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LOCALIZATIONS OF LINKED QUATERNIONIC MAPPINGS

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1. Introduction. Let G and B be abelian groups with G having exponent 2 and a distinguished element -1. In [**7**] we defined a linked quaternionic mapping to be a map $q : G \times G \to B$ satisfying the following properties:

(A) q is symmetric and bilinear

(B) $q(a, a) = q(a, -1)$ for every $a \in G$, and

(L) $q(a, b) = q(c, d)$ implies there exists $x \in G$ such that $q(a, b) = q(a, x)$ and $q(c, d) = q(c, x)$.

A form (of dimension n over q) is a symbol $\varphi = \langle a_1, \dots, a_n \rangle$ with $a_1, \dots, a_n \in G$. The determinant and Hasse invariant of such a form φ are

$$\det \varphi = \prod_i a_i \in G \quad \text{and} \quad s(\varphi) = \prod_{i<j} q(a_i, a_j) \in B.$$

Isometry of one and two dimensional forms is defined by

(1) $\langle a \rangle \simeq \langle b \rangle \Leftrightarrow a = b$ and

(2) $\langle a, b \rangle \simeq \langle c, d \rangle \Leftrightarrow ab = cd$ and $q(a, b) = q(c, d)$.

For forms of dimension $n \geqq 3$, isometry is defined inductively by

$$\langle a_1, \dots, a_n \rangle \simeq \langle b_1, \dots, b_n \rangle \Leftrightarrow \text{there exist } a, b, c_3, \dots, c_n \in G$$

such that

$$\langle a_2, \dots, a_n \rangle \simeq \langle a, c_3, \dots, c_n \rangle,$$
$$\langle b_2, \dots, b_n \rangle \simeq \langle b, c_3, \dots, c_n \rangle \quad \text{and} \quad \langle a_1, a \rangle \simeq \langle b_1, b \rangle.$$

Equivalently, $\langle a_1, \dots, a_n \rangle \simeq \langle b_1, \dots, b_n \rangle \Leftrightarrow$ there exists a finite chain from $\langle a_1, \dots, a_n \rangle$ to $\langle b_1, \dots, b_n \rangle$ each step of which consists of a change of two elements in accordance with (2).

We say that a form φ *represents* $x \in G$ if there exist $x_2, \dots, x_n \in G$ such that $\varphi \simeq \langle x, x_2, \dots, x_n \rangle$. $D(\varphi)$ denotes the set of all elements of G represented by φ. If $\varphi = \langle a_1, \dots, a_n \rangle$ and $\psi = \langle b_1, \dots, b_m \rangle$ their sum and product are

$$\varphi + \psi = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle \quad \text{and}$$
$$\varphi\psi = \langle a_1 b_1, \dots, a_n b_1, \dots, a_n b_m \rangle.$$

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By $a\varphi$ we mean $\langle a\rangle\varphi$ and we denote by $\mathbf{H}$ the binary form $\langle 1, -1\rangle$. Finally, we use the notation $\langle\langle a_1, \ldots, a_n\rangle\rangle$ to denote the n-fold Pfister form $\prod_{i=1}^{n} \langle 1, a_i\rangle$.

For more details on linked quaternionic mappings, see [**7**]. There, this abstract theory of quadratic forms was developed and a ring theoretic description of the class of Witt rings $W(q)$ was given.

The main goal of this paper is to define and study localizations of linked quaternionic mappings in relationship to the classification of quadratic forms. Section 2 is preparatory in nature. The notion of signature is defined and it is shown that every signature σ on q gives rise to a surjective ring homomorphism $\sigma : W(q) \rightarrow \mathbf{Z}$. The kernels of such maps correspond precisely to the prime ideals P of $W(q)$ with $W(q)/P \cong \mathbf{Z}$. A signature is then a generalization of the notion of an ordering on a field. For other generalizations see for example [**5**] or [**6**]. We close Section 2 with a generalization of some of the work done in [**3**]. In particular, we classify linked quaternionic mappings $q : G \times G \rightarrow \{\pm 1\}$ with trivial radical.

In Section 3 the notion of strong signature is defined and we investigate the relationship between strong signatures and signatures. For a linked quaternionic mapping q and a character f on B we notice that the map $q_f : G \times G \rightarrow \{\pm 1\}$ defined $q_f = f \circ q$ is also a linked quaternionic mapping. After studying forms over q_f and forms over the linked quaternionic mapping

$$\bar{q} : \bar{G} \times \bar{G} \rightarrow B(\bar{G} = G/\text{rad}\, q)$$

we prove Theorem 3.8, the main theorem of this paper.

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2. Signatures and the local theory. Throughout this paper $q : G \times G \rightarrow B$ will be a linked quaternionic mapping and without loss of generality we assume that the subgroup generated by the image of $q = B$. A *signature* on q will be a group homomorphism $\sigma : G \rightarrow \{\pm 1\}$ which satisfies the following conditions:

(i) $\sigma(-1) = -1$;

(ii) if $\sigma(a) = 1$ then $\sigma(b) = 1$ whenever $q(b, ab) = 1$.

PROPOSITION 2.1. *Let σ be a signature on q. σ gives rise to a surjective ring homomorphism $\sigma : W(q) \rightarrow \mathbf{Z}$ defined by*

$$\sigma(\langle a_1, \ldots, a_n\rangle) = \sum_{i=1}^{n} \sigma(a_i).$$

Proof. To show σ is well defined suppose

$$\langle a_1, \ldots, a_n\rangle \simeq \langle b_1, \ldots, b_s\rangle + m\mathbf{H}.$$

We induct on n. If $n = 2$ we must show

$$\langle a_1, a_2\rangle \simeq \langle b_1, b_2\rangle \text{ implies } \sigma(a_1) + \sigma(a_2) = \sigma(b_1) + \sigma(b_2).$$

Since $a_1a_2 = b_1b_2$, $\sigma(a_1a_2) = \sigma(b_1b_2)$. Assume first that $\sigma(a_1a_2) = 1$. Here $\sigma(a_1) = \sigma(a_2)$ and $\sigma(b_1) = \sigma(b_2)$. Since $a_1b_1 \in D(\langle 1, a_1a_2\rangle)$ it follows that $\sigma(a_1) = \sigma(b_1)$, thus

$$\sigma(a_1) + \sigma(a_2) = \sigma(b_1) + \sigma(b_2).$$

If $\sigma(a_1a_2) = -1$ then $\sigma(a_1) = -\sigma(a_2)$. Thus $\sigma(b_1) = -\sigma(b_2)$ and consequently

$$\sigma(a_1) + \sigma(a_2) = \sigma(b_1) + \sigma(b_2) = 0.$$

In general, if $\langle a_1, \ldots, a_n\rangle \simeq \langle b_1, \ldots, b_s\rangle + m\mathbf{H}$ there exist $a, b, c_3, \ldots, c_n \in G$ such that

$$\langle a_2, \ldots, a_n\rangle \simeq \langle a, c_3, \ldots, c_n\rangle,$$
$$\langle b_2, \ldots, b_3\rangle + m\mathbf{H} \simeq \langle b, c_3, \ldots, c_n\rangle \quad \text{and}$$
$$\langle a_1, a\rangle \simeq \langle b_1, b\rangle.$$

The desired result now follows by induction. It is easy to see that σ is a ring homomorphism and since $\sigma(n\langle 1\rangle) = n$, σ is surjective.

We will denote the collection of all signatures on q by $X(q)$.

PROPOSITION 2.2. *The prime ideals P of $W(q)$ with $W(q)/P \simeq \mathbf{Z}$ correspond precisely to the kernels of $\sigma : W(q) \to \mathbf{Z}$, $\sigma \in X(q)$.*

Proof. Let $\sigma \in X(q)$. By Proposition 2.1, $\sigma : W(q) \to \mathbf{Z}$ is a surjective ring homomorphism hence $W(q)/\text{Ker } \sigma \simeq \mathbf{Z}$. Conversely, suppose P is a prime ideal of $W(q)$ with $\gamma : W(q)/P \to \mathbf{Z}$ an isomorphism. Note that $\langle a\rangle\langle a\rangle = \langle 1\rangle$ for every $a \in G$ hence $\gamma(\langle a\rangle + P) = \pm 1$. Define $\sigma : G \to \{\pm 1\}$ by $\sigma(a) = \gamma(\langle a\rangle + P)$. σ is a group homomorphism. Since $\gamma(\langle 1, -1\rangle + P) = 0$ we have $\sigma(1) + \sigma(-1) = 0$ hence $\sigma(-1) = -\sigma(1) = -1$. Suppose $\sigma(a) = 1$ and $q(b, ab) = 1$. Then $\langle 1, a\rangle \simeq \langle b, ba\rangle$, thus

$$2 = \sigma(1) + \sigma(a) = \sigma(b) + \sigma(ba).$$

Consequently $\sigma(b) = 1$ and $\sigma \in X(q)$. Clearly

$$\text{Ker } \{\sigma : W(q) \to \mathbf{Z}\} = P.$$

We will denote the set

$$\{r \in G |\ q(r, x) = 1 \text{ for all } x \in G\}$$

by rad q. If rad $q = \{1\}$ we say q has a trivial radical.

LEMMA 2.3. *Suppose $q : G \times G \to \{\pm 1\}$ is a linked quaternionic mapping with a trivial radical.*

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(1) *Any 4-dimensional anisotropic form over q which represents* 1 *must be a 2-fold Pfister form.*

(2) *If* $D(\langle\langle 1, 1\rangle\rangle) \neq G$ *then* $\sigma : G \to \{\pm 1\}$ *defined by*

$$\sigma(x) = \begin{cases} 1 & \text{if } x \in D(\langle\langle 1, 1\rangle\rangle) \\ -1 & \text{otherwise} \end{cases}$$

is a signature on q.

Proof. (1) Suppose $\varphi = \langle 1, a, b, c\rangle$ is an anisotropic form over G. We may assume $q(-a, -b) = q(-a, -c) = -1$ else φ is isotropic. Consequently, $\langle b, ab\rangle \simeq \langle c, ac\rangle$ and $bc \in D(\langle 1, a\rangle)$. Write $c = bz$ for some $z \in D(\langle 1, a\rangle)$ and let $d \in G$. If $d \in D(\langle 1, z\rangle)$ then $bd \in D(\langle b, bz\rangle)$ hence $-bd \notin D(\langle 1, a\rangle)$ else φ is isotropic. Consequently

$$q(d, -z) = 1 \Rightarrow q(-a, -bd) \neq 1 \Rightarrow q(-a, d) = 1.$$

Now consider the form $b\varphi = \langle 1, z, b, ba\rangle$. A similar argument shows that

$$q(-a, d) = 1 \Rightarrow q(d, -z) = 1.$$

Therefore $q(-a, d) = q(d, -z)$ hence $az \in \text{rad } q = \{1\}$ and thus $a = z$. Consequently $\varphi = \langle 1, a, b, ab\rangle$.

(2) First note that $q(-1, -1) \neq 1$ else $D(\langle\langle 1, 1\rangle\rangle) = G$ hence $B = \{1, q(-1, -1)\}$. If $z \in G - D(\langle 1, 1\rangle)$ then $q(-1, z) = q(-1, -1)$ hence $q(-1, -z) = 1$, that is, $-z \in D(\langle 1, 1\rangle)$. The result will follow quite easily if we can show $D(\langle\langle 1, 1\rangle\rangle) = D(\langle 1, 1\rangle)$. Assume

$$x \in D(\langle\langle 1, 1\rangle\rangle) - D(\langle 1, 1\rangle)$$

and let $y \in G$. Then $-x \in D(\langle 1, 1\rangle)$ implies

$$-1 = x(-x) \in D(\langle\langle 1, 1\rangle\rangle).$$

If $y \notin D(\langle\langle 1, 1\rangle\rangle)$ then $-y \in D(\langle 1, 1\rangle)$ hence

$$y = (-1)(-y) \in D(\langle\langle 1, 1\rangle\rangle),$$

a contradiction. Consequently $D(\langle\langle 1, 1\rangle\rangle) = G$, a final contradiction.

THEOREM 2.4. *For a linked quaternionic mapping* $q : G \times G \to B$ *the following statements are equivalent:*

1. *Either* q *has a unique signature and* $|G| = 2$ *or* q *has a unique anisotropic 4-dimensional form* φ *and* $D(\varphi) = G$.
2. $B = \{\pm 1\}$ *and* q *has a trivial radical.*

Proof. $1 \Rightarrow 2$. If q has a unique signature and $G = \{\pm 1\}$ then

$$q(a, b) = \begin{cases} -1 & \text{if } a = b = -1 \\ 1 & \text{otherwise} \end{cases}$$

hence $B = \{\pm 1\}$ and rad $q = \{1\}$.

Suppose now that q has a unique anisotropic 4-dimensional form φ with $D(\varphi) = G$. We may assume $\varphi = \langle 1, b, c, d \rangle$. Consider the form $\psi = \langle 1, b, c, bc \rangle$. If ψ is isotropic then $q(-b, -c) = 1$ hence

$$\langle b, c \rangle \simeq \langle -1, -bc \rangle \quad \text{and} \quad \varphi \simeq \langle 1, -1, -bc, d \rangle,$$

a contradiction. Consequently we may assume $\varphi = \psi$. Suppose $a \in \text{rad}\, q$. Since $D(\varphi) = G$ we may write

$$\varphi \simeq \langle -a, x, y, -axy \rangle \quad \text{for some } x, y \in G.$$

Let $\varphi_1 = \langle x, y, -axy \rangle$. If $\varphi_1 + \langle -1 \rangle$ is anisotropic then

$$\varphi_1 + \langle -1 \rangle \simeq \varphi_1 + \langle -a \rangle.$$

Comparing determinants we obtain $a = 1$. Assume $\varphi_1 + \langle -1 \rangle$ is isotropic. Write

$$\varphi_1 + \langle -1 \rangle \simeq \langle 1, -1, w, -wa \rangle \quad \text{for some } w \in G.$$

Since $a \in \text{rad}\, q$, $\langle 1, -a \rangle \simeq \langle w, -wa \rangle \simeq \langle x, -xa \rangle$ hence

$$\varphi_1 + \langle -1 \rangle \simeq \langle 1, -1, x, -xa \rangle.$$

Consequently

$$\langle y, -axy \rangle \simeq \langle 1, -ax \rangle \quad \text{and}$$
$$\varphi \simeq \langle -a, x, y, -axy \rangle \simeq \langle -a, x, 1, -ax \rangle.$$

But $D(\langle 1, -a \rangle) = G$ hence φ is isotropic, a contradiction. This shows $\text{rad}\, q = \{1\}$. Since there is one and only one anisotropic 4-dimensional form (a 2-fold Pfister form) over q, B is clearly equal to $\{\pm 1\}$.

$2 \Rightarrow 1$. Since $B = \{\pm 1\}$ there exists a unique anisotropic 2-fold Pfister form φ over q. Let us first assume that $X(q) \neq \emptyset$. Here $\varphi \simeq \langle\langle 1, 1 \rangle\rangle$. Let $x \in G$, $x \neq 1$. Since $\text{rad}\, q = \{1\}$, there exists $y \in G$ such that $\langle\langle -x, -y \rangle\rangle \simeq \langle\langle 1, 1 \rangle\rangle$. Comparing signatures we find $\sigma(x) = -1$ for every $\sigma \in X(q)$. Consequently $G = \{\pm 1\}$ and clearly q has a unique signature. Now suppose $X(q) = \emptyset$. By Lemma 2.3 (1) any 4-dimensional anisotropic form over q must be of the form $c\varphi$ for some $c \in G$. To prove 1 it suffices to show $D(\varphi) = G$. Let $x \in G$ and write $\varphi = \langle\langle -a, -b \rangle\rangle$.

$$q(-ax, -bx) = q(a, b)q(-x, -ab).$$

If $q(-x, -ab) \neq 1$, then $q(-ax, -bx) = 1$ hence

$$\langle -ax, -bx \rangle \simeq \langle 1, ab \rangle.$$

It follows that $\langle -a, -b \rangle \simeq \langle x, xab \rangle$, thus $x \in D(\varphi)$. If $q(-x, -ab) = 1$, then $\langle -x, -ab \rangle \simeq \langle 1, xab \rangle$, thus $-x \in D(\langle 1, ab \rangle)$. If $q(x, -ab) = 1$ then $\langle x, -ab \rangle \simeq \langle 1, -xab \rangle$ hence $x \in D(\langle 1, ab \rangle)$ and $x \in D(\varphi)$. Con-

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sequently we may assume $q(x, -ab) \neq 1$. Now

$$q(-1, -ab) = q(-x, -ab)q(x, -ab) \neq 1,$$

thus $\langle -a, -b\rangle \simeq \langle 1, ab\rangle$ and $\varphi \simeq \langle 1, 1, ab, ab\rangle$. If $-1 \in D(\langle 1, 1\rangle)$ then $\varphi \simeq \langle -1, -1, -ab, -ab\rangle$ but $-x \in D(\langle 1, ab\rangle)$ hence $x \in D(\varphi)$. If $-1 \notin D(\langle 1, 1\rangle)$ then $q(-1, -1) = q(a, b)$ and $\varphi \simeq \langle\langle 1, 1\rangle\rangle$. If $x \notin D(\varphi)$ we can apply Lemma 2.3 (2) and obtain a contradiction.

3. Localization and classification. Let q be a linked quaternionic mapping. By a *strong signature* on q we will mean a surjective group homomorphism $\tilde{\sigma} : B \to \{\pm 1\}$ such that $\tilde{\sigma}(q(a, -b)) = 1$ whenever $\tilde{\sigma}(q(a, b)) = -1$. In the following proposition we prove that every strong signature on q gives rise to a signature on q.

PROPOSITION 3.1. *Suppose $\tilde{\sigma}$ is a strong signature on q. The mapping $\sigma : G \to \{\pm 1\}$ defined by $\sigma(a) = \tilde{\sigma}(q(a, -1))$ is a signature on q.*

Proof. Clearly σ is a group homomorphism. Since $\tilde{\sigma}$ is surjective there exist $a, b \in G$ such that $\tilde{\sigma}(q(a, b)) = -1$. But then $\tilde{\sigma}(q(a, -b)) = 1$, thus $\tilde{\sigma}(q(a, -1)) = -1$ and consequently $\tilde{\sigma}(q(-a, -1)) = 1$. It follows that $\tilde{\sigma}(q(-1, -1)) = -1$, that is, $\sigma(-1) = -1$. Now suppose $\sigma(c) = 1$ and $q(d, cd) = 1$. Since $\langle -1, -c\rangle \simeq \langle -d, -dc\rangle$, $q(-1, -c) = q(-c, -d)$. Now

$$\tilde{\sigma}(q(-1, -c)) = \tilde{\sigma}(q(-1, -1))\tilde{\sigma}(q(-1, c)) = -1 \cdot 1 = -1$$

hence

$$-1 = \tilde{\sigma}(q(-c, -d)) = \tilde{\sigma}(q(-1, -d)q(c, -d)).$$

Notice that if $\tilde{\sigma}(q(c, -d)) = -1$, then $\tilde{\sigma}(q(-c, -d)) = 1$ hence $\tilde{\sigma}(q(-1, d)) = -1$. This would contradict the fact that

$$-1 = \tilde{\sigma}(q(-1, -d)q(c, -d)).$$

Consequently

$$\tilde{\sigma}(q(c, -d)) = 1 \quad \text{and} \quad \tilde{\sigma}(q(-1, -d)) = -1.$$

In particular $\tilde{\sigma}(q(-1, d)) = 1$, that is, $\sigma(d) = 1$ as desired.

If q is the quaternionic mapping associated with a field or a semi-local ring R with $1/2 \in R$ it is known that every signature on q gives rise to a strong signature on q. This is still an open problem for arbitrary linked quaternionic mappings. We do have the following:

THEOREM 3.2. *A homomorphism $\tilde{\sigma} : B \to \{\pm 1\}$ is a strong signature on q if and only if there exist a signature σ on q such that*

$$\tilde{\sigma}(q(a, b)) = \begin{cases} -1 & \text{if } \sigma(a) = \sigma(b) = -1 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Suppose $\tilde{\sigma}$ is a strong signature on q. Consider $\sigma : G \to \{\pm 1\}$ defined by $\sigma(a) = \tilde{\sigma}(q(a, -1))$. By Proposition 3.1, σ is a signature on q. If $\sigma(a) = \sigma(b) = -1$ then

$$\tilde{\sigma}(q(a, -1)) = \tilde{\sigma}(q(b, -1)) = -1.$$

Assume $\tilde{\sigma}(q(a, b)) = 1$. Then

$$\tilde{\sigma}(q(-a, b)) = \tilde{\sigma}(q(-1, b)q(a, b)) = -1.$$

But on the other hand,

$$\tilde{\sigma}(q(-a, -b)) = \tilde{\sigma}(q(-1, -b)q(a, -1)q(a, b)) = -1$$

and consequently $\tilde{\sigma}(q(-a, b)) = 1$, a contradiction. If $\tilde{\sigma}(q(a, b)) = -1$ then

$$1 = \tilde{\sigma}(q(a, -b)) = \tilde{\sigma}(q(a, -1)q(a, b))$$

hence $\tilde{\sigma}(q(a, -1)) = -1$. Similarly

$$1 = \tilde{\sigma}(q(b, -a)) = \tilde{\sigma}(q(b, -1)q(a, b))$$

hence $\tilde{\sigma}(q(b, -1)) = -1$. It follows that $\tilde{\sigma}(q(a, b)) = -1$ if and only if $\sigma(a) = \sigma(b) = -1$ as desired. The converse is clear.

Let $\tilde{\sigma}$ be a strong signature on q. By Proposition 3.1, $\tilde{\sigma}$ induces a signature σ on q. Recall that σ in turn induces a ring homomorphism $\sigma : W(q) \to \mathbf{Z}$. We will say that two forms φ, ψ over q have the same *total strong signature* if $\sigma(\varphi) = \sigma(\psi)$ for all strong signatures $\tilde{\sigma}$ on q.

Let $\bar{G} = G/\text{rad } q$. Define $\bar{q} : \bar{G} \times \bar{G} \to B$ by $\bar{q}(\bar{a}, \bar{b}) = q(a, b)$. It is easy to see that $\bar{q}$ is also a linked quaternionic mapping.

PROPOSITION 3.3. *A homomorphism $\sigma : G \to \{\pm 1\}$ is a signature on q if and only if $\bar{\sigma} : \bar{G} \to \{\pm 1\}$ is a signature on $\bar{q}$.*

Proof. ($\Rightarrow$) $\bar{\sigma}$ is well-defined since rad $q \subseteq D(\langle 1, 1\rangle)$. Clearly, $\bar{\sigma}$ is a group homomorphism and $\bar{\sigma}(-\bar{1}) = \sigma(-1) = -1$. If $\bar{q}(\bar{d}, \overline{cd}) = 1$ and $\bar{\sigma}(\bar{d}) = 1$ then $1 = \bar{q}(\bar{c}, \overline{cd}) = q(c, cd)$. Consequently $\bar{\sigma}(\bar{c}) = \sigma(c) = 1$ since σ is a signature on q.

($\Leftarrow$) is trivial.

Remarks 3.4. (i) Since the subgroup generated by the image of q is the same as the subgroup generated by the image of $\bar{q}$, a homomorphism $\tilde{\sigma}$ is a strong signature on q if and only if $\tilde{\sigma}$ is a strong signature on $\bar{q}$.

(ii) If φ is a form over q then $s(\varphi) = s(\bar{\varphi})$.

Let q be a linked quaternionic mapping and suppose f is a character on B. Define $q_f : G \times G \to \{\pm 1\}$ by $q_f = f \circ q$. It is easy to see that q_f is a quaternionic mapping. To see that q_f is linked suppose $q_f(a, b) = q_f(c, d)$. If $q_f(a, b) = 1$ take $x = 1$. If $q_f(a, b) = -1$ then one of $x = b$, $x = d$ or $x = bd$ will work. $(\overline{q_f})$ will be called the *localization* of q to f.

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Remark 3.5. It is easy to see that a character f on B is a strong signature on q if and only if the identity map $\{\pm 1\} \to \{\pm 1\}$ is a strong signature on q_f.

Let C be a subset of characters on B with the property that $\bigcap_{f \in C} \operatorname{Ker} f = \{1\}$ (for example $C = \chi(B)$).

LEMMA 3.6. *For a linked quaternionic mapping* q

$$\operatorname{rad} q = \bigcap_{f \in C} \operatorname{rad} q_f.$$

Proof. If $r \in \operatorname{rad} q$ then $q(r, x) = 1$ for every $x \in G$. Clearly, $f(q(r, x)) = 1$ for every $f \in \chi(B)$ hence

$$r \in \bigcap_{f \in C} \operatorname{rad} q_f.$$

If $r \in \bigcap_{f \in C} \operatorname{rad} q_f$ then for every $x \in G$ and $f \in C$, $q_f(r, x) = 1$. Consequently,

$$q(r, x) \in \bigcap_{f \in C} \operatorname{Ker} f = \{1\},$$

i.e., $r \in \operatorname{rad} q$.

LEMMA 3.7. *Suppose q has a trivial radical and let C be as in Lemma* 3.6. *For two forms φ and ψ over q we have*

1. $\det(\overline{\varphi_f}) = \det(\overline{\psi_f})$ *for every $f \in C$ implies* $\det \varphi = \det \psi$.
2. $s(\overline{\varphi_f}) = s(\overline{\psi_f})$ *for every $f \in C$ implies* $s(\varphi) = s(\psi)$.

Proof. 1. Fix a character f in C. There exists $r_f \in \operatorname{rad} q_f$ such that $\det \varphi = \det \psi \cdot r_f$. Now if f' is any other character in C there is also $r_{f'} \in \operatorname{rad} q_{f'}$ such that $\det \varphi = \det \psi \cdot r_{f'}$. Consequently,

$$\det \psi \cdot r_f = \det \psi \cdot r_{f'}$$

and hence $r_{f'} = r_f$. This shows that

$$r_f \in \bigcap_{f \in C} \operatorname{rad} q_f.$$

By Lemma 3.6, $r_f \in \operatorname{rad} q = \{1\}$ and we can conclude that $\det \varphi = \det \psi$.

2. Write $\varphi = \langle a_1, \ldots, a_n \rangle$ and $\psi = \langle b_1, \ldots, b_m \rangle$. Since $s(\overline{\varphi_f}) = s(\overline{\psi_f})$ we have

$$\prod_{i<j} \bar{q}_f(\bar{a}_i, \bar{a}_j) = \prod_{i<j} \bar{q}_f(\bar{b}_i, \bar{b}_j).$$

Consequently,

$$\prod_{i<j} f(q(a_i, a_j)) = \prod_{i<j} f(q(b_i, b_j))$$

and hence

$$s(\varphi) \cdot s(\psi) \in \prod_{f \in C} \operatorname{Ker} f = \{1\},$$

i.e., $s(\varphi) = s(\psi)$.

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The following main theorem was motivated by the following examples.

Example 1. If φ and ψ are forms over the rational field $\mathbf{Q}$, then $\varphi \simeq \psi$ over $\mathbf{Q}$ if and only if $\varphi \simeq \psi$ over all p-adic fields $\mathbf{Q}_p$. Since there are only 2 quaternion algebras over $\mathbf{Q}_p$ we can view the quadratic form structure of $\mathbf{Q}_p$ as the abstract structure $(\bar{q}_f) = \bar{G} \times \bar{G} \to \{\pm 1\}$ where f is induced by the map Br $(\mathbf{Q}) \to$ Br $(\mathbf{Q}_p)$.

Example 2. Let F be a formally real field with a real closure Δ. Here again there are only 2 quaternion algebras over Δ and the map Br $(F) \to$ Br (Δ) induces $f : B \to \{\pm 1\}$ where B is the subgroup of Br (F) generated by the quaternion algebras over F. Again we can view the quadratic form structure on Δ as the abstract structure $(\overline{q_f}) : \bar{G} \times \bar{G} \to \{\pm 1\}$. A similar situation prevails if F is a semilocal ring with $1/2 \in \dot{F}$. (See [**4**]).

THEOREM 3.8. *Let q be a linked quaternionic mapping with a trivial radical and let C be as in Lemma* 3.6. *If C contains all strong signatures then the following statements are equivalent.*

1. *For any two forms φ and ψ over q, $\varphi \simeq \psi$ if and only if $(\overline{\varphi_f}) \simeq (\overline{\psi_f})$ for all $f \in C$.*

2. *Forms over q are classified by dimension, determinant, Hasse invariant and total strong signature.*

Proof. 1 $\Rightarrow$ 2. Suppose φ and ψ are forms over q with the same dimension, determinant, Hasse invariant and total strong signature and let $f \in C$. By 1, to show $\varphi \simeq \psi$ it suffices to show $(\overline{\varphi_f}) \simeq (\overline{\psi_f})$. We first assume there is a signature on $(\overline{q_f})$. By Theorem 2.4, $|\bar{G}| = 2$ and since φ and ψ have the same dimension and total strong signature over q it follows that $(\overline{\varphi_f}) \simeq (\overline{\psi_f})$. Now assume $(\overline{q_f})$ has no signatures. By Theorem 2.4 we can write

$$(\overline{\varphi_f}) - (\overline{\psi_f}) \simeq (\overline{\rho_f}) + s\overline{\mathbf{H}}$$

for some $s \in \mathbf{Z}$ and some anisotropic form $(\overline{\rho_f})$ with dim $(\overline{\rho_f}) \leqq 4$ and dim $(\overline{\rho_f})$ even. If dim $(\overline{\rho_f}) = 4$ then as in the proof of Theorem 2.4 $(1 \Rightarrow 2)$ we may write $(\overline{\rho_f}) = \langle\langle -\bar{a}, -\bar{b}\rangle\rangle$. But then by checking Hasse invariants we see that $(\overline{\rho_f}) = \langle\langle -1, 1\rangle\rangle$, a contradiction. If dim $(\overline{\rho_f}) = 2$ then a determinant comparison shows det $(\overline{\rho_f}) = -1$, a contradiction. Consequently, dim $(\overline{\rho_f}) = 0$ and $(\overline{\varphi_f}) - (\overline{\psi_f})$ is hyperbolic. By cancellation, $(\overline{\varphi_f}) \simeq (\overline{\psi_f})$.

2 $\Rightarrow$ 1. Suppose φ and ψ are forms over q with $(\overline{\varphi_f}) \simeq (\overline{\psi_f})$ for every $f \in C$. Clearly, φ and ψ have the same dimension. By Lemma 3.7, φ and ψ have the same determinant and Hasse invariant also. Let $\tilde{\sigma}$ be a strong signature on q. By Remarks 3.4 (i) and 3.5, the identity map $\{\pm 1\} \to \{\pm 1\}$ is a strong signature on $(\overline{q_{\tilde{\sigma}}})$. Now $(\overline{\varphi_{\tilde{\sigma}}})$ and $(\overline{\psi_{\tilde{\sigma}}})$ have the same total

strong signature hence

$$\tilde{\sigma}(q(a_1, -1)) + \ldots + \tilde{\sigma}(q(a_n, -1)) = \tilde{\sigma}(q(b_1, -1)) + \ldots + \tilde{\sigma}(q(a_n, -1)).$$

Hence φ and ψ have the same total strong signature. By 2, $\varphi \simeq \psi$.

Notice that in the case when q arises from a field or semi-local ring, total strong signature may be replaced by total signature in the statement of Theorem 3.8 since every strong signature on q gives rise to a signature on q.

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