Canad. Math. Bull. Vol. **42** (3), 1999 pp. 291–297

Spaces of Quasi-Measures

D. J. Grubb and Tim LaBerge

Abstract. We give a direct proof that the space of Baire quasi-measures on a completely regular space (or the space of Borel quasi-measures on a normal space) is compact Hausdorff. We show that it is possible for the space of Borel quasi-measures on a non-normal space to be non-compact. This result also provides an example of a Baire quasi-measure that has no extension to a Borel quasi-measure. Finally, we give a concise proof of the Wheeler-Shakmatov theorem, which states that if X is normal and $\dim(X) \leq 1$, then every quasi-measure on X extends to a measure.

1 Introduction

Let $\mathcal{O}$ be a collection of subsets of a set X and set $\mathcal{C} = \{X \setminus O : O \in \mathcal{O}\}$. If $\mathcal{O}$ (and hence $\mathcal{C}$) is closed under finite intersections and unions and $X, \varnothing \in \mathcal{O}$, we say $\mathcal{O} \cup \mathcal{C}$ is a quasi-algebra. If this occurs, then an $\mathcal{O}$-*quasi-measure* is a set function $\mu \colon \mathcal{O} \cup \mathcal{C} \to [0,1]$ that satisfies

1. $\mu(\varnothing) = 0$.
2. If $O, P \in \mathcal{O}$ and $O \subseteq P$, then $\mu(O) \leq \mu(P)$.
3. If $O, P \in \mathcal{O}$ and $O \cap P = \varnothing$, then $\mu(O \cup P) = \mu(O) + \mu(P)$.
4. If $O, P \in \mathcal{O}$ and $O \cup P = X$, then $\mu(O) + \mu(P) = 1 + \mu(O \cap P)$.
5. If $O \in \mathcal{O}$ and $\epsilon > 0$ is given, then there is an $C \in \mathcal{C}$ such that $C \subseteq O$ and $\mu(O \setminus C) < \epsilon$.
6. If $O \in \mathcal{O}$, then $\mu(X \setminus O) = 1 - \mu(O)$.

We will denote the collection of all $\mathcal{O}$-quasi-measures on X by $\mathrm{QM}(\mathcal{O})$. In our intended applications, X will be a completely regular topological space and $\mathcal{O}$ will be either the collection $\mathcal{V}$ of cozero subsets of X or the collection $\mathcal{U}$ of open subsets of X. In these situations, it is not difficult to see that $\mathrm{QM}(\mathcal{V})$ and $\mathrm{QM}(\mathcal{U})$ are the collections of Baire and Borel quasi-measures on X, respectively, as defined in [B1], [B2], or [W].

If $\mathcal{O}$ generates a quasi-algebra on X, we can topologize $\mathrm{QM}(\mathcal{O})$ as follows. For each $O \in \mathcal{O}$ and $\alpha \in [0,1]$, set $O^*_\alpha = \{\mu : \mu \in \mathrm{QM}(\mathcal{O}) \text{ and } \mu(O) > \alpha\}$. We use the family $\{O^*_\alpha : O \in \mathcal{O} \text{ and } \alpha \in [0,1]\}$ as a subbasis for the desired topology. For reasons explained below, we call this the w^*-topology on $\mathrm{QM}(\mathcal{O})$. Clearly, a net $\{\mu_\alpha\}$ converges to μ in the w^*-topology if and only if $\liminf \mu_\alpha(O) \geq \mu(O)$ for all $O \in \mathcal{O}$.

This notion of quasi-measure is due to Aarnes (see [A1]). Boardman introduced the ideas of Baire and Borel quasi-measures in his dissertation [B1] and paper [B2]. It is crucial to note that even if X is normal or compact, a quasi-measure need not be the restriction of a finitely additive measure. The first example of a quasi-measure that is not the restriction of a finitely additive measure was given by Aarnes in [A1]. Moreover, as can be seen from results in [W] and Fremlin [F], an $\mathcal{O}$-quasi-measure can be extended to a measure on an algebra containing $\mathcal{O}$ exactly when it is subadditive on $\mathcal{O}$.

Received by the editors August 20, 1997; revised February 18, 1998.
AMS subject classification: 28C15.

https://doi.org/10.4153/CMB-1999-035-5 Published online by Cambridge University Press

The theory of non-linear integration with respect to a Borel quasi-measure on a compact Hausdorff space X is developed by Aarnes in [A1]. (Our choice of terminology is justified by the fact that the w^*-topology induced by these integrals agrees with what we are calling the w^*-topology on $\mathrm{QM}(\mathcal{U})$.) In the same paper, Aarnes also establishes a representation theory that shows that in this case, $\mathrm{QM}(\mathcal{U})$ with the w^*-topology corresponds to the collection of quasi-linear functionals on $C(X)$ with the topology of pointwise convergence. (A functional $\rho\colon C(X) \to [0,1]$ is *quasi-linear* if $\rho(\mathbf{1}) = 1$ and ρ is linear on every singly generated norm closed subalgebra of $C(X)$.) By an Alaoglu argument, the collection of quasi-linear functionals is compact Hausdorff, so $\mathrm{QM}(\mathcal{U})$ with the w^*-topology is compact Hausdorff if X is. (Boardman established similar representation and integration results for Baire quasi-measures on completely regular spaces in [B1] and [B2].)

In [W], Wheeler showed that if X is completely regular, then the collection of Baire quasi-measures $\mathrm{QM}(\mathcal{V})$ on X corresponds to the collection of Borel quasi-measures $\mathrm{QM}(\mathcal{U})$ on βX, the Stone-Čech compactification of X. Thus, if X is completely regular, $\mathrm{QM}(\mathcal{V})$ is also compact Hausdorff. He also showed that if X is normal, every Baire quasi-measure extends uniquely to a Borel quasi-measure, so that in this case, the collection $\mathrm{QM}(\mathcal{U})$ of Borel quasi-measures is also compact Hausdorff.

We say that an $\mathcal{O}$-quasi-measure is *simple* if it takes only the values 0 and 1. We denote the collection of simple $\mathcal{O}$-quasi-measures by $\mathrm{QM}_s(\mathcal{O})$. In [A2], Aarnes uses a projective limit argument to show that if X is compact, then $\mathrm{QM}_s(\mathcal{U})$ is compact. By Wheeler's arguments, the same is true of $\mathrm{QM}_s(\mathcal{V})$ if X is completely regular and of $\mathrm{QM}_s(\mathcal{U})$ if X is normal.

Our first goal in this paper is to provide a unified, direct, topological proof of these results. Our approach is based on that of Topsoe [T]. Our second goal is to construct an example showing that if X is not normal, then the collection of Borel quasi-measures on X with the w^*-topology may be non-compact. This example is based on a construction of a Baire quasi-measure on X that has no extension to a Borel quasi-measure. These examples show that Wheeler's normality assumptions in [W] are essential. Finally, we give a concise proof of a general version of the Wheeler-Shakmatov result (see [W] and [S]) which states that if X is normal and $\dim(X) \leq 1$, then every (Baire or Borel) quasi-measure on X is subadditive, and hence extends to a measure.

2 Compactness in Spaces of Quasi-Measures

In this section, we provide a direct proof of the compactness results described above. In order to unify the proofs of these results, we will require our quasi-algebras to satisfy an additional property. By definition, if X is normal, then $\mathcal{U}$ has the property that if H and K are complements of elements of $\mathcal{U}$ and H and K are disjoint, then there are pairwise disjoint elements of $\mathcal{U}$ that separate H and K. The corresponding fact is also true for $\mathcal{V}$ in the completely regular case. Thus, if $\mathcal{O}$ generates an quasi-algebra $\mathcal{O} \cup \mathcal{C}$, we say that this quasi-algebra is *normal* if pairwise disjoint elements of $\mathcal{C}$ can be separated by pairwise disjoint elements of $\mathcal{O}$.

The following lemma is adapted from [T].

Lemma 2.1 *Suppose X is a set and that $\mathcal{O}$ is a collection of subsets of X that generates a normal quasi-algebra $\mathcal{O} \cup \mathcal{C}$. Suppose further that $\nu\colon \mathcal{O} \cup \mathcal{C} \to [0,1]$ satisfies the first four*

https://doi.org/10.4153/CMB-1999-035-5 Published online by Cambridge University Press

$\mathcal{O}$-quasi-measure axioms. Then there is an $\mathcal{O}$-quasi-measure μ such that $\mu(O) \leq \nu(O)$ for all $O \in \mathcal{O}$. Moreover, if ν takes only the values 0 and 1, then μ is simple.

Proof Given ν as above, our plan is to "regularize" ν in two steps. First, define for each $C \in \mathcal{C}$, $\tau(C) = \inf\{\nu(O) : O \in \mathcal{O} \text{ and } C \subseteq O\}$. Then the following are true when $C, D \in \mathcal{C}$:

(a) $\tau(\varnothing) = 0$
(b) If $C \subseteq D$, then $\tau(C) \leq \tau(D)$.
(c) If $C \cap D = \varnothing$, then $\tau(C \cup D) = \tau(C) + \tau(D)$.
(d) If $C \cup D = X$, then $\tau(C) + \tau(D) = 1 + \tau(C \cap D)$.

(a) and (b) are trivial, and (c) follows because $\mathcal{O} \cup \mathcal{C}$ is normal. To see (d), suppose that we have $C, D \in \mathcal{C}$ with $C \cup D = X$ and a $W \in \mathcal{O}$ such that $C \cap D \subseteq W$. Then $O = C \cup (W \setminus D) \in \mathcal{O}$, $P = D \cup (W \setminus C) \in \mathcal{O}$, $C \cap D \subseteq O \cap P$, and $O \cap P = W$, so

$$\inf_{\substack{W \in \mathcal{O} \\ C \cap D \subseteq W}} \nu(W) = \inf_{\substack{C \subseteq O \in \mathcal{O} \\ D \subseteq P \in \mathcal{O}}} \nu(O \cap P).$$

Thus,

$$\tau(C \cap D) = \inf_{\substack{C \subseteq O \in \mathcal{O} \\ D \subseteq P \in \mathcal{O}}} \nu(O \cap P) = \inf_{\substack{C \subseteq O \in \mathcal{O} \\ D \subseteq P \in \mathcal{O}}} \nu(O) + \nu(P) - 1 = \tau(C) + \tau(D) - 1.$$

Now, for $O \in \mathcal{O}$, define $\mu(O) = \sup\{\tau(C) : C \in \mathcal{C} \text{ and } C \subseteq O\}$. For $C \in \mathcal{C}$, define $\mu(C) = 1 - \mu(O)$; we claim that μ is an $\mathcal{O}$-quasi-measure. Clearly, μ satisfies axioms (1), (2), and (3).

Suppose that $O, P \in \mathcal{O}$ and that $O \cup P = X$. Because $\mathcal{O} \cup \mathcal{C}$ is normal, we can find $C, D \in \mathcal{C}$ with $C \subseteq O$, $D \subseteq P$, and $C \cup D = X$. Then whenever $C', D' \in \mathcal{C}$ and $C \subseteq C' \subseteq O$ and $D \subseteq D' \subseteq P$, we have $\tau(C') + \tau(D') = 1 + \tau(C' \cap D') \leq 1 + \mu(O \cap P)$, so $\mu(O) + \mu(P) \leq 1 + \mu(O \cap P)$. Conversely, if $C \subseteq O \cap P$, we can find $D, D' \in \mathcal{C}$ with $D \subseteq O$, $D' \subseteq P$, $D \cup D' = X$, and $C \subseteq D \cup D'$. This gives $1 + \mu(O \cap P) \leq \mu(O) + \mu(P)$, and (4) follows.

To show (5), suppose $O \in \mathcal{O}$ and that $\epsilon > 0$ is given. Pick $C \in \mathcal{C}$ with $C \subseteq O$ and $\tau(C) > \mu(O) - \epsilon/2$. Then whenever $D \in \mathcal{C}$ and $D \subseteq O \setminus C$, we have $C \cap D = \varnothing$ and $C \cup D \subseteq O$, so that $\tau(C) + \tau(D) = \tau(C \cup D) \leq \mu(O)$. This gives $\tau(D) \leq \mu(O) - \tau(C) < \epsilon/2$, so $\mu(O \setminus C) < \epsilon$.

Thus, μ is an $\mathcal{O}$-quasi-measure. By definition, whenever $O \in \mathcal{O}$, we have $\mu(O) \leq \nu(O)$ and μ is clearly simple if ν is two-valued, so the proof is complete. ■

Theorem 2.2 *Suppose $\mathcal{O}$ generates a normal quasi-algebra. Then* QM($\mathcal{O}$) *is compact Hausdorff in the w^*-topology.*

Proof We first show that QM($\mathcal{O}$) is Hausdorff. Suppose that $\mu, \nu \in$ QM($\mathcal{O}$) and that $\mu \neq \nu$. Then there is a $O \in \mathcal{O}$ and an $\alpha \in [0, 1]$ such that (without loss of generality) $\mu(O) < \alpha < \nu(O)$. Find $C \in \mathcal{C}$ with $C \subseteq O$ and $\nu(C) > \alpha$. By normality, there are $P \in \mathcal{O}$ and $D \in \mathcal{C}$ such that $C \subseteq P \subseteq D \subseteq O$. Then $\nu(P) > \alpha$, $\mu(P) \leq \mu(O) < \alpha$, and

$\nu(X \setminus D) < 1 - \alpha < \mu(X \setminus D)$. Thus, $\nu \in P_\alpha^*$ and $\mu \in (X \setminus D)_{1-\alpha}^*$. If $\tau \in (X \setminus D)_{1-\alpha}^*$, then $\tau(D) < \alpha$, so $\tau(P) < \alpha$, thus $\tau \notin P_\alpha^*$. Therefore, $P_\alpha^* \cap (X \setminus D)_{1-\alpha}^* = \varnothing$, and $\mathrm{QM}(\mathcal{O})$ is Hausdorff.

To see that $\mathrm{QM}(\mathcal{O})$ is compact, let $\{\mu_\alpha\}$ be a net. Pick an ultranet $\{\mu_{\alpha_\beta}\}$ and define $\nu(O) = \lim_\beta \mu_{\alpha_\beta}(O)$ for all $O \in \mathcal{O}$. Then ν satisfies the hypotheses of Lemma 2.1, so there is a $\mu \in \mathrm{QM}(\mathcal{O})$ such that $\mu(O) \le \nu(O)$. But then $\liminf \mu_{\alpha_\beta}(O) \ge \mu(O)$ for all $O \in \mathcal{O}$, so the ultranet $\{\mu_{\alpha_\beta}\}$ converges to μ and the proof is complete. ■

We record the most important instances of the theorem in the following corollary.

Corollary 2.3 *If X is completely regular, then $\mathrm{QM}(\mathcal{V})$ and $\mathrm{QM}_s(\mathcal{V})$ are compact Hausdorff in the w^*-topology. If X is normal, then $\mathrm{QM}(\mathcal{U})$ and $\mathrm{QM}_s(\mathcal{U})$ are compact Hausdorff in the w^*-topology.*

3 Non-Compactness in Spaces of Quasi-Measures

In this section, we show that our use of normal quasi-algebras in the previous section is essential, by constructing a non-normal space for which the space of Borel quasi-measures is not compact. We will also construct an example of a Baire quasi-measure on this space that has no extension to a Borel quasi-measure, so that Wheeler's assumption of normality in [W] is also essential.

These results contrast sharply with the situation for ordinary measures. The Bachman-Sultan Theorem (see [BS]) states that if μ is a finitely additive, zero set regular Baire measure, then μ has an extension to a finitely additive, closed set regular Borel measure. Also, since the collection of simple finitely additive Borel measures on a completely regular space X with the w^*-topology corresponds to the Wallman compactification of X, the collection of simple finitely additive Borel measures on X is always compact, although it is Hausdorff only if X is normal.

Our constructions will utilize the one-point compactification of the long line. Let ω_1 be the first uncountable ordinal. The long line is the connected space L obtained by inserting a copy of $(0,1)$ between each ordinal $\alpha \in \omega_1$ and its successor. To obtain the one-point compactification $L \oplus 1$, we adjoin the ordinal ω_1 to L. Because it has a natural order, we can use interval notation to describe subsets of $L \oplus 1$.

Set $X = \big((L \oplus 1) \times [0,1]\big) \setminus \{(\omega_1, 1)\}$, $T = L \times \{1\}$, and $R = \{\omega_1\} \times [0,1]$. Then X is not normal because R and T cannot be separated by disjoint open sets. Also $\beta X = (L \oplus 1) \times [0,1]$.

Example 3.1 *There is a Baire quasi-measure μ on X that does not extend to a Borel quasi-measure.*

Proof We will use Aarnes' method of solid set functions (see [A3]) to define a Borel quasi-measure $\bar{\nu}$ on βX. Set $p = (0,0)$ and $F = R \cup T \cup \{(\omega_1, 1)\}$. Recall that a closed or open subset of a space is *solid* if both it and its complement are connected. Define a solid set

https://doi.org/10.4153/CMB-1999-035-5 Published online by Cambridge University Press

function ν on the solid subsets of βX by

$$\nu(A) = \begin{cases} 0 & \text{if } A \cap F = \varnothing, \\ 1 & \text{if } F \subseteq A, \\ 1 & \text{if } p \in A \text{ and } A \cap F \neq \varnothing. \end{cases}$$

Then, by Theorem 5.1 of [A3], ν extends to a Borel quasi-measure $\bar{\nu}$ on βX. By the correspondence between Baire quasi-measures on X and Borel quasi-measures on βX, see [W], $\bar{\nu}$ induces a Baire quasi-measure μ on X. We claim that μ does not extend to a Borel quasi-measure on βX.

By way of contradiction, suppose that τ is Borel extension of μ. Set $U = X \setminus (R \cup T)$; we claim that $\tau(U) = \varnothing$. To see this, suppose that $K \subseteq U$ and that K is closed. Then K is compact, so there is a zero set Z such that $K \subseteq Z \subseteq U$. Clearly, $\mu(Z) = 0$. Since τ is monotone and extends μ, $\tau(K) = 0$. By inner regularity, $\tau(U) = 0$ also. We also claim that $\tau(T) = 0 = \tau(R)$. This follows from additivity on finite pairwise disjoint unions of closed sets and the fact that $\mu\big((L \oplus 1) \times \{0\}\big) = 1 = \mu(\{0\} \times [0,1])$. But then $\tau(X) = \tau(R) + \tau(U) + \tau(T) = 0$, a contradiction. So μ does not have a Borel extension. ■

Example 3.2 *There is a space X such that* $\mathrm{QM}(\mathcal{U})$ *is non-compact in the w^*-topology.*

Proof Let X be as in the previous example. For each $\alpha \in \omega_1$, define a Borel quasi-measure $\bar{\nu}_\alpha$ on βX as follows. Set $p = (0,0)$ and $F_\alpha = (\{\alpha\} \times [0,1]) \cup ([0,\alpha] \times \{1\})$. Define $\bar{\nu}_\alpha$ on βX by extending the solid set function

$$\bar{\nu}_\alpha(A) = \begin{cases} 0 & \text{if } A \cap F_\alpha = \varnothing, \\ 1 & \text{if } F_\alpha \subseteq A, \\ 1 & \text{if } p \in A \text{ and } A \cap F_\alpha \neq \varnothing. \end{cases}$$

to all closed subsets of βX. Let μ_α be the Baire quasi-measure on X induced by $\bar{\nu}_\alpha$. Since the support of μ_α is compact, we can extend μ_α to a Borel quasi-measure ν_α on X.

Clearly, $\{\nu_\alpha : \alpha \in \omega_1\}$ with the obvious ordering is a net, we claim that it has no convergent subnet. Suppose, by way of contradiction, that ν is the limit of a subnet $\{\nu_{\alpha_\beta}\}$. We first show that $U = X \setminus (R \cup T)$ satisfies $\nu(U) = 0$. If K is any zero set in U, then there is an $\gamma \in \omega_1$ such that $K \subseteq [0,\gamma] \times [0,1]$. Arguing as before, if $\alpha_\beta > \gamma$, we have $\nu_{\alpha_\beta}(K) = 0$. By the definition of convergence in the w^*-topology, $\nu(K) = 0$. By regularity, $\nu(U) = 0$. Since each $\nu_\alpha(R) = 0 = \nu_\alpha(T)$, another application of the definition of w^*-convergence gives $\nu(R) = 0 = \nu(T)$. But then $\nu(X) = \nu(R) + \nu(U) + \nu(T) = 0$, a contradiction. Thus, X is not compact in the w^*-topology. ■

4 Quasi-Measures and Dimension Theory

In [W] and [S], Wheeler and Shakmatov establish a remarkable connection between the existence of quasi-measures and classical dimension theory: suppose X is normal and let $\dim(X)$ denote the Čech-Lebesgue covering dimension of X. Then $\dim(X) \leq 1$ implies that

https://doi.org/10.4153/CMB-1999-035-5 Published online by Cambridge University Press

every (Baire or Borel) quasi-measure on X is subadditive, and hence extends to a measure. In this section, we present a concise proof of a slightly more general version of this result. We will need the following generalization of Čech-Lebesgue covering dimension.

Definition 4.1 *Suppose $\mathcal{O}$ generates a quasi-algebra on X. We say that the $\mathcal{O}$-covering dimension of X is at most* 1 *(and write $\mathcal{O}\text{-dim}(X) \leq 1$) if whenever $O_1, O_2, O_3 \in \mathcal{O}$ and $O_1 \cup O_2 \cup O_3 = X$, then there are $O_1', O_2', O_3' \in \mathcal{O}$ with $O_1' \subseteq O_1$, $O_2' \subseteq O_2$, and $O_3' \subseteq O_3$; $O_1' \cup O_2' \cup O_3' = X$; and $O_1' \cap O_2' \cap O_3' = \varnothing$.*

Clearly, this definition could be extended to define $\mathcal{O}$-covering dimension for any non-negative integer, although we will not require such generality here. We call the collection of O_i''s in the definition of $\mathcal{O}$-covering dimension a *refinement* of the O_i's.

Lemma 4.2 *Suppose μ is an $\mathcal{O}$-quasi-measure on X. Let $O_1, O_2, O_3 \in \mathcal{O}$ with $O_1 \cup O_2 \cup O_3 = X$ and $O_1 \cap O_2 \cap O_3 = \varnothing$. Then $\mu(O_1) + \mu(O_2) + \mu(O_3) \geq 1$.*

Proof Let $O_1, O_2, O_3 \in \mathcal{O}$ be as above. We then have each O_i $(i = 1, 2, 3)$ is the pairwise disjoint union of the sets $X \setminus (O_j \cup O_k)$, $O_i \cap O_j$, and $O_i \cap O_k$, where $1 \leq j < k \leq 3$ and $j \neq i \neq k$. This gives the following three inequalities:

$$\mu(O_1) \geq \mu\big(X \setminus (O_2 \cup O_3)\big) + \mu(O_1 \cap O_2),$$

$$\mu(O_2) \geq \mu\big(X \setminus (O_3 \cup O_1)\big) + \mu(O_2 \cap O_3), \quad \text{and}$$

$$\mu(O_3) \geq \mu\big(X \setminus (O_1 \cup O_2)\big) + \mu(O_3 \cap O_1).$$

The six sets on the right hand side of these inequalities are pairwise disjoint and have union X, so the sum of their measures is one. This gives $\mu(O_1) + \mu(O_2) + \mu(O_3) \geq 1$, as desired. ■

Theorem 4.3 *Suppose $\mathcal{O}$ generates a quasi-algebra on X and that $\mathcal{O}\text{-dim}(X) \leq 1$. Then every $\mathcal{O}$-quasi-measure on X is subadditive on $\mathcal{O}$.*

Proof We prove the contrapositive. Suppose that μ is an $\mathcal{O}$-quasi-measure on X and that μ is not subadditive on $\mathcal{O}$. Then there are $O_1, O_2 \in \mathcal{O}$ such that $\mu(O_1 \cup O_2) > \mu(O_1) + \mu(O_2)$. Find an $O_3 \in \mathcal{O}$ such that $X \setminus (O_1 \cup O_2) \subseteq O_3$ and $\mu(O_1) + \mu(O_2) + \mu(O_3) < 1$. Then any refinement $\{O_1', O_2', O_3'\}$ of the O_i's satisfies $\mu(O_1') + \mu(O_2') + \mu(O_3') < 1$, so by the lemma, $O_1' \cap O_2' \cap O_3' \neq \varnothing$. Thus, it is not the case that $\mathcal{O}\text{-dim}(X) \leq 1$. ■

Corollary 4.4 *If $\mathcal{O}\text{-dim}(X) \leq 1$, then every $\mathcal{O}$-quasi-measure on X extends to a finitely additive measure on an algebra containing $\mathcal{O}$.*

Proof Let μ be a $\mathcal{O}$-quasi-measure on X. Since μ is subadditive on $\mathcal{O}$, we have that $\mu(K) + \mu(L) - 1 \leq \mu(K \cap L)$ for $K, L \in \mathcal{C}$ by considering complements. Now, the proof of Lemma 3 of [F] shows that $\mu(K) \leq \mu(L) + \mu_*(K \setminus L)$ for $K, L \in \mathcal{C}$, where $\mu_*(A) = \sup\{\mu(K') : K' \subseteq A\}$.

https://doi.org/10.4153/CMB-1999-035-5 Published online by Cambridge University Press

We may now define $\sum = \{A \subseteq X : \mu(K) \leq \mu_*(A \cap K) + \mu_*(K \setminus A)$ for all $K \in \mathcal{C}\}$. By standard techniques, $\sum$ is an algebra and μ_* is a finitely additive measure on $\sum$ extending μ. ■

Corollary 4.5 *If X is normal and* $\dim(X) \leq 1$, *then every Borel quasi-measure on X extends to a finitely additive measure on the Borel algebra.*

References

[A1] J. Aarnes, *Quasi-states and quasi-measures.* Adv. Math. **86**(1991), 41–67.

[A2] ———, *Pure quasi-states and extremal quasi-measures.* Math. Ann. **295**(1993), 575–588.

[A3] ———, *Construction of non-subadditive measures and discretization of Borel measures.* Fund. Math. **147**(1995), 213–237.

[BS] G. Bachman and A. Sultan, *On regular extensions of measures.* Pacific J. Math. **86**(1980), 389–395.

[B1] J. Boardman, *Quasi-measures on completely regular spaces.* Doctoral dissertation, Northern Illinois University, 1994.

[B2] ———, *Quasi-measures on completely regular spaces.* Rocky Mountain J. Math. (2) **27**(1997), 447–470.

[F] D. Fremlin, *On quasi-measures.* Preprint.

[S] D. Shakmatov, *Linearity of quasi-states on commutative C^*-algebras of stable rank 1.* Preprint.

[T] F. Topsoe, *Compactness in spaces of measures.* Studia Math. **36**(1970), 195–212.

[W] R. F. Wheeler, *Quasi-measures and dimension theory.* Topology Appl. **66**(1995), 75–92.

Department of Mathematical Sciences
Northern Illinois University
DeKalb, IL 60115
USA
email: grubb@math.niu.edu
Website: http://www.math.niu.edu/˜grubb/

Department of Mathematics
Centenary College
Shreveport, LA 71134
USA
email: laberget@beta.centenary.edu
Website: http://www.centenary.edu/˜laberget/

https://doi.org/10.4153/CMB-1999-035-5 Published online by Cambridge University Press