ON THE EXACTNESS OF THE ECKMANN-HILTON HOMOTOPY SEQUENCE

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The theorem that the homotopy sequence is exact splits into six statements. Scherk ([4]) obviates the use of homotopy extension in the proof of one of these statements. The purpose of this note is to show that the method can be adapted to give a direct proof of the corresponding statement in the theorem that the Eckmann-Hilton homotopy sequence ([1]) is exact. The note is based on Eckmann's exposition ([2]). We are concerned with the proof of <u>b2</u>, pp. 34-35. Eckmann's notation is used; in particular all base-points are denoted by the symbol o , all constant maps by the symbol 0

Suppose we have a mapping of pairs

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B_1 \\ {\scriptstyle\alpha}\downarrow & \Rightarrow & \downarrow{\scriptstyle\beta} \\ CA & \xrightarrow[f_2]{} & B_2 \end{array}$$

where $\alpha : A \to CA$ is the natural injection of A into CA. And suppose that the mapping of pairs

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B_1 \\ {\scriptstyle\alpha}\downarrow & \Rightarrow & \downarrow \\ CA & \xrightarrow[0]{} & o \end{array}$$

is homotopic to 0 .

Let $F : A \times I \to B_1$ be a homotopy between f_1 and 0, and let $\gamma : A \times I \to CA$ be the identification mapping. We consider $G : A \times I \times I \to B_2$ defined by

$$G(a, s, t) = \begin{cases} f_2\gamma(a, -\phi) & \text{if } \phi \leq 0 \\ \beta F(a, \phi) & \text{if } \phi \geq 0 \end{cases}$$

where $\phi = t - s - st$.

G is well defined and continuous for

$$f_2\gamma(a, 0) = f_2\alpha(a) = \beta f_1(a) = \beta F(a, 0) .$$

And

$$G(o, s, t) = o , \quad G(a, 1, t) = f_2\gamma(a, 1) = f_2(o) = o .$$

Hence ([3], Lemma 3.4, p. 109) there is a continuous function $H : CA \times I \to B_2$ such that $G(a, s, t) = H(\gamma(a, s), t)$. Let $g : CA \to B_2$ be given by $g(c) = H(c, 1)$. (F, H) is a homotopy between (f_1, f_2) and $(0, g)$.

For consider the diagram

$$\begin{array}{ccc} A \times I & \xrightarrow{\ F\ } & B_1 \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \beta} \\ CA \times I & \xrightarrow[\ H\ ]{} & B_2 \end{array}$$

where $\delta(a, t) = (\alpha(a), t)$.

If $(a, t) \in A \times I$, $H\delta(a, t) = G(a, 0, t) = \beta F(a, t)$

and so the diagram is commutative.

F is a homotopy between f_1 and 0; and if $c \in CA$,

https://doi.org/10.4153/CMB-1966-081-8 Published online by Cambridge University Press

$$H(c, 0) = G(a, s, 0) = f_2\gamma(a, s) = f_2(c) , \qquad H(c, 1) = g(c)$$

so that H is a homotopy between f_2 and g .

Finally consider

$$\begin{array}{ccc} A & \xrightarrow{\ \ 0\ \ } & o \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow \\ CA & \xrightarrow[\ \ g\ \ ]{} & B_2 \end{array} .$$

$$g\alpha(a) = G(a, 0, 1) = \beta F(a, 1) = \beta(o) = o .$$

Thus the homotopy class of $(0, g)$ is an element of $\Pi_1(A, B_2)$. Hence the class of (f_1, f_2) belongs to the image in $\Pi_1(A, \beta)$ of $\Pi_1(A, B_2)$ by J .

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https://doi.org/10.4153/CMB-1966-081-8 Published online by Cambridge University Press