dimensional Berger sphere as a naturally reductive space

$$SU(4)/SU(3).$$

The associated infinitesimal model is always regular, and we get a 1-parameter family of metrics.

Case $(\mathfrak{g},\mathfrak{h})=(\mathfrak{so}(7),\mathfrak{g}_2)$. There is, up to conjugation, only one subalgebra $\mathfrak{g}_2\subset\mathfrak{so}(7)$, and the corresponding infinitesimal model is regular. There is only a 1-parameter family of metrics and the corresponding naturally reductive space $SO(7)/G_2$ is isometric to S^7 with a round metric.

3.2 Classification of Type I in dimension 8

In the second column of Table 4, we list all candidates of compact semi-simple Lie algebras $\mathfrak{g}$ of dimension $8\le k\le 16$. We have already shown that $\mathfrak{g}$ can only have dimension less than or equal to 13 or the dimension of $\mathfrak{g}$ is 16. In the third column of Table 4, we list all Lie algebras of dimension $\dim(\mathfrak{g})-8$ that satisfy $\mathrm{rank}(\mathfrak{h})\le\min(\mathrm{rank}(\mathfrak{g}),\mathrm{rank}(\mathfrak{so}(8))\le 4$.

$\dim(\mathfrak{g})$	$\mathfrak{g}$	$\mathfrak{h}$
8	$\mathfrak{su}(3)$	$\{0\}$
9	$\mathfrak{su}(2)^3$	$\mathbb{R}$
10	$\mathfrak{so}(5)$	$\mathbb{R}^2$
11	$\mathfrak{su}(3)\oplus\mathfrak{su}(2)$	$\mathfrak{su}(2),\ \mathbb{R}^3$
12	$\mathfrak{su}(2)^4$	$\mathfrak{su}(2)\oplus\mathbb{R}, \mathbb{R}^4$
13	$\mathfrak{so}(5)\oplus\mathfrak{su}(2)$	$\mathfrak{su}(2)\oplus\mathbb{R}^2$
16	$\mathfrak{so}(5)\oplus\mathfrak{su}(2)^2$	$\mathfrak{su}(3),\ \mathfrak{su}(2)^2\oplus\mathbb{R}^2$
16	$\mathfrak{su}(3)^2$	$\mathfrak{su}(3),\ \mathfrak{su}(2)^2\oplus\mathbb{R}^2$

Table 4: Candidates for 8-dimensional Type I naturally reductive pairs.

The pairs $(\mathfrak{g},\mathfrak{h})$ that are excluded by Lemma 3.5 are

$$(\mathfrak{su}(3)\oplus\mathfrak{su}(2),\mathbb{R}^3),\quad (\mathfrak{su}(2)^4,\mathfrak{su}(2)\oplus\mathbb{R}),\quad (\mathfrak{su}(2)^4,\mathbb{R}^4),$$
$$(\mathfrak{so}(5)\oplus\mathfrak{su}(2),\mathfrak{su}(2)\oplus\mathbb{R}^2),\quad (\mathfrak{so}(5)\oplus\mathfrak{su}(2)^2,\mathfrak{su}(2)^2\oplus\mathbb{R}^2),$$
$$(\mathfrak{su}(3)^2,\mathfrak{su}(2)^2\oplus\mathbb{R}^2).$$

For the pair $(\mathfrak{so}(5)\oplus\mathfrak{su}(2)^2,\mathfrak{su}(3))$ there does not exist an injective Lie algebra homomorphism from $\mathfrak{su}(3)$ to $\mathfrak{so}(5)\oplus\mathfrak{su}(2)^2$. The remaining cases are

$$\big(\mathfrak{su}(3),\{0\}\big),\quad \big(\mathfrak{su}(2)^3,\mathbb{R}\big),\quad \big(\mathfrak{so}(5),\mathbb{R}^2\big),$$
$$\big(\mathfrak{su}(3)\oplus\mathfrak{su}(2),\mathfrak{su}(2)\big),\quad \big(\mathfrak{su}(3)^2,\mathfrak{su}(3)\big).$$

We will discuss them case by case below.

Case $(\mathfrak{g},\mathfrak{h})=(\mathfrak{su}(3),\{0\})$. The pair $(\mathfrak{su}(3),\{0\})$ is always regular. The simply connected naturally reductive space for this case is $SU(3)$ with some bi-invariant metric. In other words, we have a 1-parameter family of naturally reductive structures.

Case $(\mathfrak{g},\mathfrak{h})=(\mathfrak{su}(2)^3,\mathbb{R})$. Let x_1,x_2,x_3 be as in (3.1). The $\operatorname{ad}(\mathfrak{g})$-invariant nondegenerate symmetric bilinear form is necessarily positive definite and is given by $\overline{g}=\frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)}\oplus\frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}\oplus\frac{-1}{8\lambda_3^2}B_{\mathfrak{su}(2)}$, so we can assume that $0<\lambda_1\le\lambda_2\le\lambda_3$. Every subalgebra $\mathbb{R}\subset\mathfrak{su}(2)^3$ is conjugate to one given by

$$\mathbb{R}_{a_1,a_2,a_3}=\operatorname{span}\{(a_1x_1,a_2x_1,a_3x_1)\},$$

with $a_1,a_2,a_3\ge 0$. If $\lambda_1=\lambda_2<\lambda_3$, then we can conjugate $\mathfrak{h}$ such that $a_1\le a_2$. Similarly, if $\lambda_1<\lambda_2=\lambda_3$, then we can arrange that $a_2\le a_3$. Lastly if $\lambda_1=\lambda_2=\lambda_3$, then we can arrange that $a_1\le a_2\le a_3$. Under these conditions none of these are conjugate to each other. From Lemma 3.2 we see that the naturally reductive decomposition is irreducible if and only if all a_1,a_2,a_3 are non-zero. Clearly, the connected subgroup of $SU(2)^3$ with this Lie algebra is a closed subgroup if and only if $\frac{a_2}{a_1}=q_1\in\mathbb{Q}$ and $\frac{a_3}{a_1}=q_2\in\mathbb{Q}$. If it is closed, then we obtain a 3-parameter family of naturally reductive structures on $SU(2)^3/S^1_{q_1,q_2}$, where $S^1_{q_1,q_2}$ is the connected subgroup

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with $\mathrm{Lie}(S^1_{q_1,q_2}) = \mathfrak{h}$. Note that $(\mathfrak{su}(2),\mathbb{R})$ is a symmetric pair with $(\mathfrak{sl}(2,\mathbb{R}),\mathbb{R})$ its dual symmetric pair. We obtain the partial dual spaces by replacing one $\mathfrak{su}(2)$-factor by $\mathfrak{sl}(2,\mathbb{R})$. If we replaced the j-th $\mathfrak{su}(2)$-summand by $\mathfrak{sl}(2,\mathbb{R})$, then the restriction $\overline{g}^*|_{\mathfrak{h}\times\mathfrak{h}}$ is negative definite if and only if $\sum_{i=1}^3(-1)^{\delta_{ij}}(\frac{a_i}{\lambda_i})^2 < 0$.

Case $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{so}(5),\mathbb{R}^2)$. The subalgebra $\mathbb{R}^2 \subset \mathfrak{so}(5)$ has to be the maximal torus. In particular, these spaces are always regular. The simply connected naturally reductive space for this case is $SO(5)/(SO(2)\times SO(2))$, where $SO(2)\times SO(2)$ is embedded block diagonally. The metric is induced from any negative multiple of the Killing form of $\mathfrak{so}(5)$. In other words, we have a 1-parameter family of naturally reductive structures.

Case $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{su}(3)\oplus\mathfrak{su}(2),\mathfrak{su}(2))$. Up to conjugation, there are two injective Lie algebra homomorphisms $\mathfrak{su}(2)\to\mathfrak{su}(3)\oplus\mathfrak{su}(2)$ such that conditions (3) and (4) from the beginning of this section are satisfied. For the inclusion in the second factor, there is only the identity. For the inclusion in $\mathfrak{su}(3)$ there are two choices, namely, the standard inclusion, denoted by i_{st} and the other given by the 3-dimensional irreducible representation of $\mathfrak{su}(2)$, denoted by i_{ir}. For both inclusions, the infinitesimal model is regular. The simply connected homogeneous spaces are

$$\big(SU(3)\times SU(2)\big)/(i_{\mathrm{st}}\times\mathrm{id})\big(SU(2)\big) \quad \text{and} \quad \big(SU(3)\times SU(2)\big)/(i_{\mathrm{ir}}\times\mathrm{id})\big(SU(2)\big),$$

where we denote the corresponding group homomorphism of i_{st} and i_{ir} also by i_{st} and i_{ir}, respectively. There is a 2-parameter family of $\mathrm{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear forms: $\overline{g} = \frac{-\lambda_1}{12}B_{\mathfrak{su}(3)} \oplus \frac{-\lambda_2}{8}B_{\mathfrak{su}(2)}$. The normal homogeneous spaces correspond to $\lambda_1,\lambda_2 > 0$. For the non-normal homogeneous spaces, we have $\lambda_1 > 0$ and $\lambda_2 < 0$. Furthermore, we require that the condition $\lambda_1+\lambda_2 < 0$ holds for the first space and $4\lambda_1+\lambda_2 < 0$ for the second space.

For the space $(SU(3)\times SU(2))/(i_{\mathrm{ir}}\times\mathrm{id})(SU(2))$ there is a non-compact partial dual space

$$\big(SL(3,\mathbb{R})\times SU(2)\big)/(i_{\mathrm{ir}}\times\mathrm{id})\big(SU(2)\big).$$

The $\mathrm{ad}(\mathfrak{g}^*)$-invariant non-degenerate symmetric bilinear forms are

$$\overline{g}^* = \frac{\lambda_1}{12}B_{\mathfrak{sl}(3,\mathbb{R})} \oplus \frac{-\lambda_2}{8}B_{\mathfrak{su}(2)}.$$

In order to obtain a positive definite metric on our space, require that $\lambda_1,\lambda_2 > 0$ and $-4\lambda_1+\lambda_2 < 0$.

Case $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{su}(3)^2,\mathfrak{su}(3))$. There are two possible conjugacy classes of the subalgebra $\mathfrak{su}(3)$, namely $\mathfrak{su}(3)\times\{0\}$ and the diagonal $\mathfrak{su}(3)_\Delta$. The first case clearly does not satisfy condition (4). Therefore, the subalgebra $\mathfrak{h}$ has to be the diagonal subalgebra. The $\mathrm{ad}(\mathfrak{g})$-invariant metrics are given by $\overline{g} = -\lambda_1 B_{\mathfrak{su}(3)} \oplus -\lambda_2 B_{\mathfrak{su}(3)}$, with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. By permuting the two $\mathfrak{su}(3)$-factors, we can assume that $\lambda_1 \geq \lambda_2$. The normal homogeneous spaces correspond to $\lambda_1,\lambda_2 > 0$. Note that for $\lambda_1 = \lambda_2$ and $\lambda_1 > 0$ we obtain a symmetric space. For the non-normal homogeneous spaces, we require that the signature of $\overline{g}$ is $(8,8)$ and that $\overline{g}|_{\mathfrak{h}\times\mathfrak{h}}$ is negative definite. This is the case if and only if $\lambda_1+\lambda_2 < 0$ and $\lambda_1 > 0 > \lambda_2$. All the naturally reductive structures are regular and irreducible. For every case, the homogeneous space is isometric to $SU(3)$ with some bi-invariant metric.

The pair $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair. The $\mathrm{ad}(\mathfrak{g}^*)$-invariant non-degenerate symmetric bilinear forms for the dual pair $(\mathfrak{sl}(3,\mathbb{C}), \mathfrak{su}(3))$ are all a multiple of the Killing form of $\mathfrak{sl}(3,\mathbb{C})$, and thus all induce a symmetric structure. Consequently, there are no non-symmetric partial dual naturally reductive structures.

This concludes the classification of all 7- and 8-dimensional naturally reductive spaces of Type I. We summarize the discussion from Subsection 3.1 and Section 3.2 as the following result.

Theorem 3.6 *All 7- and 8-dimensional compact simply connected naturally reductive spaces of Type I are presented in Table 5 and Table 6, respectively. Furthermore, the dimension of the parameter space of naturally reductive structures is indicated and whether non-compact partial dual spaces exist or not.*

4 Classification of Type II

By Theorem 2.4 we can construct every infinitesimal model of a naturally reductive decomposition of Type II as a $(\mathfrak{k}, B)$-extension of a naturally reductive decomposition of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^n,$$

where $\mathfrak{h} \oplus \mathfrak{m}$ is semi-simple and $\mathfrak{g}$ is the transvection algebra of this naturally reductive decomposition. In this section we will construct all 7 and 8 dimensional irreducible $(\mathfrak{k}, B)$-extensions of all naturally reductive decomposition of the above form with $\mathfrak{h} \oplus \mathfrak{m}$ compact. We use the partial duality to obtain all other spaces; see Remark 2.8. For every case we will mention if there exist partial dual spaces.

We start by finding all possible candidates for the canonical base spaces of irreducible Type II spaces. From this list, we construct all possible irreducible $(\mathfrak{k}, B)$-extensions. To guarantee that there are no duplicates in our list, we use Lemma 2.14 and Proposition 2.16. Note that to classify the naturally reductive spaces of Type II in some dimension k, we need the classification of all naturally reductive spaces of Type I up to dimension $k-1$.

Remark 4.1 We want all of our $(\mathfrak{k}, B)$-extensions to be irreducible. If there exists an irreducible $(\mathfrak{k}, B)$-extension of a naturally reductive transvection algebra as in (2.9), then Lemma 2.12 in particular implies $\mathfrak{s}(\mathfrak{h}_i \oplus \mathfrak{m}_i) \neq \{0\}$ for every $i = 1, \dots, p$. In particular, this excludes the possibility that $\mathfrak{h}_i \oplus \mathfrak{m}_i$ is an irreducible symmetric decomposition that is not hermitian symmetric.

If $\mathfrak{k}$ is abelian and $\mathfrak{k} = \mathfrak{k}_1$, then by Lemma 2.14(i), we require that $\pi_{\mathfrak{m}}(\mathcal{Z}(\mathfrak{b}_1)) = \{0\}$. We need this condition in order for the canonical base space to be the base space we start with. Note that $\pi_{\mathfrak{m}}(\mathfrak{k}_1) = \{0\}$ if and only if $\mathfrak{k}_1 \subset \mathcal{Z}(\mathfrak{h})$. Thus, for the $(\mathfrak{k}, B)$-extension to be irreducible and satisfy Lemma 2.14(i) we require that $\mathcal{Z}(\mathfrak{h}_i) \neq \{0\}$ for every $i = 1, \dots, p$.

4.1 Classification of Type II in Dimension 7

First, we argue that all possible canonical base spaces of irreducible naturally reductive decompositions of Type II with a compact semi-simple factor are given in (4.1).

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This is done by systematically excluding all other possibilities.

$$
\begin{array}{llll}
 & \mathbb{R}^6, & \mathbb{R}^4, & S^2\times\mathbb{R}^4,\\
 & \mathbb{C}P^2\times\mathbb{R}^2, & S^2\times S^2\times\mathbb{R}^2, & Sp(2)/(SU(2)\times S^1),\\
(4.1) & SO(5)/(SO(3)\times SO(2)), & SU(3)/(S^1\times S^1), & SU(4)/S(U(1)\times U(3)),\\
 & \mathbb{C}P^2\times S^2, & S^2\times S^2\times S^2. &
\end{array}
$$

Remark 4.2 Even though we write all above base spaces as globally homogeneous spaces, we actually treat the family of naturally reductive decompositions to which they belong, which can also contain non-regular decomposition, *i.e.*, strictly locally homogeneous spaces. The parameter values for which the locally naturally reductive structures are regular have to be considered case by case.

The Euclidean factor cannot be $\mathbb{R}^5$, because then the Lie algebra $\mathfrak{k}\subset\mathfrak{so}(5)$ is two-dimensional and its linear action on $\mathbb{R}^5$ has a vector on which it acts trivially. From Lemma 2.12, we see that such any $(\mathfrak{k}, B)$-extension results in a reducible naturally reductive space.

Suppose that the Euclidean factor is $\mathbb{R}^3$, the Lie algebra $\mathfrak{k}\subset\mathfrak{so}(3)$ has to be equal to $\mathfrak{so}(3)$ in order not to have a vector on which it acts trivially. This means that the semi-simple factor of the base space has to be 1-dimensional, which is not possible.

Suppose that the Euclidean factor is $\mathbb{R}^2$. If the dimension of the semi-simple factor is two, then $\dim(\mathfrak{s}(\mathfrak{g}))\leq 2$ and thus we cannot construct an irreducible $(\mathfrak{k}, B)$-extension of dimension 7. If the dimension of the semi-simple factor is three, then $\dim(\mathfrak{k}) = 2$, and thus $\mathfrak{k}$ is abelian. The semi-simple factor is either $SU(2)$ or the symmetric space $(SU(2)\times SU(2))/SU(2)_\Delta$. The last case is excluded by Remark 4.1. For $SU(2)$, the algebra $\mathfrak{k}_1$ is 1-dimensional, and thus we see that Lemma 2.14(i) cannot be satisfied. If the dimension of the semi-simple factor is four, then the semi-simple factor has to be a hermitian symmetric space by [13]. There are only two compact homogeneous spaces that allow a hermitian symmetric structure; these are $S^2\times S^2$ and $\mathbb{C}P^2$.

For all other 7-dimensional naturally reductive spaces of Type II, the base space has only a semi-simple factor. It is easy to check that every 7-dimensional $(\mathfrak{k}, B)$-extension of any naturally reductive space of Type I of dimension less than or equal to 4 is reducible. This leaves us with the 5- and 6-dimensional cases. The only compact spaces of Type I in dimension 5 with $\dim(\mathfrak{s}(\mathfrak{g}))\geq 2$ are $S^2\times SU(2)$ and $(SU(2)\times SU(2))/S^1$. However, we see for any 2-dimensional $\mathfrak{k}\subset\mathfrak{s}(\mathfrak{g})$ that condition (i) of Lemma 2.14 is not satisfied in both cases and thus they are excluded. The nearly Kähler spaces $G_2/SU(3)$ and $(SU(2)\times SU(2)\times SU(2))/SU(2)_\Delta$ can be excluded, because for both $\mathfrak{s}(\mathfrak{g}) = \{0\}$ holds. Similarly $((SU(2)\times SU(2))/SU(2)_\Delta)\times((SU(2)\times SU(2))/SU(2)_\Delta)$ satisfy $\mathfrak{s}(\mathfrak{g}) = \{0\}$. The spaces $SU(2)\times((SU(2)\times SU(2))/SU(2)_\Delta$, and $SU(2)\times SU(2)$ can be excluded, because they do not satisfy Lemma 2.14(i) for any $\mathfrak{k}\subset\mathfrak{s}(\mathfrak{g})$. All other 6-dimensional naturally reductive spaces of Type I are possible.

A classification of naturally reductive decompositions of Type II in dimension 7 is readily obtained in the following steps. From the list of possible canonical base spaces in (4.1) we have to make all irreducible $(\mathfrak{k}, B)$-extensions such that Lemma 2.14(i)

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and (ii) are satisfied. Lemma 2.12 tells us exactly when the constructed spaces are irreducible. We also have to filter out all isomorphic spaces. Proposition 2.16 makes this quite easy in all the occurring cases. How to obtain a globally homogeneous naturally reductive space from these data is described in [16]. To make the classification complete we just need to check for every case if partial dual naturally reductive spaces exist. We will not discuss every case, because some cases are very similar. We attempt to cover all the different 'types' of $(\mathfrak{k}, B)$-extensions in the selected cases below. The final result is formulated in Theorem 4.5.

Remark 4.3 From Lemma 2.15, we know that $\mathfrak{k}_1 = \{0\}$ implies $\ker(R|_{\mathrm{ad}(\mathfrak{h})+\psi(\mathfrak{k})}) = \{0\}$. Thus, in particular, Lemma 2.14(i) and (ii) are automatically satisfied. Therefore, we will only check the conditions of Lemma 2.14 when $\mathfrak{k}_1 \neq \{0\}$.

Before we start, let us introduce some notation.

Notation 4.4 Below $e_{ij} \in \mathfrak{so}(n)$ is the matrix whose only non-zero entries are its ij-th and ji-th entry, which are -1 and 1, respectively. Let B_{Λ^2} be the metric on $\mathfrak{so}(n)$ be defined by $B_{\Lambda^2}(x, y) := -\frac{1}{2}\mathrm{tr}(xy)$. In the following, we use the contraction with the metric on $\mathfrak{m}$ to make the identification $\Lambda^2\mathfrak{m} \cong \mathfrak{so}(\mathfrak{m})$; *i.e.*, e_{ij} is identified with $e_i \wedge e_j$. The curvature tensor then becomes a symmetric map $R\colon \mathfrak{so}(\mathfrak{m}) \to \mathfrak{so}(\mathfrak{m})$ with respect to B_{Λ^2}.

The representation $\varphi\colon \mathfrak{k} \to \mathfrak{so}(\mathfrak{m} \oplus \mathbb{R}^n)$, from Definition 2.13, uniquely determines the algebra $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g})$, and below we will always describe $\mathfrak{k}$ through $\varphi(\mathfrak{k})$.

The canonical base space is $\mathbb{R}^6$. Consider the canonical base space $\mathbb{R}^6$. The Lie algebra $\mathfrak{k}$ is 1-dimensional. Let k be a unit vector in $\mathfrak{k}$. Then there is an orthonormal basis $e_1, \dots, e_6$ of $\mathbb{R}^6$ and constants $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\varphi(k) = c_1 e_{12} + c_2 e_{34} + c_3 e_{56} \in \mathfrak{so}(6).$$

It is clear from Lemma 2.12 that the spaces are irreducible precisely when $c_1, c_2, c_3 \in \mathbb{R}\backslash\{0\}$. Therefore, from now on, we suppose that $c_1, c_2, c_3 \in \mathbb{R}\backslash\{0\}$. The $(\mathfrak{k}, B)$-extensions describe naturally reductive structures on the 7-dimensional Heisenberg group, as explained in [16]. We get a 3-parameter family of naturally reductive structures on the 7-dimensional Heisenberg group. We can ensure that $0 < c_1 \leq c_2 \leq c_3$ by choosing a different basis of $\mathbb{R}^6$. When we do this, all the described naturally reductive structures are non-isomorphic.

The canonical base space is $\mathbb{R}^4$. The Lie algebra $\mathfrak{k}$ has to be $\mathfrak{su}(2)$, and the representation $\varphi\colon \mathfrak{su}(2) \to \mathfrak{so}(4)$ has to be the irreducible 4-dimensional representation in order for the $(\mathfrak{k}, B)$-extension to be irreducible. The $(\mathfrak{k}, B)$-extension will yield a naturally reductive structure on the quaternionic Heisenberg group. The choice of an invariant metric B on $\mathfrak{k}$ gives us a 1-parameter family of naturally reductive structures. This family of naturally reductive structures is quite interesting and is investigated in [2]. In [9], it was proved that the Heisenberg groups and the quaternionic-Heisenberg groups are the only groups of type H for which the natural left invariant metric is naturally reductive.

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The canonical base space is $S^2 \times \mathbb{R}^4$. Let h, e_1, e_2 be an orthonormal basis of $\mathfrak{su}(2)$ with respect to $\frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)}$. The transvection algebra of the base space is given by

$$\mathfrak{g} = \mathfrak{su}(2) \oplus_{L.a.} \mathbb{R}^4 = \mathfrak{h} \oplus \mathfrak{m} \oplus_{L.a.} \mathbb{R}^4,$$

where $\mathfrak{h} := \operatorname{span}\{h\}$ and $\mathfrak{m} := \operatorname{span}\{e_1, e_2\}$. The $\operatorname{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ is given by $\overline{g} = \frac{-1}{8\lambda_1^2}B_{\mathfrak{su}(2)} \oplus B_{\mathrm{eucl}}$. We have $\mathfrak{s}(\mathfrak{g}) = \operatorname{span}\{h\} \oplus \mathfrak{so}(4)$. Let $k \in \mathfrak{k}$ be a unit vector. Then there is an orthonormal basis of $\mathbb{R}^4$ such that

$$\varphi(k) = c_1 \operatorname{ad}(h)|_{\mathfrak{m}} + c_2 e_{34} + c_3 e_{56},$$

with $c_1, c_2, c_3 \in \mathbb{R}$. All these spaces are irreducible precisely when $c_1, c_2, c_3 \in \mathbb{R}\backslash\{0\}$ by Lemma 2.12. Therefore, from now on, we suppose that $c_1, c_2, c_3 \in \mathbb{R}\backslash\{0\}$. We have $\mathfrak{k} = \mathfrak{k}_2$ and from [16, Sec. 2.3] we know that the $(\mathfrak{k}, B)$-extension defines a naturally reductive structure on $S^2 \times H^5$, where H^5 denotes the 5-dimensional Heisenberg group. On this homogeneous space, we obtain a 4-parameter family of naturally reductive structures, with c_1, c_2, c_3 and $\lambda_1 > 0$ as parameters. By an automorphism of $\mathfrak{g}$, we can arrange that $c_2 \geq c_3 > 0$ and $c_1 > 0$. When we do this, none of these naturally reductive structures are isomorphic. Note that we can replace the semi-simple factor $S^2 = SU(2)/S^1$ by its non-compact dual symmetric space, $SL(2,\mathbb{R})/S^1$.

The canonical base space is $\mathbb{C}P^2 \times S^2$. Let $h_1, h_2, h_3, h_4, e_1, e_2, e_3, e_4$ be an orthonormal basis of $\mathfrak{su}(3)$ with respect to $-1/12\lambda_1^2 B_{\mathfrak{su}(3)}$ such that h_1, h_2, h_3, h_4 span the Lie algebra of the isotropy group $S(U(2) \times U(1)) \subset SU(3)$ with h_4 spanning the center. Let h_5, e_5, e_6 be an orthonormal basis of $\mathfrak{su}(2)$ with respect to $-1/8\lambda_2^2 B_{\mathfrak{su}(2)}$. The transvection algebra of the base space is given by $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} := \operatorname{span}\{h_1, \dots, h_5\}$ and $\mathfrak{m} := \operatorname{span}\{e_1, \dots, e_6\}$. The $\operatorname{ad}(\mathfrak{g})$-invariant non-degenerate symmetric bilinear form is $\overline{g} = \frac{-1}{12\lambda_1^2}B_{\mathfrak{su}(3)} \oplus \frac{-1}{8\lambda_2^2}B_{\mathfrak{su}(2)}$. The algebra $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g}) = \operatorname{span}\{h_4, h_5\}$ is 1-dimensional. Let $k \in \mathfrak{k}$ be a unit vector. Then $\varphi(k) = c_1 \operatorname{ad}(h_4)|_{\mathfrak{m}} + c_2 \operatorname{ad}(h_5)|_{\mathfrak{m}}$. The curvature of the $(\mathfrak{k}, B)$-extension is given by

$$R = -\sum_{i=1}^{5} \operatorname{ad}(h_i)|_{\mathfrak{m}} \odot \operatorname{ad}(h_i)|_{\mathfrak{m}} + \varphi(k) \odot \varphi(k).$$

From Lemma 2.15, we have $\ker(R|_{\operatorname{ad}(\mathfrak{h}\oplus\mathfrak{k})}) = \ker(R|_{\operatorname{ad}(\mathfrak{z}(\mathfrak{h}\oplus\mathfrak{k}))}) = \ker(R|_{\operatorname{ad}(\mathfrak{z}(\mathfrak{h}))})$. We need to check when $R|_{\operatorname{ad}(\mathfrak{z}(\mathfrak{h}))}$ has trivial kernel. Note that the center of $\mathfrak{h}$ is given by $\operatorname{span}\{h_4, h_5\}$. Let $\omega_1, \omega_2 \in \mathfrak{so}(\mathfrak{m})$ be such that $B_{\Lambda^2}(\omega_1, h_j) = \delta_{4j}$ and $B_{\Lambda^2}(\omega_2, h_j) = \delta_{5j}$ for $j = 1, \dots, 5$. Then

$$R(\omega_1) = (-1 + c_1^2)\operatorname{ad}(h_4)|_{\mathfrak{m}} + c_1 c_2 \operatorname{ad}(h_5)|_{\mathfrak{m}},$$
$$R(\omega_2) = c_1 c_2 \operatorname{ad}(h_4)|_{\mathfrak{m}} + (-1 + c_2^2)\operatorname{ad}(h_5)|_{\mathfrak{m}}.$$

We see that $R|_{\operatorname{ad}(\mathfrak{z}(\mathfrak{h}))}$ has rank 2 precisely when $c_1^2 + c_2^2 \neq 1$. In other words, the base space is equal to the canonical base space if and only if $c_1^2 + c_2^2 \neq 1$. By Lemma 2.12, the $(\mathfrak{k}, B)$-extension is reducible precisely when either $c_1 = 0$ or $c_2 = 0$. Suppose that the $(\mathfrak{k}, B)$-extension is irreducible. With an automorphism of $\mathfrak{g}$ we can always arrange that $c_1 > 0$ and $c_2 > 0$. Under this condition, none of the described $(\mathfrak{k}, B)$-extensions are isomorphic. The $(\mathfrak{k}, B)$-extension is regular if and only if the connected subgroup H_0 with Lie subalgebra $\mathfrak{h}_0 = \mathfrak{k}^\perp \subset \mathfrak{h}$ is closed in $SU(3) \times SU(2)$;

see [16]. We have $\mathfrak{h}_0 = \operatorname{span}\{c_2 h_4 - c_1 h_5\}$. We see that H_0 is closed precisely when $q = c_2\lambda_1/\sqrt{3}c_1\lambda_2 \in \mathbb{Q}$. The $(\mathfrak{k}, B)$-extension describes a naturally reductive structure on $(SU(3)\times SU(2))/(SU(2)\times S^1_q)$, where $SU(2)$ is the standard subgroup of $SU(3)$ and S^1_q is the subgroup with Lie subalgebra $\mathfrak{h}_0$. To obtain all of these naturally reductive structures on the fixed homogeneous space $(SU(3)\times SU(2))/(SU(2)\times S^1_q)$, we start by defining $\mathfrak{h}_0 := \operatorname{Lie}(S^1_q)$ and $\mathfrak{k} := \mathfrak{h}_0^\perp \subset \mathfrak{h}$ with respect to $\overline{g}$. We have a 1-parameter family of $\operatorname{ad}(\mathfrak{k})$-invariant metrics on $\mathfrak{k}$. Together with the parameters λ_1, λ_2, this gives us a 3-parameter family of naturally reductive structures on $(SU(3)\times SU(2))/(SU(2)\times S^1_q)$. Note that we can replace $SU(3)/S(U(2)\times U(1))$ by its non-compact dual $SU(2,1)/S(U(2)\times U(1))$, and we can also replace S^2 by its non-compact dual.

4.2 Classification of Type II in Dimension 8

First we argue that all possible canonical base spaces of irreducible naturally reductive decompositions of Type II with a compact semi-simple factor are given in (4.2). This is done by systematically excluding all other possibilities. A point space is denoted by $\{*\}$.

$$
\begin{array}{ll}
\mathbb{R}^6, & \mathbb{R}^5, \\
\mathbb{R}^4, & S^2\times\mathbb{R}^4 \\
SU(2)\times\mathbb{R}^4, & \mathbb{C}P^2\times\mathbb{R}^2, \\
S^2\times S^2\times\mathbb{R}^2, & (SU(2)\times SU(2))/S^1_q\times\mathbb{R}^2, \\
SU(3)/SU(2)_{\mathrm{st}}\times\mathbb{R}^2, & SU(2)\times S^2\times\mathbb{R}^2, \\
SU(3)/S^1_q, & (SU(3)\times SU(2))/(SU(2)_{\mathrm{st}}\times S^1_q), \\
(SU(3)\times SU(2))/(SU(2)_\Delta\times S^1), & SU(2)^3/(S^1_{q_1}\times S^1_{q_2}), \\
S^2\times(SU(2)\times SU(2))/S^1_q, & SU(3)/(S^1\times S^1), \\
S^2\times S^2\times S^2, & S^2\times\mathbb{C}P^2 \\
\{*\}. &
\end{array}
\tag{4.2}
$$

Just as for the 7-dimensional case, Remark 4.2 also applies here.

The Euclidean factor cannot be $\mathbb{R}^7$, because then $\dim(\mathfrak{k}) = 1$ and the linear action of $\mathfrak{k}$ on $\mathbb{R}^7$ has a vector on which it acts trivially, and by Lemma 2.12, any such $(\mathfrak{k}, B)$-extension is reducible.

If the Euclidean factor is $\mathbb{R}^6$, then the semi-simple factor needs to have dimension zero and $\dim(\mathfrak{k}) = 2$.

If the Euclidean factor is $\mathbb{R}^5$ and the semi-simple factor is 2-dimensional, then $\dim(\mathfrak{k}) = 1$. Just as for $\mathbb{R}^7$, we see that the linear action of $\mathfrak{k}$ on $\mathbb{R}^5$ has a vector on which it acts trivially and by Lemma 2.12, any such $(\mathfrak{k}, B)$-extension is reducible. Thus, also for $\mathbb{R}^5$, the semi-simple factor has to be zero dimensional.

Suppose that the Euclidean factor is $\mathbb{R}^4$. The semi-simple factor can be 0-, 2-, or 3-dimensional. If it is 2-dimensional, then it is S^2. If it is 3-dimensional, then it either is the symmetric space $(SU(2)\times SU(2))/SU(2)$ or the Lie group $SU(2)$. The first case is excluded by Remark 4.1.

If the Euclidean factor is $\mathbb{R}^3$, then $\mathfrak{k}$ has to contain $\mathfrak{so}(3)$ in order for the linear representation of $\mathfrak{k}$ on $\mathbb{R}^3$ not to have a vector on which it acts trivially. We see that

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if the semi-simple factor is 0-dimensional, then we cannot construct an irreducible 8-dimensional $(\mathfrak{k}, B)$-extension. The only other possibility is that the semi-simple factor is 2-dimensional. In this case, we immediately see by Lemma 2.12 that any such $(\mathfrak{so}(3), B)$-extension is reducible.

Suppose that the Euclidean factor is $\mathbb{R}^2$. The semi-simple factor can either be 3-, 4-, or 5-dimensional, because if the semi-simple factor is 2-dimensional, then $\dim(\mathfrak{s}(\mathfrak{g})) \le 2$ and thus we cannot make an irreducible 8-dimensional $(\mathfrak{k}, B)$-extension from this. Suppose that the semi-simple factor is 5-dimensional. We see that there are three possibilities: $(SU(2) \times SU(2))/S^1_q$, $SU(3)/SU(2)$, and $SU(2) \times S^2$. Suppose that the semi-simple factor is 4-dimensional. If it is irreducible, then it can only be $\mathbb{C}P^2$. If it is reducible, then it can only be $S^2 \times S^2$. Suppose that the semi-simple factor is 3-dimensional. From Remark 4.1, we see that the semi-simple factor has to be equal to $SU(2)$ and $\mathfrak{s}(\mathfrak{g}) = \mathfrak{su}(2) \oplus \mathfrak{so}(2)$. The Lie algebra $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{g}) = \mathfrak{su}(2) \oplus \mathfrak{so}(2)$ is a 3-dimensional subalgebra. Hence $\mathfrak{k} = \mathfrak{su}(2) \subset \mathfrak{s}(\mathfrak{g})$, and thus $\mathfrak{k}$ acts trivially on $\mathbb{R}^2$. Therefore, by Lemma 2.12, any such $(\mathfrak{k}, B)$-extension is reducible.

Only base spaces with no Euclidean part remain. Now we discuss the case for which the base space has an irreducible 3-dimensional factor. There are only two compact irreducible 3-dimensional naturally reductive spaces of Type I: $SU(2)$ and the symmetric space $(SU(2) \times SU(2))/SU(2)$. The symmetric space is excluded by Remark 4.1. If we have $SU(2)$ as a 3-dimensional factor, then $\mathfrak{k}$ has to be at least 3-dimensional; see Lemma 2.14(i). The only possibility for a base space is $SU(2) \times S^2$, but just as for the case $SU(2) \times \mathbb{R}^2$, any 8-dimensional $(\mathfrak{k}, B)$-extension of this space is reducible. We conclude that if there is no Euclidean factor, then the semi-simple factor can not contain a 3-dimensional factor.

If the base space is 7-dimensional, then $\dim(\mathfrak{k}) = 1$, and thus $\mathfrak{k}$ is abelian. By Remark 4.1 we require that $\mathcal{Z}(\mathfrak{h}_i) \neq \{0\}$ for every $i = 1, \ldots, p$. We noted above that there cannot be a 3-dimensional factor; hence, the 7-dimensional space either is irreducible or it is a product of a 5-dimensional irreducible space and a 2-dimensional space. Consequently, all possible spaces are

$$SU(3)/S^1_q, \quad (SU(3) \times SU(2))/(SU(2) \times S^1_q),$$
$$(SU(3) \times SU(2))/(SU(2)_\Delta \times S^1), \quad SU(2)^3/(S^1_{q_1} \times S^1_{q_2}),$$
$$(SU(2) \times SU(2))/S^1_q \times S^2.$$

For a 6-dimensional base space, $\mathfrak{k}$ is abelian, and thus by Remark 4.1, we need that $\mathcal{Z}(\mathfrak{h}_i) \neq \{0\}$ for every $i = 1, \ldots, p$. There are no 3-dimensional factors by the above argument. We can easily see that all possibilities are $SU(3)/(S^1 \times S^1)$, $\mathbb{C}P^2 \times S^2$, and $S^2 \times S^2 \times S^2$.

We check that every 5-dimensional irreducible naturally reductive space of Type I satisfies $\dim(\mathfrak{s}(\mathfrak{g})) \le 2$ and thus we cannot make an 8-dimensional irreducible $(\mathfrak{k}, B)$-extension from this. Every reducible 5-dimensional space of Type I contains a 3-dimensional factor and thus can be excluded by the above discussion. Similarly for every 4-dimensional space of Type I we have $\dim(\mathfrak{s}(\mathfrak{g})) \le 2$, and thus we cannot make an irreducible 8-dimensional $(\mathfrak{k}, B)$-extension of this.

The Lie algebra $\mathfrak{su}(3)$ has dimension 8 and is a compact simple Lie algebra. Therefore, a point space is also a possible base space.

We proceed as in the 7-dimensional case. Also we do not discuss every case separately, because of the large similarities between them.

The canonical base space is $\mathbb{R}^5$. The Lie algebra $\mathfrak{k}$ has to be 3-dimensional and in order to have a 5-dimensional representation without vectors on which $\mathfrak{k}$ acts trivially. The only possibility is $\mathfrak{k} = \mathfrak{su}(2)$, and the representation of $\mathfrak{k}$ is the 5-dimensional irreducible representation of $\mathfrak{su}(2)$. Let k_1, k_2, k_3 be an orthonormal basis of $\mathfrak{su}(2)$ with respect to $B = -\frac{1}{2\lambda^2} B_{\mathfrak{su}(2)}$. We choose a basis such that

$$\varphi(k_1) = \lambda(\sqrt{3}e_{13} - e_{24} - e_{35}), \quad \varphi(k_2) = \lambda(-\sqrt{3}e_{12} + e_{34} - e_{25}),$$
$$\varphi(k_3) = \lambda(e_{23} + 2e_{45}).$$

The $(\mathfrak{k}, B)$-extension defines a naturally reductive structure on an 8-dimensional 2-step nilpotent Lie group, as described in [16, Sec. 2.2]. On this homogeneous space, we obtain a 1-parameter family of naturally reductive structures, with $\lambda > 0$ as parameter.

The canonical base space is $(SU(2) \times SU(2))/S^1_\alpha \times \mathbb{R}^2$. The Lie algebra $\mathfrak{k}$ is 1-dimensional. Let $k \in \mathfrak{k}$ be a unit vector. To keep the notation concise, we consider $\mathfrak{su}(2) \cong \mathfrak{sp}(1) \subset \mathfrak{gl}(1, \mathbb{H})$. We denote by i, j, k the imaginary quaternions, *i.e.*, $i^2 = j^2 = k^2 = ijk = -1$. The non-degenerate symmetric bilinear form on $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ is given by $-\frac{1}{8\lambda_1^2} B_{\mathfrak{sp}(1)} \oplus -\frac{1}{8\lambda_2^2} B_{\mathfrak{sp}(1)}$, where $B_{\mathfrak{sp}(1)}$ denotes the Killing form of $\mathfrak{sp}(1)$. Let

$$e_1 := (\lambda_1 j, 0), \quad e_3 := (0, \lambda_2 j), \quad e_5 := (\alpha^2\lambda_1^2 + \lambda_2^2)^{-1/2}\left(\lambda_1^2 \alpha i, -\lambda_2^2 i\right),$$
$$e_2 := (\lambda_1 k, 0), \quad e_4 := (0, \lambda_2 k), \quad h := \frac{\lambda_1\lambda_2}{\sqrt{\alpha^2\lambda_1^2 + \lambda_2^2}}(i, \alpha i),$$

where $e_1, \dots, e_5$ is an orthonormal basis of $\mathfrak{m} := h^\perp$ with respect to the metric above and $\alpha \in \mathbb{R}\backslash\{0\}$. Let $\{e_6, e_7\}$ be an orthonormal basis of $\mathbb{R}^2$. For $k \in \mathfrak{k}$ a unit vector, we have

$$\varphi(k) = c_1 \mathrm{ad}(h)|_{\mathfrak{m}} + c_2 \mathrm{ad}(e_5)|_{\mathfrak{m}} + c_3 e_{67},$$

where e_6, e_7 is an orthonormal basis of $\mathbb{R}^2$. The $(\mathfrak{k}, B)$-extension is reducible precisely when $c_3 = 0$ or $c_1 = c_2 = 0$. If $c_1 \neq 0$, then the $(\mathfrak{k}, B)$-extension defines a naturally reductive structure on

$$(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha,$$

where the image of $\mathrm{Lie}(\mathbb{R}_\alpha)$ in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is spanned by h and in $\mathrm{Lie}(H^3)$ by the center; see [16, Sec. 2.3] for more details. Using an automorphism of $\mathfrak{g}$ we can arrange that $c_1, c_2, c_3 \geq 0$. Under these extra assumptions all the naturally reductive structures are non-isomorphic. This $(\mathfrak{k}, B)$-extension is regular for all values of α even though the base space is only regular when $\alpha \in \mathbb{Q}$. For every $\alpha \in \mathbb{R}\backslash\{0\}$, we obtain in this way a 5-parameter family of naturally reductive structures with $\lambda_1, \lambda_2 > 0$ and c_1, c_2, c_3 as parameters.

If $c_1 = 0$, then the naturally reductive structure is only regular when $\alpha = q \in \mathbb{Q}$. In this case, the $(\mathfrak{k}, B)$-extension defines a naturally reductive structure on

$$\big(SU(2) \times SU(2)\big)/S^1_q \times H^3.$$

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On this homogeneous space we obtain a 4-parameter family of naturally reductive structures, with $\lambda_1, \lambda_2 > 0$ and c_2, c_3 as parameters.

For both spaces, we can replace one S^2 factor by its symmetric dual $SL(2,\mathbb{R})/S^1$.

The canonical base space is $SU(3)/(S^1 \times S^1)$. We pick the following orthonormal basis with respect to $\overline{g} = \frac{-1}{12\lambda^2} B_{\mathfrak{su}(3)}$ of $\mathfrak{h} := \mathrm{Lie}(S^1 \times S^1)$:

$$h_1 := \begin{pmatrix} i\lambda & 0 & 0 \\ 0 & -i\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad h_2 := \begin{pmatrix} \frac{-i\lambda}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{-i\lambda}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2i\lambda}{\sqrt{3}} \end{pmatrix}.$$

In this case, we have $\varphi(\mathfrak{k}) = \mathrm{ad}(\mathfrak{h})|_{\mathfrak{m}}$. The only freedom is in the choice of a metric B on $\mathfrak{k}$. For $x_1, x_2, x_3 \in \mathbb{R}$, we define a quadratic form on $\mathcal{Z}(\mathfrak{u}(3))$ by

$$\begin{pmatrix} ia & 0 & 0 \\ 0 & ib & 0 \\ 0 & 0 & ic \end{pmatrix} \longmapsto \frac{x_1 a^2 + x_2 b^2 + x_3 c^2}{\lambda^2}.$$

Restricting this to $\mathfrak{h}$ gives us the following symmetric bilinear form in the basis (h_1, h_2):

$$B_{x_1,x_2,x_3} := \begin{pmatrix} x_1 + x_2 & \frac{1}{\sqrt{3}}(-x_1 + x_2) \\ \frac{1}{\sqrt{3}}(-x_1 + x_2) & \frac{1}{3}(x_1 + x_2 + 4x_3) \end{pmatrix}.$$

This is positive definite if and only if its trace and determinant are positive; *i.e.*,

$$\frac{3}{4}\mathrm{tr}(B_{x_1,x_2,x_3}) = x_1 + x_2 + x_3 > 0,$$
$$\frac{3}{4}\det(B_{x_1,x_2,x_3}) = x_1x_2 + x_2x_3 + x_1x_3 > 0.$$

This parametrizes exactly all metric tensors on $\mathfrak{h}$. From Proposition 2.16, we know that two of these metrics induce an isomorphic naturally reductive structure precisely when they are conjugate by an automorphism of $\mathfrak{su}(3)$, which preserves $\mathfrak{h}$, *i.e.*, an element of the normalizer $N_{\mathfrak{su}(3)}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{su}(3)$. Two metrics are conjugate by an element of $N_{\mathfrak{su}(3)}(\mathfrak{h})$ if and only if they are conjugate by an element of the Weyl group of $\mathfrak{su}(3)$. The Weyl group of $\mathfrak{su}(3)$ is isomorphic to S_3, and the action of the Weyl group on $\mathfrak{h}$ is given by permuting the diagonal entries. Therefore, the induced Weyl group action on the metrics B_{x_1,x_2,x_3} simply permutes the indices. We see that under the conditions $x_3 \geq x_2 \geq x_1$ every S_3-orbit of these metrics is parametrized exactly ones. We still need to know when Lemma 2.14(ii) is satisfied. The curvature of the $(\mathfrak{k}, B)$-extension is given by

$$R = -\mathrm{ad}(h_1)|_{\mathfrak{m}} \odot \mathrm{ad}(h_1)|_{\mathfrak{m}} - \mathrm{ad}(h_2)|_{\mathfrak{m}} \odot \mathrm{ad}(h_2)|_{\mathfrak{m}} + \sum_{i,j=1}^{2} (B^{-1})_{ij}\mathrm{ad}(h_i)|_{\mathfrak{m}} \odot \mathrm{ad}(h_j)|_{\mathfrak{m}}.$$

In the basis (h_1, h_2), this becomes

$$R = 6\lambda^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + 6\lambda^2 \det(B)^{-1} \begin{pmatrix} \frac{1}{3}(x_1 + x_2 + 4x_3) & \frac{1}{\sqrt{3}}(x_1 - x_2) \\ \frac{1}{\sqrt{3}}(x_1 - x_2) & x_1 + x_2 \end{pmatrix}.$$

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This has full rank if and only if $x_1x_2 + x_2x_3 + x_1x_3 - x_1 - x_2 - x_3 + \frac{3}{4} \neq 0$. Under this condition the canonical base space is equal to $SU(3)/(S^1 \times S^1)$ by Lemma 2.14. The $(\mathfrak{k}, B)$-extension is always regular and irreducible. Under the above conditions, we obtain a 4-parameter family of naturally reductive structures on $SU(3)$, with $\lambda > 0$ and x_1, x_2, x_3 as parameters. None of these structures are isomorphic under the condition $x_3 \geq x_2 \geq x_1$.

The canonical base space is $\{*\}$. We write $\mathfrak{g} = \{0\}$ for the 0-dimensional Lie algebra. Let $\mathfrak{k} = \mathfrak{su}(3)$ and let $B = \frac{-1}{\lambda^2} B_{\mathfrak{su}(3)}$. Let $x_1, \ldots, x_8$ be an orthonormal basis of $\mathfrak{su}(3)$ with respect to B. The torsion and curvature are given by

$$T(x, y, z) = 2B([x, y], z) \quad \text{and} \quad R = \sum_{i=1}^{8} \operatorname{ad}(x_i) \odot \operatorname{ad}(x_i).$$

The infinitesimal model is always irreducible and regular and defines a 1-parameter family of naturally reductive structures on $\mathbb{R}^8$ with $\lambda > 0$ as parameter; see [16].

We summarize the classification of all 7- and 8-dimensional naturally reductive spaces of Type II in the following theorem.

Theorem 4.5 *All 7- and 8-dimensional simply connected naturally reductive spaces of Type II for which the semi-simple factor of the canonical base space is compact are presented in Tables 7 and 8, respectively. Furthermore, for every space, the canonical base space is listed, the dimension of the parameter space of naturally reductive structures of Type II and whether partial dual spaces exist or not.*

Notation 4.6 In Tables 7 and 8, H^n denotes the n-dimensional Heisenberg group and QH^7 denotes the 7-dimensional quaternionic Heisenberg group. The subscripts $q_i \in \mathbb{Q}$ and $\alpha \in \mathbb{R}$ denote parameters that determine the subgroup; see Section 3 for the details. Lastly, for $\varphi\colon \mathfrak{k} \to \mathfrak{so}(n)$, a Lie algebra representation $Nil(\varphi)$ denotes a naturally reductive structure on the 2-step nilpotent Lie group as described in [8] and [16, Sec. 2.2].

A Tables

Tables 5 and 6 are referred to in Theorem 3.6, and they contain all compact simply connected naturally reductive spaces of Type I in dimensions 7 and 8. In the first column Lie(G) is the transvection algebra of the naturally reductive space. The second column indicates if there exist non-compact partial dual spaces. The third column indicates the number of parameters of naturally reductive structures of Type I there exist on the space.

Tables 7 and 8 are referred to in Theorem 4.5. They contain all simply connected naturally reductive spaces of Type II for which the semi-simple factor of the canonical base space is compact. The second column gives the canonical base space and the third column indicates if partial dual spaces exist. In the fourth column, the number of parameters of naturally reductive structures of Type II are given.

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G/H	dual	# param.
$SU(3)/S^1_q$	✗	1
$SU(2)^3/(S^1_{q_1} \times S^1_{q_2})$	✓	3
$SO(5)/SO(3)_{\mathrm{ir}}$	✗	1
$SO(5)/SO(3)_{\mathrm{st}}$	✗	1
$Sp(2)/Sp(1)_{\mathrm{st}}$	✗	1
$(SU(3) \times SU(2))/(SU(2) \times S^1_q)$	✓	2
$(SU(3) \times SU(2))/(SU(2)_\Delta \times S^1)$	✓	2
$(\mathrm{Spin}(5) \times SU(2))/(SU(2)_\Delta \times SU(2))$	✓	2
$SU(4)/SU(3)$	✗	1
$\mathrm{Spin}(7)/G_2$	✗	1

Table 5: 7-dimensional naturally reductive spaces of Type I.

G/H	dual	# param.
$SU(3)$	✗	1
$SU(2)^3/S^1_{q_1,q_2}$	✓	3
$SO(5)/(SO(2) \times SO(2))$	✗	1
$(SU(3) \times SU(2))/SU(2)_{\mathrm{st} \times \mathrm{id}}$	✗	2
$(SU(3) \times SU(2))/SU(2)_{\mathrm{ir} \times \mathrm{id}}$	✓	2
$(SU(3) \times SU(3))/SU(3)_\Delta$	✓	2

Table 6: 8-dimensional naturally reductive spaces of Type I.

G/H	canonical base space	dual	# param.
H^7	$\mathbb{R}^6$	✗	3
QH^7	$\mathbb{R}^4$	✗	1
$S^2 \times H^5$	$S^2 \times \mathbb{R}^4$	✓	4
$S^2 \times S^2 \times H^3$	$S^2 \times S^2 \times \mathbb{R}^2$	✓	5
$\mathbb{C}P^2 \times H^3$	$\mathbb{C}P^2 \times \mathbb{R}^2$	✓	3
$Sp(2)/Sp(1)_{\mathrm{st}}$	$Sp(2)/(SU(2) \times S^1)$	✗	2
$SU(3)/S^1_q$	$SU(3)/(S^1 \times S^1)$	✗	2
$SO(5)/SO(3)_{\mathrm{st}}$	$SO(5)/(SO(3) \times SO(2))$	✓	2
$SU(4)/SU(3)$	$SU(4)/S(U(1) \times U(3))$	✓	2
$(SU(3) \times SU(2))/(SU(2) \times S^1_q)$	$\mathbb{C}P^2 \times S^2$	✓	3
$SU(2)^3/(S^1_{q_1} \times S^1_{q_2})$	$S^2 \times S^2 \times S^2$	✓	4

Table 7: 7-dimensional naturally reductive spaces of Type II.

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G/H	canonical base space	dual	# param.
$Nil(\mathbb{R}^2 \to \mathfrak{so}(6))$	$\mathbb{R}^6$	✗	5
$Nil(\varphi_{\mathrm{ir}} : \mathfrak{so}(3) \to \mathfrak{so}(5))$	$\mathbb{R}^5$	✗	1
$Nil(\mathfrak{u}(2) \to \mathfrak{so}(4))$	$\mathbb{R}^4$	✗	2
$(SU(2) \times Nil(\mathbb{R}^2 \to \mathfrak{so}(4)))/\mathbb{R}$	$S^2 \times \mathbb{R}^4$	✓	6
$SU(2) \times H^5$	$S^2 \times \mathbb{R}^4$	✓	5
$SU(2) \times H^5$	$SU(2) \times \mathbb{R}^4$	✗	4
$SU(3)/SU(2)_{\mathrm{st}} \times H^3$	$\mathbb{C}P^2 \times \mathbb{R}^2$	✓	4
$(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha$	$S^2 \times S^2 \times \mathbb{R}^2$	✓	6
$(SU(2) \times SU(2))/S^1_q \times H^3$	$S^2 \times S^2 \times \mathbb{R}^2$	✓	5
$(SU(2) \times SU(2) \times H^3)/\mathbb{R}_\alpha$	$(SU(2) \times SU(2))/S^1_\alpha \times \mathbb{R}^2$	✓	5
$(SU(2) \times SU(2))/S^1_q \times H^3$	$(SU(2) \times SU(2))/S^1_q \times \mathbb{R}^2$	✓	4
$SU(3)/SU(2)_{\mathrm{st}} \times H^3$	$SU(3)/SU(2)_{\mathrm{st}} \times \mathbb{R}^2$	✗	3
$SU(2) \times S^2 \times H^3$	$SU(2) \times S^2 \times \mathbb{R}^2$	✓	5
$SU(3)$	$SU(3)/S^1_q$	✗	3
$SU(2)^3/S^1_{q_3}$	$SU(2)^3/(S^1_{q_1} \times S^1_{q_2})$	✓	4
$SU(2)^3/S^1_{q_1,q_2}$	$(SU(2) \times SU(2))/S^1_q \times S^2$	✓	4
$(SU(3) \times SU(2))/SU(2)_{\mathrm{st}\times\mathrm{id}}$	$(SU(3) \times SU(2))/(SU(2)_\Delta \times S^1)$	✓	3
$(SU(3) \times SU(2))/SU(2)_{\mathrm{st}}$	$(SU(3) \times SU(2))/(SU(2)_{\mathrm{st}} \times S^1_q)$	✓	3
$SU(3)$	$SU(3)/(S^1 \times S^1)$	✗	4
$SU(3)/SU(2)_{\mathrm{st}} \times SU(2)$	$\mathbb{C}P^2 \times S^2$	✓	5
$SU(2)^3/S^1_{q_1,q_2}$	$S^2 \times S^2 \times S^2$	✓	6
$\mathbb{R}^8$	$\{*\}$	✗	1

Table 8: 8-dimensional naturally reductive spaces of Type II.

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References

[1] I. Agricola, A. C. Ferreira, and T. Friedrich, *The classification of naturally reductive homogeneous spaces in dimensions $n \leq 6$*. Differential Geom. Appl. 39(2015), 59–92. https://doi.org/10.1016/j.difgeo.2014.11.005

[2] I. Agricola, A. C. Ferreira, and R. Storm, *Quaternionic Heisenberg groups as naturally reductive homogeneous spaces*. Int. J. Geom. Methods Mod. Phys. 12(2015), 1560007, 10. https://doi.org/10.1142/S0219887815600075

[3] W. Ambrose and I. M. Singer, *On homogeneous Riemannian manifolds*. Duke Math. J. 25(1958), 647–669.

[4] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*. Graduate Texts in Mathematics, 98, Springer, New York, 1985. https://doi.org/10.1007/978-3-662-12918-0

[5] E. Cartan, *Sur une classe remarquable d'espaces de Riemann*. Bull. Soc. Math. France 54(1926), 214–264. http://www.numdam.org/item?id=BSMF_1926__54__214_0

[6] J. E. D'Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*. Mem. Amer. Math. Soc. 18(1979), 215, iii + 72.
[7] T. Friedrich, I. Kath, A. Moroianu, and U. Semmelmann, *On nearly parallel G_2-structures*. J. Geom. Phys. 23(1997), 259–286. https://doi.org/10.1016/S0393-0440(97)80004-6
[8] C. S. Gordon, *Naturally reductive homogeneous Riemannian manifolds*. Canad. J. Math. 37(1985), 467–487. https://doi.org/10.4153/CJM-1985-028-2
[9] A. Kaplan, *On the geometry of groups of Heisenberg type*. Bull. London Math. Soc. 15(1983), 35–42. https://doi.org/10.1112/blms/15.1.35
[10] S. Klaus, *Diploma Thesis (in German): Einfach-zusammenhängende kompakte homogene Räume bis zur Dimension 9*. 06 1988. https://doi.org/10.13140/2.1.4088.8324
[11] B. Kostant, *On differential geometry and homogeneous spaces. I, II*. Proc. Natl Acad. Sci. USA 42(1956), 258–261, 354–357.
[12] O. Kowalski, *Counterexample to the "second Singer's theorem"*. Ann. Global Anal. Geom. 8(1990), 2, 211–214. https://doi.org/10.1007/BF00128004
[13] O. Kowalski and L. Vanhecke, Four-dimensional naturally reductive homogeneous spaces. *Rend. Sem. Mat. Univ. Politec. Torino*, (Special Issue): 223–232 (1984), 1983. Conference on differential geometry on homogeneous spaces (Turin, 1983).
[14] O. Kowalski and L. Vanhecke, *Classification of five-dimensional naturally reductive spaces*. Math. Proc. Cambridge Philos. Soc. 97(1985), 3, 445–463. https://doi.org/10.1017/S0305004100063027
[15] K. Nomizu, *Invariant affine connections on homogeneous spaces*. Amer. J. Math. 76(1954), 33–65.
[16] R. Storm, *A new construction of naturally reductive spaces*. Transform. Groups 23(2018), 2, 527–553. https://doi.org/10.1007/s00031-017-9446-5
[17] R. Storm, *Structure theory of naturally reductive spaces*. Differential Geom. Appl. 64(2019), 174–200. http://www.sciencedirect.com/science/article/pii/S0926224518302316
[18] F. Tricerri, *Locally homogeneous Riemannian manifolds*. Rend. Sem. Mat. Univ. Politec. Torino 50(1993), 4, 411–426. 1992. Differential geometry (Turin, 1992).
[19] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*. London Mathematical Society Lecture Note Series, 83, Cambridge University Press, Cambridge, 1983. https://doi.org/10.1017/CBO9781107325531
[20] K. Tsukada, *Totally geodesic hypersurfaces of naturally reductive homogeneous spaces*. Osaka J. Math. 33(1996), 3, 697–707. http://projecteuclid.org/euclid.ojm/1200787097
[21] B. Wilking, *The normal homogeneous space* $(\mathrm{SU}(3) \times \mathrm{SO}(3))/\mathrm{U}^{\bullet}(2)$ *has positive sectional curvature*. Proc. Amer. Math. Soc. 127(1999), 4, 1191–1194. https://doi.org/10.1090/S0002-9939-99-04613-4

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