J. Austral. Math. Soc. (*Series A*) **51** (1991), 324–330

DIOPHANTINE APPROXIMATION BY CONTINUED FRACTIONS

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(Received 29 November 1989; revised 3 May 1990)

Communicated by J. H. Loxton

Abstract

Let ξ be an irrational number with simple continued fraction expansion

$$\xi = [a_0; a_1, \dots, a_i, \dots],$$

p_i/q_i be its ith convergent. Let $M_i = [a_{i+1}; a_i, \dots, a_1] + [0; a_{i+2}, a_{i+3}, \dots]$. In this paper we prove that $M_{n-1} < r$ and $M_n < R$ imply $M_{n+1} > 1/(r^{-1} + a_{n+1}\sqrt{1 - 4/(rR)} - a_{n+1}^2 R^{-1})$, which generalizes a previous result of the author.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 11 J 04, 11 A 55.

1. Introduction

Let ξ be an irrational number with simple continued fraction expansion $\xi = [a_0; a_1, \dots, a_i, \dots]$, and p_i/q_i be its ith convergent. Let $M_i = [a_{i+1}; a_i, \dots, a_1] + [0; a_{i+2}, a_{i+3}, \dots]$. In a recent paper [9], the present author proved the following conjugate property of the triplet (M_{n-1}, M_n, M_{n+1}), which implies the classical results of Borel and Segre [5] on a symmetric and a symmetric Diophantine approximations.

THEOREM 1. *Let* $r > a_{n+1}$. *Then*

(i) $M_n > r$ *implies* $\min(M_{n-1}, M_{n+1}) < 4r/(r^2 - a_{n+1}^2)$;

(ii) $M_n < r$ *implies* $\max(M_{n-1}, M_{n+1}) > 4r/(r^2 - a_{n+1}^2)$;

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(iii) $M_n = r$ *implies*

$$\min(M_{n-1}, M_{n+1}) < 4r/(r^2 - a_{n+1}^2) < \max(M_{n-1}, M_{n+1}).$$

The essence of Theorem 1 is using the magnitude of M_n to estimate the magnitudes of M_{n-1} and M_{n+1}. It is very natural to pose two related questions: how to use M_{n-1}, M_{n+1} to estimate M_n and how to use M_{n-1}, M_n to estimate M_{n+1}? Apparently these questions are more complicated because there are two parameters involved.

In this paper we solve these two questions and show that Theorem 1 is a corollary of our results.

2. Preliminaries

Since

$$M_i = [a_{i+1}; a_i, \dots, a_1] + [0; a_{i+2}, a_{i+3}, \dots],$$

letting $P = [a_{n+2}; a_{n+3}, \dots]$ and $Q = [a_n; a_{n-1}, \dots, a_1]$, we have

$$M_{n-1} = Q + \frac{1}{a_{n+1} + P^{-1}}, \tag{1}$$

$$M_n = a_{n+1} + P^{-1} + Q^{-1}, \tag{2}$$

and

$$M_{n+1} = P + \frac{1}{a_{n+1} + Q^{-1}}. \tag{3}$$

It is well known [2, 3, 4] that

$$\xi - \frac{p_i}{q_i} = \frac{(-1)^i}{M_i q_i^2}. \tag{4}$$

3. Main results

THEOREM 2. *Let* ξ *be an irrational number such that*

$$\xi = [a_0; a_1, \dots, a_i, \dots].$$

If r, R *are two real numbers such that* $r > 1$, $R > 1$ *and* $rR > 4$, *then* $M_{n-1} < r$ *and* $M_n < R$ *imply*

$$M_{n+1} > \frac{1}{\frac{1}{r} + a_{n+1}\sqrt{1 - \frac{4}{Rr} - \frac{a_{n+1}^2}{R}}}. \tag{5}$$

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PROOF. Since $M_{n-1} < r$, by (1) we have $Q^{-1} > 1/(r - 1/(a_{n+1} + P^{-1}))$. By (2) we have

$$R > a_{n+1} + P^{-1} + \frac{1}{r - \frac{1}{a_{n+1}+P^{-1}}} = \frac{r(a_{n+1} + P^{-1})^2}{r(a_{n+1} + P^{-1}) - 1},$$

$$r(a_{n+1} + P^{-1})^2 - Rr(a_{n+1} + P^{-1}) + R < 0,$$

$$a_{n+1} + P^{-1} < \frac{1}{2}\left(R + \sqrt{R^2 - \frac{4R}{r}}\right),$$

and

(6) $$P > 2/\left(R + \sqrt{R^2 - \frac{4R}{r}} - 2a_{n+1}\right).$$

From $M_n < R$ and (2) we have

(7) $$a_{n+1} + Q^{-1} < R - P^{-1}.$$

By (3), we then have

(8) $$M_{n+1} > P + \frac{1}{R - P^{-1}}.$$

By (7), we have $RP > P(a_{n+1} + P^{-1} + Q^{-1}) > a_{n+1}P + 1 > 2$.
This implies that the right-hand side of (8) is an increasing function of P. By (6) and (8) we then have

$$M_{n+1} > 1/\left(\frac{1}{r} + a_{n+1}\sqrt{1 - \frac{4}{Rr} - \frac{a_{n+1}^2}{R}}\right).$$

REMARK 1. In Theorem 2, if we reverse the directions of the inequality signs in the proof, we have a conjugate theorem.

THEOREM 2′. *Let ξ, r, R be given as in Theorem 2. Then $M_{n-1} > r$ and $M_n > R$ imply*

$$M_{n+1} < 1/\left(\frac{1}{r} + a_{n+1}\sqrt{1 - \frac{4}{Rr} - \frac{a_{n+1}^2}{R}}\right).$$

REMARK 2. The conditions $M_{n-1} < r$ and $M_n < R$ in Theorem 2 can be changed to be $M_{n-1} \le r$ and $M_n \le R$ but $M_{n-1} = r$ and $M_n = R$ do not hold simultaneously. A similar result is true for Theorem 2′.

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REMARK 3. If we interchange the roles of M_{n+1} and M_{n-1} in the proof of Theorem 2, and use equation (3) instead of equation (1), the conclusion of the theorem becomes $M_n < R$ and $M_{n+1} < r$ implying

$$M_{n-1} > 1/\left(\frac{1}{r} + a_{n+1}\sqrt{1 - \frac{4}{Rr} - \frac{a_{n+1}^2}{R}}\right).$$

As in Remark 2, the inequalities $M_n < R$ and $M_{n+1} < r$ may be replaced by $M_n \le R$ and $M_{n+1} \le r$, provided that $M_n = R$ and $M_{n+1} = r$ do not hold simultaneously.

COROLLARY 1. *Let r, r' be two real numbers such that $r > 1, r' > 1$. Then $M_{n-1} < r$ and $M_{n+1} < r'$ imply*

$$M_n > \frac{\frac{1}{r} + \frac{1}{r'} + \sqrt{a_{n+1}^2 + \frac{4}{rr'}}}{1 - a_{n+1}^{-2}\left(\frac{1}{r} - \frac{1}{r'}\right)^2}. \tag{9}$$

PROOF. Let H be the right-hand side of inequality (9). It is easy to check the following equality.

$$\left(a_{n+1}^{-2}\left(\frac{1}{r} - \frac{1}{r'}\right)^2 - 1\right)H^2 + 2\left(\frac{1}{r} + \frac{1}{r'}\right)H + a_{n+1}^2 = 0. \tag{10}$$

We consider two possible cases.

CASE 1. $r \le r'$. Rewrite (10) as follows:

$$\left(a_{n+1}^{-1}\left(\frac{1}{r} - \frac{1}{r'}\right)H + a_{n+1}\right)^2 = H^2 - \frac{4H}{r'}. \tag{11}$$

Since $r > 1, r' > 1$, we have $0 < 1 - a_{n+1}^{-2}(1/r - 1/r')^2 \le 1$. From $r \le r'$ we have

$$r'\left(\frac{1}{r} + \frac{1}{r'} + \sqrt{a_{n+1}^2 + \frac{4}{rr'}}\right) > \frac{r'}{r} + 1 + \sqrt{\frac{4r'}{r}} \ge 4.$$

Hence $Hr' > 4$ and (11) becomes

$$a_{n+1}^{-1}\left(\frac{1}{r} - \frac{1}{r'}\right)H + a_{n+1} = H\sqrt{1 - \frac{4}{Hr'}},$$

$$r = 1/\left(\frac{1}{r'} + a_{n+1}\sqrt{1 - \frac{4}{Hr'} - \frac{a_{n+1}^2}{H}}\right).$$

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If $M_n \leq H$, by $M_{n+1} < r'$ and Remark 3, we have

$$M_{n-1} > 1\Big/\left(\frac{1}{r'} + a_{n+1}\sqrt{1 - \frac{4}{Hr'} - \frac{a_{n+1}^2}{H}}\right) = r,$$

contradicting the assumption that $M_{n-1} < r$. Hence $M_n > H$.

CASE 2. $r > r'$. By a similar method we can prove that

$$r' = 1\Big/\left(\frac{1}{r} + a_{n+1}\sqrt{1 - \frac{4}{Hr} - \frac{a_{n+1}^2}{H}}\right).$$

By Theorem 2, $M_n \leq H$ and $M_{n-1} < r$ imply $M_{n+1} > r'$, contradicting the assumption that $M_{n+1} < r'$. Hence $M_n > H$.

Now we discuss a special case of Corollary 1. By (1) and (3), if $M_{n-1} = M_{n+1}$ we have $(P - Q)(a_{n+1}PQ + P + Q) = 0$. Hence $P = Q$. But P is an infinite continued fraction, and hence irrational, while Q is finite and hence rational so that $P \neq Q$. Therefore $M_{n-1} = r$ and $M_{n+1} = r$ cannot hold simultaneously. Corollary 1 may be varied in the same way as Theorem 1, as described in Remarks 1, 2 and 3. By these variations and setting $r = r'$, we obtain the following result.

COROLLARY 2. *Let* $r > 1$ *be a real number. Then*

(i) $M_{n-1} \leq r$ *and* $M_{n+1} \leq r$ *imply*

$$M_n > \frac{2}{r} + \sqrt{a_{n+1}^2 + \frac{4}{r^2}};$$

(ii) $M_{n-1} \geq r$ *and* $M_{n+1} \geq r$ *imply*

$$M_n < \frac{2}{r} + \sqrt{a_{n+1}^2 + \frac{4}{r^2}}.$$

REMARK 4. Theorem 1 is a simple corollary to Corollary 2 because $\min(M_{n-1}, M_{n+1}) \geq 4r/(r^2 - a_{n+1}^2)$ implies, by (ii), that $M_n < r$ and $\max(M_{n-1}, M_{n+1}) \leq 4r/(r^2 - a_{n+1}^2)$ implies, by (i), that $M_n > r$. The contrapositives of these two statements imply the three parts of Theorem 1.

We give another application of Corollary 2.

In [7], the present author proved that if τ is a real number such that $1 \leq \tau < 2 + \sqrt{5}$, then among any three consecutive convergents p_i/q_i ($i = n-1, n, n+1$) of an irrational number ξ, at least one satisfies the following inequality:

$$(12) \qquad \frac{-1}{\sqrt{a_{n+1}^2 + 4\tau}\quad q_i^2} < \xi - \frac{p_i}{q_i} < \frac{\tau}{\sqrt{a_{n+1}^2 + 4\tau}\cdot q_i^2}.$$

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We show that the restriction $\tau < 2+\sqrt{5}$ may be dropped.

THEOREM 3. *Let* $\tau \geq 1$ *be a real number. Then among three consecutive convergents* p_i/q_i $(i = n-1, n, n+1)$ *of an irrational number* ξ, *at least one satisfies inequality* (12).

PROOF. By (4) we need only prove that either there is an odd index i among $n-1, n, n+1$ such that $M_i > \sqrt{a_{n+1}^2 + 4\tau}$, or there is an even index i such that $M_i > \sqrt{a_{n+1}^2 + 4\tau}/\tau$.

We discuss two possible cases.

CASE 1. n is odd. Then $n-1$ and $n+1$ are even. If one of $M_{n-1}, M_{n+1} > \sqrt{a_{n+1}^2 + 4\tau}/\tau$, then (12) holds by (4). If both $M_{n-1}, M_{n+1} \leq \sqrt{a_{n+1}^2 + 4\tau}/\tau$, we may assume $\sqrt{a_{n+1}^2 + 4\tau}/\tau > 1$ because $\sqrt{a_{n+1}^2 + 4\tau}/\tau \leq 1$ implies

$$M_{n+1} > a_{n+2} \geq 1 \geq \sqrt{a_{n+1}^2 + 4\tau}/\tau.$$

Letting

$$r = \sqrt{a_{n+1}^2 + 4\tau}/\tau$$

in Corollary 2(i), we have $M_n > \sqrt{a_{n+1}^2 + 4\tau}$.

CASE 2. n is even. Then $n-1$ and $n+1$ are odd.

If one of $M_{n-1}, M_{n+1} > \sqrt{a_{n+1}^2 + 4\tau}$, then (12) holds by (4).

If both $M_{n-1}, M_{n+1} \leq \sqrt{a_{n+1}^2 + 4\tau}$, then letting $r = \sqrt{a_{n+1}^2 + 4\tau} > 1$ in Corollary 2(i), we have

$$M_n > \frac{2 + \sqrt{a_{n+1}^4 + 4a_{n+1}^2\tau + 4}}{\sqrt{a_{n+1}^2 + 4\tau}}. \tag{13}$$

Since $\tau \geq 1$, the right-hand side of (13) is greater than $\sqrt{a_{n+1}^2 + 4\tau}/\tau$. The proof is complete.

REMARK 5. An alternative proof of Theorem 3 can be found in [1].

Acknowledgement

The author thanks the referee sincerely for his valuable suggestions to improve this paper.

https://doi.org/10.1017/S1446788700034273 Published online by Cambridge University Press

References

[1] C. Kraaikamp, 'On the approximation by continued fractions', preprint.
[2] W. J. LeVeque, *Topics in Number Theory* I, II, (Addison-Wesley Publ. Co., 1956).
[3] O. Perron, *Die Lehre von den Kettenbruchen* I, II, 3rd ed., (Teubner, Leipzig, 1954).
[4] W. M. Schmidt, *Diophantine Approximation*, (Lecture Notes in Math. 785, Springer-Verlag, 1980).
[5] B. Segre, 'Lattice points in infinite domains and asymmetric Diophantine approximation', *Duke J. Math.* **12** (1945), 337–365.
[6] J. Tong, 'The conjugate property of the Borel theorem on Diophantine approximation', *Math. Z.* **184** (1983), 151–153.
[7] ——, 'A theorem on approximation of irrational numbers by simple continued fractions', *Proc. Edinburgh Math. Soc.* **31** (1988), 197–204.
[8] ——, 'Segre's theorem on asymmetric Diophantine approximation', *J. Number Theory* **28** (1988), 116–118.
[9] ——, 'The conjugate property for Diophantine approximation of continued fractions', *Proc. Amer. Math. Soc.* **105** (1989), 535–539.

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https://doi.org/10.1017/S1446788700034273 Published online by Cambridge University Press