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THE DENSITY OF SUBSEQUENCES

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Let $A = \{a_1, a_2, \ldots\}$ be an increasing sequence of positive integers whose upper density, θ, is less than 1, Then the counting function of the subsequence

$$\{a_1 = q, a_q = r, a_r = s, \ldots\}$$

does not exceed $\left[\log_{\sqrt{\theta}}(1/n)\right] - 1$.

1. Introduction

In this paper we develop a method to calculate precisely the asymptotic density of a subsequence in terms of the density of its indices and of the original sequence. This technique also allows us to estimate its Schnirelmann density.

2. Preliminary considerations

Let $A = \{a_1, a_2, \ldots\}$ be a strictly increasing sequence of positive integers, and let $A(n)$ count the elements of A which are less than or equal to n. Let $B = \{b_1, b_2, \ldots\}$ also be a strictly increasing sequence of positive integers. Define $A_B = \{a_{b_1}, a_{b_2}, \ldots\}$ to be the subsequence of A whose indices are determined by the elements of B. If

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$i < j$, then $b_i < b_j$ since B is increasing. Since $b_i < b_j$, $a_{b_i} < a_{b_j}$ because A is increasing. So A_B is strictly increasing and $A_B(n)$ is defined.

Define lub $A(n)/n$ as the *upper density* of A , glb $A(n)/n$ as the *Schnirelmann density* of A [1] and $\lim_{n\to\infty} A(n)/n$ as the asymptotic density of A .

3. Density relations

LEMMA 1. *If* θ, δ *and* θ_δ *are the upper densities of* A, B *and* A_B *respectively, then* $\theta_\delta \le \theta \cdot \delta$.

Proof. Let $t = B[A(n)]$. Then $b_t \le A(n) < b_{t+1}$. This implies $a_{b_t} \le n < a_{b_{t+1}}$. So $t \le A_B(n) < t + 1$. Therefore $B[A(n)] = t = A_B(n)$. If $A(n) \ne 0$, then $\theta \cdot \delta \ge (A(n)/n) \cdot (B[A(n)]/A(n)) = A_B(n)/n$. If $A(n) = 0$, then $\theta \cdot \delta > 0 = B(0)/n = B[A(n)]/n = A_B(n)/n$. Therefore $\theta \cdot \delta \ge \theta_\delta$.

REMARK 1. Observe that if α, β and α_β are defined to be the Schnirelmann densities of A, B and A_B , we then conclude that $\alpha_\beta \ge \alpha \cdot \beta$. Let $A + B = \{a_i+b_j \mid a_i \text{ is in } A;\ b_j \text{ is in } B\}$. A is a *basis* if there is a positive integer k such that $A + \ldots + A$ (k times) is the set of positive integers. Schnirelmann has shown that A is a basis when $\alpha > 0$ [1].

So we conclude that if α and β are positive, A_B is a basis. In particular, if $\alpha > 0$, A_A is a basis.

REMARK 2. Observe that if Δ, Γ and Δ_Γ are defined to be the asymptotic densities of A, B and A_B respectively, we then conclude that Δ_Γ exists provided Δ and Γ do and that $\Delta_\Gamma = \Delta \cdot \Gamma = \Gamma_\Delta$, the asymptotic density of B_A . In particular, since $6/\pi^2$ is the asymptotic

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dnesity of the sequence of square-free integers [2], we see that $36/\pi^4$ is the asymptotic density of the sequence of square-free numbers with square-free index.

Let A^0 be the sequence of positive integers. Let $A^1 = A$ and $A^2 = A_A$. In general define A^l to be those elements of A^{l-1} whose indices are exactly the elements of A . This operation is associative; that is, of $k \geq 0$, then A^l is the set of elements of A^{l-k} whose indices are exactly the elements in A^k . For example, if E is the set of even numbers, then $E^l = \{2^l, 2^l \cdot (2), 2^l \cdot (3), \ldots\}$.

LEMMA 2. *If* $\theta[l]$ *is the upper density of* A^l *, then* $\theta[l] \leq \theta^l$.

Proof. Proceed by induction over l . The lemma is true for $l = 1$. Assume $\theta[l] \leq \theta^l$ for some $l \geq 1$. Since $A^{l+1} = A_{A^l}$, $\theta[l+1) = \theta_{\theta[l]}$. But $\theta_{\theta[l]} \leq \theta \cdot \theta^l = \theta^{l+1}$ by Lemma 1.

Designate by a_1^l the first element of A^l . This symbol does not mean exponentiation; however, it obeys the same laws under composition. For example, $a^m_{\left(a_1^n\right)} = a_1^{m+n}$ and if A^n is considered the original sequence, $\left(a_1^n\right)^m = a_1^{mn}$. In the discussion below exponents are used only with upper densities, never with an element of a sequence.

Define $Aa_1 = \left\{a_1, a_1^2, a_1^3, \ldots\right\}$. If $a_1 > 1$, then Aa_1 is increasing and $Aa_1(n)$ is defined. In the sequel θ will denote the upper density of A . Note that if $\theta < 1$ then $a_1 > 1$.

THEOREM. $Aa_1(n) \leq \log_{\sqrt{\theta}}(1/n) - 1$ *provided* $\theta < 1$ *and* $n \geq a_1$.

Proof. Fix n . Proceed by induction on $Aa_1(n)$. Suppose $Aa_1(n) = 1$. Then $a_1 \leq n$ and $1 \leq A(n)$. So $1/n \leq A(n)/n \leq \theta = (\sqrt{\theta})^2$.

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Therefore $2 \le \log_{\sqrt{\theta}}(1/n)$ since $\sqrt{\theta} < 1$. Or

$$Aa_1(n) = 1 \le \log_{\sqrt{\theta}}(1/n) - 1 .$$

Let $Aa_1(n) = k$ and assume the theorem true for all sequences B such that $1 \le Bb_1(n) < k$. Set $B = A^2$. If $Bb_1(n) = 0 < 1$, then $n < b_1 = a_1^2$. Since $a_1 \le n$ by hypothesis, $a_1 \le n < a^2$, which implies $Aa_1(n) = 1$, which has been considered. If $k = Aa_1(n) \le Bb_1(n) = t$, then $a_1^{2k} = b_1^k \le b_1^t \le n < a_1^{k+1}$, which implies $a_1^{k-1} < 1$ since A is strictly increasing. Since A consists only of positive integers, $k - 1 \le 0$ or $Aa_1(n) \le 1$. Therefore $Aa_1(n) = 1$, which has been considered. So if $B = A^2$, we can assume $1 \le Bb_1(n) < k$.

Also $\theta[2] = \theta_\theta \le \theta^2 < \theta < 1$. Since $Bb_1(n) \ge 1$, $b_1 \le n$ as well. Using the induction hypothesis, we can assume $Bb_1(n) \le \log_{\sqrt{\theta[2]}}(1/n) - 1$. But $\log_{\sqrt{\theta[2]}}(1/n) - 1 \le \log_{\sqrt{\theta^2}}(1/n) - 1 = \left(\log_{\sqrt{\theta}}(1/n)\right)/2 - 1$. Depending on whether $Aa_1(n)$ is even or odd, we have

(i) $2Bb_1(n) = Aa_1(n)$, or

(ii) $2Bb_1(n) + 1 = Aa_1(n)$.

If (i), then $Aa_1(n) = 2Bb_1(n) \le \log_{\sqrt{\theta}}(1/n) - 2 < \log_{\sqrt{\theta}}(1/n) - 1$.

If (ii), then

$$Aa_1(n) = 2Bb_1(n) + 1 \le \log_{\sqrt{\theta}}(1/n) - 2 + 1 = \log_{\sqrt{\theta}}(1/n) - 1 .$$

COROLLARY. *The asymptotic density of* Aa_1 *is zero if the upper density of* A *is less than* 1.

Proof. Dividing the inequality in the above theorem by n, we obtain $Aa_1(n)/n \le \left(\log_{\sqrt{\theta}}(1/n)\right)/n - 1/n$. Taking the limit as n approaches infinite, we have our result.

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