Proceedings of the Edinburgh Mathematical Society (2025) **68**, 63–79
doi:10.1017/S001309152400052X

RESULTS ON THE NON-VANISHING OF DERIVATIVES OF L-FUNCTIONS OF VECTOR-VALUED MODULAR FORMS

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(Received 19 May 2023)

Abstract We prove the existence of a vector-valued cusp form for the full modular group for which the nth derivative of its L-function does not vanish under certain conditions. As an application, we generalize our result to Kohnen's plus space and prove an analogous result for Jacobi forms.

Keywords: Vector-valued modular forms; L-functions; Critical strip; Jacobi forms; non-vanishing

2020 Mathematics subject classification: Primary 11F66; 11F50
Secondary 11F30

1. Introduction

The theory of L-functions plays a crucial role in both number theory and arithmetic geometry. L-Functions exhibit natural connections with various mathematical subjects including number fields, automorphic forms, Artin representations, Shimura varieties, abelian varieties and intersection theory. The central values of L-functions and their derivatives reveal important connections to the geometric and arithmetic properties of Shimura varieties such as the Gross-Zagier formula, Colmez's conjecture and the averaged Colmez formula. On the other hand, vector-valued modular forms are important generalizations of elliptic modular forms that arise naturally in the theory of Jacobi forms, Siegel modular forms and Moonshine. Important foundational results on vector-valued modular forms were establihsed by Knopp and Mason [13, 14]. Being an important tool to tackle classical problems in the theory of modular forms, Selberg used these forms to give an estimation for the Fourier coefficients of the classical modular forms [20]. Borcherds in [4] and [6] used vector-valued modular forms associated with Weil representations to provide a description of the Fourier expansion of various theta liftings. Some applications of vector-valued modular forms stand out in high-energy physics by mainly providing a method of differential equations in order to construct the modular multiplets and also revealing the simple structure of the modular invariant mass models [18]. Other applications concerning vector-valued modular forms of half-integer weight seem to provide a simple solution to the Riemann–Hilbert problem for representations of the modular group [2]. So it is only natural to study the L-functions of vector-valued modular forms and their properties as a buildup that aligns with the development of a Hecke theory to the space of vector-valued modular forms. L-Functions of vector-valued modular forms have been investigated in connection with sign changes of Fourier coefficients, Jacobi forms and Hecke theory (for examples, see [3, 7, 11]). Moreover, they lead us to investigate a Gross–Kohnen–Zagier theorem in higher dimensions [5].

In [17], we show that averages of L-functions associated with vector-valued cusp forms do not vanish when the average is taken over the orthogonal basis of the space of vector-valued cusp forms. To illustrate, we let $\{f_{k,1}, \ldots, f_{k,d_k}\}$ be an orthogonal basis of the space $S_{k,\chi,\rho}$ of vector-valued cusp forms with Fourier coefficients $b_{k,l,j}(n)$, where χ is a multiplier system of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\mathrm{SL}_2(\mathbb{Z})$ and $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{C})$ is an m-dimensional unitary complex representation. We also let $t_0 \in \mathbb{R}, \epsilon > 0$ and $1 \leq i \leq m$. Then, there exists a constant $C(t_0, \epsilon, i) > 0$ such that for $k > C(t_0, \epsilon, i)$, the function

$$\sum_{l=1}^{d_k} \frac{< L^*(f_{k,l}, s), \mathbf{e}_i >}{(f_{k,l}, f_{k,l})} b_{k,l,i}(n_{i,0})$$

does not vanish at any point $s = \sigma + it_0$, with $\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon$, where $< L^*(f_{k,l}, s), \mathbf{e}_i >$ denotes the ith component of $L^*(f_{k,l}, s)$ and $n_{i,0}$ is the number given in equation (3.2).

Kohnen, Sengupta and Weigel in [16] proved a nonvanishing result for the derivatives of L-functions in the critical strip for elliptic modular forms on the full group. In [19], the second author generalized their result to modular forms of half-integer weight on the plus space. In this paper, we show analogous results for the averages of the derivatives of L-functions for the orthogonal basis of the space of vector-valued cusp forms in the critical strip. In particular, given $k \in \frac{1}{2}\mathbb{Z}$, χ a multiplier system of weight k on $\mathrm{SL}_2(\mathbb{Z})$, $t_0 \in \mathbb{R}, \epsilon > 0, 1 \leq i \leq m$ and n a positive integer, we show that there exists a constant $C(t_0, \epsilon, i, n) > 0$ such that for $k > C(t_0, \epsilon, i, n)$, the function

$$\sum_{l=1}^{d_k} \frac{b_{k,l,i}(n_{i,0})}{(f_{k,l}, f_{k,l})} \frac{d^n}{ds^n} < L^*(f_{k,l}, s), \mathbf{e}_i >$$

does not vanish at any point $s = \sigma + it$, with $t = t_0$, $\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon$. From this, we show that there exists a constant $C(t_0, \epsilon, n)$ such that for $k > C(t_0, \epsilon, n)$ and any $s = \sigma + it$, with $t = t_0$, $\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon$, $\frac{k}{2} + \epsilon < \sigma < \frac{k+1}{2}$, there exists $f \in S_{k,\chi,\rho}$ such that $\frac{d^n}{ds^n} L^*(f, s) \neq 0$.

https://doi.org/10.1017/S001309152400052X Published online by Cambridge University Press

The isomorphism between the space of Jacobi forms of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$ and the space of vector-valued modular cusp forms with a specific multiplier system and a given Weil representation depending on m leads to an analogous result for Jacobi forms. We also give a similar result for cusp forms in the plus space.

2. The Kernel function

In this section, we define the kernel function $R_{k,s,i}$ and determine its Fourier expansion. The kernel function being a cusp form will play an important role in determining the coefficients of a given cusp form in terms of L-functions when the given cusp form is written in terms of the orthogonal basis. So, let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $k \in \frac{1}{2}\mathbb{Z}$ and χ a unitary multiplier system of weight k on Γ, i.e. $\chi : \Gamma \to \mathbb{C}$ satisfies the following conditions:

(1) $|\chi(\gamma)| = 1$ for all $\gamma \in \Gamma$.
(2) χ satisfies the consistency condition

$$\chi(\gamma_3)(c_3\tau + d_3)^k = \chi(\gamma_1)\chi(\gamma_2)(c_1\gamma_2\tau + d_1)^k(c_2\tau + d_2)^k,$$

where $\gamma_3 = \gamma_1\gamma_2$ and $\gamma_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \Gamma$ for $i = 1, 2$ and 3.

Let m be a positive integer and $\rho : \Gamma \to \mathrm{GL}(m, \mathbb{C})$ an m-dimensional unitary complex representation. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ denote the standard basis of $\mathbb{C}^m$. For a vector-valued function $f = \sum_{j=1}^{m} f_j \mathbf{e}_j$ on $\mathbb{H}$ and $\gamma \in \Gamma$, define a slash operator by

$$(f|_{k,\chi,\rho}\gamma)(\tau) := (c\tau + d)^{-k}\chi^{-1}(\gamma)\rho^{-1}(\gamma)f(\gamma\tau).$$

The definition of the vector-valued modular forms is given as follows.

Definition 2.1. *A vector-valued modular form of weight $k \in \frac{1}{2}\mathbb{Z}$, multiplier system χ and type ρ on Γ is a sum $f = \sum_{j=1}^{m} f_j \mathbf{e}_j$ of functions holomorphic in $\mathbb{H}$ satisfying the following conditions:*

(1) *$f|_{k,\chi,\rho}\gamma = f$ for all $\gamma \in \Gamma$.*
(2) *For each $1 \le j \le m$, each function f_j has a Fourier expansion of the form*

$$f_i(\tau) = \sum_{n+\kappa_j \ge 0} a_j(n) e^{2\pi i(n+\kappa_j)\tau}.$$

Here and throughout the paper, κ_j is a certain positive number with $0 \le \kappa_j < 1$.

The space of all vector-valued modular forms of weight k, multiplier system χ and type ρ on Γ is denoted by $M_{k,\chi,\rho}$. There is a subspace $S_{k,\chi,\rho}$ of vector-valued cusp forms for which we require that each $a_j(n) = 0$ when $n + \kappa_j$ is non-positive.

Following [12], we now define the L-function of a vector-valued cusp form. For a vector-valued cusp form $f(\tau) = \sum_{j=1}^{m} \sum_{n+\kappa_j>0} a_j(n) \mathrm{e}^{2\pi i(n+\kappa_j)\tau} \mathrm{e}_j \in S_{k,\chi,\rho}$, we see that $a_j(n) = O(n^{k/2})$ for every $1 \le j \le m$ as $n \to \infty$ by the same argument for elliptic modular forms.

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Then, the vector-valued L-function defined by

$$L(f,s) := \sum_{j=1}^{m} \sum_{n+\kappa_j>0} \frac{a_j(n)}{(n+\kappa_j)^s} \mathbf{e}_j$$

converges absolutely for $\mathrm{Re}(s) \gg 0$. This has an integral representation

$$\frac{\Gamma(s)}{(2\pi)^s} L(f,s) = \int_0^\infty f(iv) v^s \frac{\mathrm{d}v}{v}.$$

From this, we see that it has an analytic continuation to $\mathbb{C}$ and a functional equation given by

$$L^*(f,s) = i^k \chi(S)\rho(S) L^*(f,k-s),$$

where $L^*(f,s) = \frac{\Gamma(s)}{(2\pi)^s} L(f,s)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Let i be an integer with $1 \leq i \leq m$. Define

$$p_{s,i}(\tau) := \tau^{-s} \mathbf{e}_i.$$

For $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$, we define the kernel function by

$$R_{k,s,i} := \gamma_k(s) \sum_{\gamma \in \Gamma} p_{s,i}|_{k,\chi,\rho}\gamma,$$

where $\gamma_k(s) := \frac{1}{2} \mathrm{e}^{\pi i s/2} \Gamma(s)\Gamma(k-s)$. Then, this series converges absolutely uniformly whenever $\tau = u+iv$ satisfies $v \geq \epsilon, v \leq 1/\epsilon$ for a given $\epsilon > 0$ and s varies over a compact set. Moreover, it is a vector-valued cusp form in $S_{k,\chi,\rho}$.

We write $< \cdot, \cdot >$ for the standard scalar product on $\mathbb{C}^m$, i.e.

$$\left\langle \sum_{j=1}^{m} \lambda_j \mathbf{e}_j, \sum_{j=1}^{m} \mu_j \mathbf{e}_j \right\rangle = \sum_{j=1}^{m} \lambda_j \overline{\mu_j}.$$

Then, for $f, g \in S_{k,\chi,\rho}$, we define the Petersson scalar product of f and g by

$$(f,g) := \int_{\mathcal{F}} < f(\tau), g(\tau) > v^k \frac{\mathrm{d}u\mathrm{d}v}{v^2},$$

where $\mathcal{F}$ is the standard fundamental domain for the action of Γ on $\mathbb{H}$. Then, by [17, Lemma 3.1], we have

$$(f, R_{k,\bar{s},i}) = c_k < L^*(f,s), \mathbf{e}_i >, \tag{2.1}$$

where $c_k := \frac{(-1)^{k/2}\pi(k-2)!}{2^{k-2}}$.

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We can also compute the Fourier expansion of $R_{k,s,i}$.

Lemma 2.2. [17, Lemma 3.2] *The function $R_{k,s,i}$ has the Fourier expansion*

$$R_{k,s,i}(\tau)=\sum_{j=1}^{m}\sum_{n+\kappa_j>0} r_{k,s,i,j}(n)e^{2\pi i(n+\kappa_j)\tau},$$

where $r_{k,s,i,j}(n)$ is given by

$$\begin{aligned}r_{k,s,i,j}(n)=&\,\delta_{i,j}(2\pi)^s\Gamma(k-s)(n+\kappa_i)^{s-1}\\&+\chi^{-1}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)_{j,i}(-1)^{k/2}(2\pi)^{k-s}\Gamma(s)(n+\kappa_j)^{k-s-1}\\&+\frac{(-1)^{k/2}}{2}(2\pi)^k(n+\kappa_j)^{k-1}\frac{\Gamma(s)\Gamma(k-s)}{\Gamma(k)}\sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)=1,ac>0}}c^{-k}\left(\frac{c}{a}\right)^s\\&\times\Bigg(e^{2\pi i(n+\kappa_j)d/c}e^{\pi is}\chi^{-1}\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\right)_{j,i}\ {}_1F_1(s,k;-2\pi in/(ac))\\&+e^{-2\pi i(n+\kappa_j)d/c}e^{-\pi is}\chi^{-1}\left(\left(\begin{smallmatrix}-a&b\\c&-d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix}-a&b\\c&-d\end{smallmatrix}\right)\right)_{j,i}\ {}_1F_1(s,k;2\pi in/(ac))\Bigg),\end{aligned}$$

where ${}_1F_1(\alpha,\beta;z)$ is Kummer's degenerate hypergeometric function.

3. The main theorem

In this section, we give the main theorem for the existence of at least one L-function whose derivative does not vanish.

Theorem 3.1. *Let $k\in\frac{1}{2}\mathbb{Z}$, and let χ be a multiplier system of weight k on $\mathrm{SL}_2(\mathbb{Z})$. Let $t_0\in\mathbb{R},\epsilon>0$ and n a positive integer. Then, there exists a constant $C(t_0,\epsilon,n)$ such that for $k>C(t_0,\epsilon,n)$ and any $s=\sigma+it$ with $t=t_0$, $\frac{k-1}{2}<\sigma<\frac{k}{2}-\epsilon$, $\frac{k}{2}+\epsilon<\sigma<\frac{k+1}{2}$, there exists $f\in S_{k,\chi,\rho}$ such that $\frac{d^n}{ds^n}L^*(f,s)\neq 0$.*

Proof. We follow the argument in the proof of Theorem 3.1 in [16]. Suppose that $\{f_{k,1},\dots,f_{k,d_k}\}$ is an orthogonal basis of $S_{k,\chi,\rho}$ with Fourier expansions

$$f_{k,l}(\tau)=\sum_{j=1}^{m}\sum_{n+\kappa_j>0} b_{k,l,j}(n)\mathrm{e}^{2\pi i(n+\kappa_j)\tau}\ (1\le l\le d_k)$$

and d_k is the dimension of $\dim S_{k,\chi,\rho}$.

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For each $1 \le i \le m$, by equation (2.1), we have

$$R_{k,s,i} = c_k \sum_{l=1}^{d_k} \frac{< L^*(f_{k,l}, s), \mathbf{e}_i >}{(f_{k,l}, f_{k,l})} f_{k,l}. \tag{3.1}$$

Let

$$n_{i,0} := \begin{cases} 1 & \text{if} \kappa_i = 0, \\ 0 & \text{if} \kappa_i \neq 0. \end{cases} \tag{3.2}$$

If we take the first Fourier coefficients of ith component function on both sides of equation (3.1), then by Lemma 2.2 we have

$$\begin{aligned} &(2\pi)^s \Gamma(k-s)(n_{i,0}+\kappa_i)^{s-1} \\ &+ \chi^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)_{i,i} (-1)^{k/2}(2\pi)^{k-s}\Gamma(s)(n_{i,0}+\kappa_i)^{k-s-1} \\ &+ \frac{(-1)^{k/2}}{2}(2\pi)^k (n_{i,0}+\kappa_i)^{k-1} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1, ac>0}} c^{-k}\left(\frac{c}{a}\right)^s \\ &\quad \times \Big(\mathrm{e}^{2\pi i(n_{i,0}+\kappa_j)d/c} \mathrm{e}^{\pi i s} \chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)_{i,i} \, {}_1f_1(s,k;-2\pi i n_{i,0}/(ac)) \\ &\qquad + \mathrm{e}^{-2\pi i(n_{i,0}+\kappa_i)d/c} \mathrm{e}^{-\pi i s} \chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d \end{smallmatrix}\right)\right)_{i,i} \, {}_1f_1(s,k;2\pi i n_{i,0}/(ac)) \Big) \\ &= c_k \sum_{l=1}^{d_k} \frac{< L^*(f_{k,l}, s), \mathbf{e}_i >}{(f_{k,l}, f_{k,l})} b_{k,l,i}(n_{i,0}), \end{aligned} \tag{3.3}$$

where

$${}_1f_1(\alpha, \beta; z) := \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)} \, {}_1F_1(\alpha, \beta; z).$$

We assume that

$$\sum_{l=1}^{d_k} \frac{b_{k,l,i}(n_{i,0})}{(f_{k,l}, f_{k,l})} \frac{d^n}{\mathrm{d}s^n} < L^*(f_{k,l}, s), \mathbf{e}_i > \tag{3.4}$$

is zero. If we take the nth derivative with respect to s on both sides in equation (3.3), then we have

$$\begin{aligned} &\frac{d^n}{ds^n}\left[(2\pi)^s \Gamma(k-s)(n_{i,0}+\kappa_i)^{s-1}\right] \\ &= -\frac{d^n}{ds^n}\left[\chi^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) \rho^{-1}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)_{i,i} (-1)^{k/2}(2\pi)^{k-s}\Gamma(s)(n_{i,0}+\kappa_i)^{k-s-1}\right] \end{aligned} \tag{3.5}$$

$$- \frac{d^n}{ds^n}\left[\frac{(-1)^{k/2}}{2}(2\pi)^k(n_{i,0}+\kappa_i)^{k-1}\sum_{(c,d)\in\mathbb{Z}^2 (c,d)=1, ac>0} c^{-k}\left(\frac{c}{a}\right)^s\right.$$
$$\times\left(\mathrm{e}^{2\pi i(n_{i,0}+\kappa_j)d/c}\mathrm{e}^{\pi i s}\chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)_{i,i}\ {}_1f_1(s,k;-2\pi i n_{i,0}/(ac))\right.$$
$$\left.\left.+\,\mathrm{e}^{-2\pi i(n_{i,0}+\kappa_i)d/c}\mathrm{e}^{-\pi i s}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)_{i,i}\ {}_1f_1(s,k;2\pi i n_{i,0}/(ac))\right)\right].$$

Then, the left-hand side of equation (3.5) is equal to

$$\begin{aligned}
&\frac{1}{n_{i,0}+\kappa_i}\sum_{\nu=0}^{n}\frac{d^\nu}{ds^\nu}[(2\pi(n_{i,0}+\kappa_i))^s]\frac{d^{n-\nu}}{ds^{n-\nu}}\Gamma(k-s)\\
&\quad=\frac{(2\pi(n_{i,0}+\kappa_i))^s}{n_{i,0}+\kappa_i}\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}(\log(2\pi(n_{i,0}+\kappa_i)))^\nu\Gamma^{(n-\nu)}(k-s)\\
&\quad=(2\pi)^s(n_{i,0}+\kappa_i)^{s-1}(\log(2\pi(n_{i,0}+\kappa_i)))^n\Gamma(k-s)\\
&\qquad+(2\pi)^s(n_{i,0}+\kappa_i)^{s-1}\sum_{\nu=0}^{n-1}(-1)^{n-\nu}\binom{n}{\nu}(\log(2\pi(n_{i,0}+\kappa_i)))^\nu\Gamma^{(n-\nu)}(k-s).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\frac{1}{(2\pi)^s(n_{i,0}+\kappa_i)^{s-1}\Gamma(k-s)}\cdot\frac{d^n}{ds^n}\left[(2\pi)^s\Gamma(k-s)(n_{i,0}+\kappa_i)^{s-1}\right]\\
&\quad=(\log(2\pi(n_{i,0}+\kappa_i)))^n+\sum_{\nu=0}^{n-1}(-1)^{n-\nu}\binom{n}{\nu}(\log(2\pi(n_{i,0}+\kappa_i)))^\nu\frac{\Gamma^{(n-\nu)}(k-s)}{\Gamma(k-s)}.
\end{aligned}$$

Let $\psi(s):=\frac{\Gamma'(s)}{\Gamma(s)}$. Then, one can see that $\frac{\Gamma^{(n)}(s)}{\Gamma(s)}$ is a polynomial $P(\psi,\psi^{(1)},\ldots,\psi^{(n-1)})$ with integral coefficients and it contains the term ψ^n, which is the highest power of ψ occurring in P. It is known that ψ satisfies the following asymptotic formulas:

$$\psi(s)\sim\log(s)-\frac{1}{2s}-\sum_{\nu=1}^{\infty}\frac{B_{2\nu}}{2\nu s^{2\nu}}$$

and

$$\psi^{(n)}(s)\sim(-1)^{n-1}\left(\frac{(n-1)!}{s^n}+\frac{n!}{2s^{n+1}}+\sum_{\nu=0}^{\infty}B_{2\nu}\frac{(2\nu+n-1)!}{(2\nu)!s^{2\nu+n}}\right)$$

for $s\to\infty$ in $|\arg(s)|<\pi$, where B_n denotes the nth Bernoulli number (for example, see [1, 6.3.18 and 6.4.11]). Let $s=\frac{k}{2}-\delta+it_0$ ($\epsilon<\delta<\frac{1}{2}$). Then, the leading term of

$\frac{\Gamma^{(n-\nu)}(k-s)}{\Gamma(k-s)}$ for $0 \le \nu \le n-1$ is $(\log(\frac{k}{2}+\delta-it_0))^{n-\nu}$ as $k \to \infty$ and $\psi^{(n)}(s) = o\left(\frac{1}{|s|^n}\right)$ as $|s| \to \infty$ in $|\arg(s)| < \pi$ for $n \in \mathbb{N}$. Therefore, we have

$$\sum_{\nu=0}^{n-1}(-1)^{n-\nu}\binom{n}{\nu}(\log(2\pi(n_{i,0}+\kappa_i)))^{\nu}\frac{\Gamma^{(n-\nu)}(k-s)}{\Gamma(k-s)} = Q\left(\log(\frac{k}{2}+\delta-it_0)\right)+o(1)$$

as $k \to \infty$, where Q is a polynomial of degree n and its highest coefficient is $(-1)^n$.

For the first term on the right-hand side of equation (3.5), we have

$$\begin{aligned}
&\frac{d^n}{ds^n}\left[\chi^{-1}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)_{i,i}(-1)^{k/2}(2\pi)^{k-s}\Gamma(s)(n_{i,0}+\kappa_i)^{k-s-1}\right]\\
&\quad=\chi^{-1}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)_{i,i}(-1)^{k/2}\frac{(2\pi(n_{i,0}+\kappa_i))^k}{n_{i,0}+\kappa_i}\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{d^\nu}{ds^\nu}\\
&\qquad\times[(2\pi(n_{i,0}+\kappa_i))^{-s}]\frac{d^{n-\nu}}{ds^{n-\nu}}\Gamma(s)\\
&\quad=\chi^{-1}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)_{i,i}(-1)^{k/2}\frac{(2\pi(n_{i,0}+\kappa_i))^{k-s}}{n_{i,0}+\kappa_i}\\
&\qquad\times\sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}\log(2\pi(n_{i,0}+\kappa_i))^{\nu}\Gamma^{(n-\nu)}(s).
\end{aligned}$$

If we divide this by $(2\pi)^s(n_{i,0}+\kappa_i)^{s-1}\Gamma(k-s)$, then we have

$$\begin{aligned}
&\chi^{-1}\left(\left(0\ -11\ 0\right)\right)\rho^{-1}\left(\left(0\ -11\ 0\right)\right)_{i,i}(-1)^{k/2}\frac{(2\pi(n_{i,0}+\kappa_i))^{k-2s}}{(n_{i,0}+\kappa_i)^2}\\
&\times\sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}\log(2\pi(n_{i,0}+\kappa_i))^{\nu}\frac{\Gamma^{(n-\nu)}(s)}{\Gamma(s)}\cdot\frac{\Gamma(s)}{\Gamma(k-s)}.
\end{aligned}\tag{3.6}$$

Let $s = \frac{k}{2}-\delta+it_0$ ($\epsilon < \delta < \frac{1}{2}$). Then, by [1, 6.1.23 and 6.1.47], we have

$$\left|\frac{\Gamma(s)}{\Gamma(k-s)}\right| = \left|\frac{k}{2}+it_0\right|^{-2\delta}\cdot\left|1+O\left(\frac{1}{|\frac{k}{2}+it_0|}\right)\right|,$$

where the O constant is absolute, uniformly in $\epsilon < \delta < \frac{1}{2}$. On the other hand, the highest-order term in $\frac{\Gamma^{(n-\nu)}(s)}{\Gamma(s)}$ is $\left(\psi\left(\frac{k}{2}-\delta+it_0\right)\right)^{n-\nu}$. This behaves like $(\log(\frac{k}{2}-\delta+it_0))^{n-\nu}$ for $0 \le \nu < n$ as $k \to \infty$. Thus, we can see that all terms in the sum in equation (3.6) go to zero as $k \to \infty$.

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The second term on the right-hand side of equation (3.5) is equal to

$$
\begin{aligned}
&-\frac{(-1)^{k/2}}{2}(2\pi)^k(n_{i,0}+\kappa_i)^{k-1}\sum_{(c,d)\in\mathbb{Z}^2 (c,d)=1,ac>0} c^{-k}\frac{d^n}{ds^n}\Bigg[\left(\frac{c}{a}\right)^s \\
&\times\Bigg(\mathrm{e}^{2\pi i(n_{i,0}+\kappa_j)d/c}\mathrm{e}^{\pi is}\chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)_{i,i}\ {}_1f_1(s,k;-2\pi in_{i,0}/(ac)) \\
&\quad+\mathrm{e}^{-2\pi i(n_{i,0}+\kappa_i)d/c}\mathrm{e}^{-\pi is}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)_{i,i}\ {}_1f_1(s,k;2\pi in_{i,0}/(ac))\Bigg)\Bigg] \\
&=-\frac{(-1)^{k/2}}{2}(2\pi)^k(n_{i,0}+\kappa_i)^{k-1}\sum_{(c,d)\in\mathbb{Z}^2 (c,d)=1,ac>0} c^{-k}\sum_{\nu=0}^{n}\binom{n}{\nu}\left(\frac{c}{a}\right)^s\left(\log\left(\frac{c}{a}\right)\right)^\nu \qquad (3.7)\\
&\times\frac{d^{n-\nu}}{ds^{n-\nu}}\Bigg[\Bigg(\mathrm{e}^{2\pi i(n_{i,0}+\kappa_j)d/c}\mathrm{e}^{\pi is}\chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)_{i,i}\ {}_1f_1(s,k;-2\pi in_{i,0}/(ac)) \\
&\quad+\mathrm{e}^{-2\pi i(n_{i,0}+\kappa_i)d/c}\mathrm{e}^{-\pi is}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)_{i,i}\ {}_1f_1(s,k;2\pi in_{i,0}/(ac))\Bigg)\Bigg].
\end{aligned}
$$

In the above equation, the derivative in the last two lines is equal to

$$
\begin{aligned}
&\sum_{w=0}^{n-\nu}\binom{n-\nu}{w}\Bigg\{e^{2\pi i(n_{i,0}+\kappa_j)d/c}\chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)_{i,i} \\
&\qquad\times\frac{d^w}{ds^w}\left[e^{\pi is}\right]\frac{d^{n-\nu-w}}{ds^{n-\nu-w}}\left[{}_1f_1(s,k;-2\pi in_{i,0}/(ac))\right] \\
&\quad+\mathrm{e}^{-2\pi i(n_{i,0}+\kappa_i)d/c}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)_{i,i} \\
&\qquad\times\frac{d^w}{ds^w}\left[\mathrm{e}^{-\pi is}\right]\frac{d^{n-\nu-w}}{ds^{n-\nu-w}}\left[{}_1f_1(s,k;2\pi in_{i,0}/(ac))\right]\Bigg\} \\
&\sum_{w=0}^{n-\nu}\binom{n-\nu}{w}\Bigg\{(\pi i)^w\,\mathrm{e}^{2\pi i(n_{i,0}+\kappa_j)d/c}\mathrm{e}^{\pi is}\chi^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)_{i,i} \\
&\qquad\times\frac{d^{n-\nu-w}}{ds^{n-\nu-w}}\left[{}_1f_1(s,k;-2\pi in_{i,0}/(ac))\right] \\
&\quad+(-\pi i)^w\,e^{-2\pi i(n_{i,0}+\kappa_i)d/c}e^{-\pi is}\chi^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)\rho^{-1}\left(\left(\begin{smallmatrix} -a & b \\ c & -d\end{smallmatrix}\right)\right)_{i,i} \\
&\qquad\times\frac{d^{n-\nu-w}}{ds^{n-\nu-w}}\left[{}_1f_1(s,k;2\pi in_{i,0}/(ac))\right]\Bigg\}.
\end{aligned}
$$

By [1, 13.2.1], for $\mathrm{Re}(\beta)>\mathrm{Re}(\alpha)>0$, we have

$$
{}_1f_1(\alpha,\beta;z)=\int_0^1\mathrm{e}^{zu}u^{\alpha-1}(1-u)^{\beta-\alpha-1}\mathrm{d}u.
$$

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Therefore, for any $n \in \mathbb{Z}_{\geq 0}$, we obtain

$$\begin{aligned}
&\frac{d^n}{ds^n}\left[\,{}_1f_1\left(s,k;\pm\frac{2\pi i n_{i,0}}{ac}\right)\right] \\
&= \int_0^1 \mathrm{e}^{\pm\frac{2\pi i n_{i,0}}{ac}u}\,\frac{d^n}{ds^n}\left[u^{s-1}(1-u)^{k-s-1}\right]du \\
&= \int_0^1 \mathrm{e}^{\pm\frac{2\pi i n_{i,0}}{ac}u}\left(\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(\log(u))^j(\log(1-u))^{n-j}\right)u^{s-1}(1-u)^{k-s-1}\mathrm{d}u.
\end{aligned}$$

Since $\log(u) = o(u^{-\epsilon'})$ for any $\epsilon' > 0$ as $u \to 0$, we see that

$$\left|\frac{d^n}{ds^n}\left[\,{}_1f_1\left(s,k;\pm\frac{2\pi i n_{i,0}}{ac}\right)\right]\right| \leq K_n,$$

where K_n is a constant depending only on n.

Let $s = \frac{k}{2} - \delta + it_0$ $(\epsilon < \delta < \frac{1}{2})$. Then, the series in equation (3.7) is

$$\begin{aligned}
\ll \sum_{a=1}^{\infty}\sum_{c=1}^{\infty} a^{-\frac{k}{2}+\delta}c^{-\frac{k}{2}-\delta}\Bigg(2\left|\log\left(\frac{c}{a}\right)\right|^n \mathrm{e}^{\pi|t_0|}K_0 \\
+ 2\mathrm{e}^{\pi|t_0|}\sum_{\nu=0}^{n-1}\binom{n}{\nu}\left|\log\left(\frac{c}{a}\right)\right|^{\nu}\sum_{w=0}^{n-\nu}\binom{n-\nu}{w}\left(\frac{\pi}{2}\right)^w K_{n-\nu-w}\Bigg).
\end{aligned}$$

This can be estimated in terms of the Riemann zeta function and a positive constant factor $B(t_0, n)$ depending only on t_0 and n. If we divide the second term on the right-hand side of equation (3.5) by $(2\pi)^s(n_{i,0} + \kappa_i)^{s-1}\Gamma(k - s)$, then the absolute value is

$$\ll \frac{(2\pi(n_{i,0} + \kappa_i))^{\frac{k}{2}+\delta}}{\Gamma(\frac{k}{2} + \delta - it_0)}B(t_0, n)$$

and this goes to 0 as $k \to \infty$ uniformly in $\delta \in (\epsilon, \frac{1}{2})$ by Stirling's formula.

In conclusion, if we divide both sides of equation (3.5) by $(2\pi)^s(n_{i,0} + \kappa_i)^{s-1}\Gamma(k - s)$, then the right-hand side goes to zero as $k \to \infty$, but the absolute value of the left-hand side is

$$\gg |\log(\frac{k}{2} + \delta - it_0)|^n$$

as $k \to \infty$. This is a contradiction.

Therefore, there exists a constant $C(t_0, \epsilon, i, n) > 0$ such that for $k > C(t_0, \epsilon, i, n)$, the function in equation (3.4) does not vanish at any point $s = \sigma + it$, with $t = t_0$, $\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon$. From this, we get the desired result by using the functional equation of $L^*(f, s)$ $(f \in S_{k,\chi,\rho})$. □

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Remark 3.2.

(1) In [15, 16], one takes the average over the basis consisting of normalized Hecke eigenforms for the case of classical modular forms. In fact, the basis is an orthogonal basis, and the results in [15, 16] hold for any orthogonal basis of the space of cusp forms.
(2) The vector-valued cusp form f in Theorem 3.1 depends on n in general.

4. The case of $\Gamma_0(N)$

Now, we consider the case of an elliptic modular form of integral weight on the congruence subgroup $\Gamma_0(N)$. By using Theorem 3.1, we can extend a result in [16] to the case of $\Gamma_0(N)$. To illustrate, let N be a positive integer and let k be a positive even integer. Let $\Gamma = \Gamma_0(N)$, and let $S_k(\Gamma)$ be the space of cusp forms of weight k on Γ. Let $\{\gamma_1, \ldots, \gamma_m\}$ be the set of representatives of $\Gamma \setminus \mathrm{SL}_2(\mathbb{Z})$, with $\gamma_1 = I$. For $f \in S_k(\Gamma)$, we define a vector-valued function $\tilde{f} : \mathbb{H} \to \mathbb{C}^m$ by $\tilde{f} = \sum_{j=1}^m f_j \mathbf{e}_j$ and

$$f_j = f|_k \gamma_j \ (1 \le j \le m),$$

where $(f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))(z) := (cz+d)^{-k} f(\gamma z)$. Then, $\tilde{f}$ is a vector-valued modular form of weight k and the trivial multiplier system with respect to ρ on $\mathrm{SL}_2(\mathbb{Z})$, where ρ is a certain m-dimensional unitary complex representation such that $\rho(\gamma)$ is a permutation matrix for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and is an identity matrix if $\gamma \in \Gamma$. Then, the map $f \mapsto \tilde{f}$ induces an isomorphism between $S_k(\Gamma)$ and $S_{k,\rho}$, where $S_{k,\rho}$ denotes the space of vector-valued cusp forms of weight k and trivial multiplier system with respect to ρ on $\mathrm{SL}_2(\mathbb{Z})$.

For $\tilde{f}, \tilde{g} \in S_{k,\rho}$, we define a Petersson inner product by

$$(\tilde{f}, \tilde{g}) := \int_{\mathcal{F}} < \tilde{f}, \tilde{g} > y^k \frac{\mathrm{d}x \mathrm{d}y}{y^2}.$$

Note that if $f, g \in S_k(\Gamma)$ such that f and g are orthogonal, then $\tilde{f}$ and $\tilde{g}$ is also orthogonal.

Corollary 4.1. *Let k be a positive even integer with $k > 2$. Let N and n be positive integers and $\Gamma = \Gamma_0(N)$. Let $t_0 \in \mathbb{R}, \epsilon > 0$. Then, there exists a constant $C(t_0, \epsilon, n) > 0$ such that for $k > C(t_0, \epsilon, n)$, there exists a basis element $f \in S_k(\Gamma)$ satisfying*

$$\frac{d^n}{ds^n} L^*(\widetilde{f}, s) \neq 0$$

at any point $s = \sigma + it_0$, with

$$\frac{k-1}{2} < \sigma < \frac{k}{2} - \epsilon \ \ \textit{and} \ \ \frac{k}{2} + \epsilon < \sigma < \frac{k+1}{2}.$$

5. The case of Jacobi forms

We now consider the case of Jacobi forms. Let k be a positive even integer and m be a positive integer. From now, we use the notation $\tau = u + iv \in \mathbb{H}$ and $z = x + iy \in \mathbb{C}$. We review basic notions of Jacobi forms (for more details, see [9, Section 3.1] and [10, Section 5]). Let F be a complex-valued function on $\mathbb{H} \times \mathbb{C}$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), X = (\lambda, \mu) \in \mathbb{Z}^2$, we define

$$(F|_{k,m}\gamma)(\tau, z) := (c\tau + d)^{-k} \mathrm{e}^{-2\pi i m \frac{cz^2}{c\tau+d}} F(\gamma(\tau, z))$$

and

$$(F|_m X)(\tau, z) := \mathrm{e}^{2\pi i m(\lambda^2\tau + 2\lambda z)} F(\tau, z + \lambda\tau + \mu),$$

where $\gamma(\tau, z) = (\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d})$.

We now give the definition of a Jacobi form.

Definition 5.1. *A Jacobi form of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic function F on $\mathbb{H} \times \mathbb{C}$, satisfying*

(1) *$F|_{k,m}\gamma = F$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,*
(2) *$F|_m X = F$ for every $X \in \mathbb{Z}^2$,*
(3) *F has the Fourier expansion of the form*

$$F(\tau, z) = \sum_{\substack{l,r \in \mathbb{Z} \\ 4ml - r^2 \geq 0}} a(l, r) e^{2\pi i l\tau} e^{2\pi i r z}. \tag{5.1}$$

We denote by $J_{k,m}$ the space of all Jacobi forms of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$. If a Jacobi form satisfies the condition $a(l, r) \neq 0$ only if $4ml - r^2 > 0$, then it is called a Jacobi cusp form. We denote by $S_{k,m}$ the space of all Jacobi cusp forms of weight k and index m on $\mathrm{SL}_2(\mathbb{Z})$.

Let F be a Jacobi cusp form $F \in S_{k,m}$ with its Fourier expansion equation (5.1). We define the partial L-functions of F by

$$L(F, j, s) := \sum_{\substack{n>0 \\ n+j^2 \equiv 0 \pmod{4m}}} \frac{a\left(\frac{n+j^2}{4m}, j\right)}{\left(\frac{n}{4m}\right)^s}$$

for $1 \leq j \leq 2m$. This L-function was studied in [3, 8]. Moreover, F can be written as

$$F(\tau, z) = \sum_{1 \leq j \leq 2m} F_j(\tau)\theta_{m,j}(\tau, z) \tag{5.2}$$

with uniquely determined holomorphic functions $F_j : \mathbb{H} \to \mathbb{C}$ and functions in $\{F_j |\ 1 \leq j \leq 2m\}$ have the Fourier expansions

$$F_j(\tau) = \sum_{\substack{n>0 \\ n+j^2 \equiv 0 \pmod{4m}}} a\left(\frac{n+j^2}{4m}, j\right) \mathrm{e}^{2\pi i n\tau/(4m)},$$

where the theta series $\theta_{m,j}$ is defined by

$$\theta_{m,j}(\tau, z) := \sum_{\substack{r\in\mathbb{Z} \\ r\equiv j \pmod{2m}}} \mathrm{e}^{2\pi i r^2\tau/(4m)} \mathrm{e}^{2\pi i r z}$$

for $1 \leq j \leq 2m$.

We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic group. The elements of $\mathrm{Mp}_2(\mathbb{R})$ are pairs $(\gamma, \phi(\tau))$, where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and ϕ denotes a holomorphic function on $\mathbb{H}$, with $\phi(\tau)^2 = c\tau + d$. Throughout this paper, following [21], we use the convention that $\sqrt{\tau}$ is chosen so that $\arg(\sqrt{\tau}) \in (-\pi/2, \pi/2]$. The map

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = (\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau + d})$$

defines a locally isomorphic embedding of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{Mp}_2(\mathbb{R})$. Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. It is well-known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by $\widetilde{T}$ and $\widetilde{S}$, where $\widetilde{T}$ and $\widetilde{S}$ are the lifts of the standard generators T and S of $\mathrm{SL}_2(\mathbb{Z})$, respectively. We define a $2m$-dimensional unitary complex representation $\widetilde{\rho}_m$ of $\mathrm{Mp}_2(\mathbb{Z})$ by

$$\widetilde{\rho}_m(\widetilde{T})\mathbf{e}_j = \mathrm{e}^{-2\pi i j^2/(4m)}\mathbf{e}_j$$

and

$$\widetilde{\rho}_m(\widetilde{S})\mathbf{e}_j = \frac{i^{\frac{1}{2}}}{\sqrt{2m}} \sum_{j'=1}^{2m} \mathrm{e}^{2\pi i j j'/(2m)}\mathbf{e}_{j'},$$

Let χ be a multiplier system of weight $\frac{1}{2}$ on $\mathrm{SL}_2(\mathbb{Z})$. We define a map $\rho_m : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_{2m}(\mathbb{C})$ by

$$\rho_m(\gamma) = \chi(\gamma)\widetilde{\rho}_m(\widetilde{\gamma})$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The map ρ_m gives a $2m$-dimensional unitary representation of $\mathrm{SL}_2(\mathbb{Z})$.

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_{2m}\}$ denote the standard basis of $\mathbb{C}^{2m}$. For $F \in S_{k,m}$, we define a vector-valued function $\tilde{F} : \mathbb{H} \to \mathbb{C}^{2m}$ by $\tilde{F} = \sum_{j=1}^{2m} F_j\mathbf{e}_j$, where F_j is defined by the theta expansion in equation (5.2). Then, the map $F \mapsto \tilde{F}$ induces an isomorphism between $S_{k,m}$ and $S_{k-\frac{1}{2},\bar{\chi},\rho_m}$.

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Let $L^*(F,j,s) := \frac{\Gamma(s)}{(2\pi)^s} L(F,j,s)$. Then, we have the following corollary.

Corollary 5.2. *Let k be a positive even integer with $k>2$. Let m and n be positive integers. Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$.*

(1) *Let j be a positive integer with $1 \le j \le 2m$. Then, there exists a constant $C(t_0,\epsilon,j,n) > 0$ such that for any $k > C(t_0,\epsilon,j,n)$ and any $s = \sigma + it_0$ with*

$$\frac{2k-3}{4} < \sigma < \frac{2k-1}{4} - \epsilon,$$

there exists a Jacobi cusp form $F \in S_{k,m}$ such that

$$\frac{d^n}{ds^n} L^*(F,j,s) \neq 0.$$

(2) *There exists a constant $C(t_0,\epsilon,n) > 0$ such that for any $k > C(t_0,\epsilon,n)$ and any $s = \sigma + it_0$ with*

$$\frac{2k-3}{4} < \sigma < \frac{2k-1}{4} - \epsilon \quad \text{and} \quad \frac{2k-1}{4} + \epsilon < \sigma < \frac{2k+1}{4},$$

there exist a Jacobi cusp form $F \in S_{k,m}$ and $j \in \{1,\ldots,2m\}$ such that

$$\frac{d^n}{ds^n} L^*(F,j,s) \neq 0.$$

Remark 5.3. Note that $\rho_m(-I)$ is not equal to the identity matrix in $\mathrm{GL}_{2m}(\mathbb{C})$. Instead, we have

$$\rho_m(-I)\mathbf{e}_j = i\mathbf{e}_{2m-j}.$$

By a similar argument, we prove the same result as in Theorem 3.1 for the representation ρ_m.

6. The case of Kohnen plus space

Let k be a positive even integer. By [10, Theorem 5.4], there is an isomorphism ϕ between $S_{k,1}$ and $S^+_{k-\frac{1}{2}}$, where $S^+_{k-\frac{1}{2}}$ denotes the space of cusp forms in the plus space of weight $k - \frac{1}{2}$ on $\Gamma_0(4)$. Moreover, this isomorphism is compatible with the Petersson scalar products.

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Let f be a cusp form in $S^+_{k-\frac{1}{2}}$ with Fourier expansion $f(\tau) = \sum_{\substack{n>0 \\ n\equiv 0,3 \pmod 4}} c(n)e^{2\pi i n\tau}$. Then, the L-function of f is defined by

$$L(f,s) := \sum_{\substack{n>0 \\ n\equiv 0,3 \pmod 4}} \frac{c(n)}{n^s}.$$

For $1 \leq j \leq 2$, let c_j be defined by

$$c_j(n) := \begin{cases} c(n) & \text{if } n \equiv -j^2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $c(n) = c_1(n) + c_2(n)$ for all n. With this, we consider partial sums of $L(f,s)$ by

$$L(f,j,s) := \sum_{\substack{n>0 \\ n\equiv 0,3 \pmod 4}} \frac{c_j(n)}{n^s}$$

for $1 \leq j \leq 2$.

Suppose that F is a Jacobi cusp form in $S_{k,1}$. By the theta expansion in equation (5.2), we have a corresponding vector-valued modular form $(F_1(\tau), F_2(\tau))$. Then, the isomorphism ϕ from $S_{k,1}$ to $S^+_{k-\frac{1}{2}}$ is given by

$$\phi(F) = \sum_{j=1}^{2} F_j(4\tau).$$

From this, we see that

$$L(f,j,s) = \frac{1}{4^s} L(F,j,s).$$

We have the following corollary regarding the partial sums of $L(f,s)$ for $f \in S^+_{k-\frac{1}{2}}$.

Corollary 6.1. *Let k be a positive even integer with $k > 2$. Let n be a positive integer. Let $t_0 \in \mathbb{R}$ and $\epsilon > 0$.*

(1) *Let j be a positive integer with $1 \leq j \leq 2$. Then, there exists a constant $C(t_0, \epsilon, j, n) > 0$ such that for any $k > C(t_0, \epsilon, j, n)$ and any $s = \sigma + it_0$ with*

$$\frac{2k-3}{4} < \sigma < \frac{2k-1}{4} - \epsilon,$$

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there exists a cusp form $f \in S^+_{k-\frac{1}{2}}$ *such that*

$$\frac{d^n}{ds^n}\left[4^s L^*(f,j,s)\right] \neq 0.$$

(2) *There exists a constant* $C(t_0,\epsilon,n) > 0$ *such that for any* $k > C(t_0,\epsilon,n)$ *and any* $s = \sigma + it_0$, *with*

$$\frac{2k-3}{4} < \sigma < \frac{2k-1}{4} - \epsilon \quad \textit{and} \quad \frac{2k-1}{4} + \epsilon < \sigma < \frac{2k+1}{4},$$

there exist a cusp form $f \in S^+_{k-\frac{1}{2}}$ *and* $j \in \{1,\ldots,2\}$ *such that*

$$\frac{d^n}{ds^n}\left[4^s L^*(f,j,s)\right] \neq 0.$$

Funding Statement. The first author was supported by the National Research Foundation of Korea grant funded by the Ministry of Science and ICT (MSIT) (No. RS-2024-00346031).

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