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FACTORISATION OF LIPSCHITZ FUNCTIONS ON ZERO DIMENSIONAL GROUPS

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Let G denote a locally compact metrisable zero dimensional group with left translation invariant metric d. The Lipschitz spaces are defined by

$$\text{Lip}(\alpha;\ r) = \left\{f \in L^r(G) \ :\ \|{}_af-f\|_r = O\left(d(a,\ 0)^\alpha\right),\ a \to 0\right\},$$

where ${}_af : x \to f(ax)$ and $\alpha > 0$; when $r = \infty$ the members of $\text{Lip}(\alpha;\ r)$ are taken to be continuous. For a suitable choice of metric it is shown that $\text{Lip}(\alpha;\ q) \subset L^p_* * L^q(G)$, where $1 \le p \le 2$, $\alpha > q^{-1}$, p, q are conjugate indices and $L^p_*(G) = \{f : f^* \in L^p(G),\ (f^*(x) = f(x^{-1}))\}$. It is also shown that for G infinite the range of values of α cannot be extended.

The problem of factorising Lipschitz functions defined on real Euclidean space or the circle group has been considered by Hahn [4], Lohoué [7] and Uno [8]. Subsequently Uno [9] has proved that for compact metrisable zero dimensional abelian groups satisfying a certain boundedness condition (the so-called bounded Vilenkin groups) $\text{Lip}(\alpha;\ q) \subset L^p * L^q(G)$, where $1 \le p \le 2$, $\alpha > q^{-1}$ and p, q are conjugate indices, that is,

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$p^{-1} + q^{-1} = 1$ (with the usual convention if $p = 1$). The proof that Uno gives makes use of an ordering on the dual group, first introduced by Vilenkin [10].

We show that this result holds for any locally compact metrisable zero dimensional group G , the proof using only some simple properties of a certain bounded approximate unit for $L^1(G)$ (see Lemma 1 below).

Throughout G will denote a locally compact metrisable zero dimensional group with right Haar measure λ . We take a neighbourhood basis (V_n) at the identity consisting of a strictly decreasing sequence of compact open subgroups of G (for the existence of such a basis see [5, (7.7)]; when G is compact the V_n are taken to be normal), (β_n) to be any strictly decreasing sequence of positive numbers tending to zero, and d defined on $G \times G$ by

$$d(x, y) = \begin{cases} \beta_{n+1} , & y^{-1}x \in V_n \backslash V_{n+1} , \\ \beta_1 , & y^{-1}x \notin V_1 , \\ 0 , & x = y \end{cases}$$

(see [11, Section 2]). It is easily verified that d is a left translation invariant metric on G compatible with the given topology. We follow [11] and put $\beta_n = \lambda(V_n)$. This choice of metric agrees with that usually taken when G is a product of finite cyclic groups, and includes that considered by Uno [9]. The Lipschitz spaces $\mathrm{Lip}(\alpha; r)$ will be defined as in the abstract with respect to the above metric. It should be noted that our choice of the strictly decreasing sequence (V_n) is arbitrary.

We define $k_n = \lambda(V_n)^{-1}\xi_n$, where ξ_n denotes the characteristic function of the set V_n . Clearly $k_n \geq 0$ and $\int_G k_n d\lambda = 1$. Furthermore, by [5, (20.15)], (k_n) is a bounded approximate unit for $L^1(G)$. We require two lemmas.

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LEMMA 1.

$$\left(k_{m+1}-k_m\right) * \left(k_{n+1}-k_n\right) = \begin{cases} 0 & , \quad m \neq n , \\ k_{n+1} - k_n , & m = n . \end{cases}$$

Proof. First note that

$$k_m * k_n(x) = \left(\lambda\left(V_m\right)\lambda\left(V_n\right)\right)^{-1}\lambda\left(V_m x \cap V_n\right) .$$

Now suppose that $m \geq n$. For $x \in V_n$, $V_m x \cap V_n = V_m x$ and $k_m * k_n(x) = \lambda\left(V_n\right)^{-1}$. For $x \notin V_n$, $V_m x$ and V_n are disjoint, and $k_m * k_n(x) = 0$. Hence $k_m * k_n = k_n$ for $m \geq n$, from which the lemma follows.

LEMMA 2. *Let* $f \in \mathrm{Lip}(\alpha ; r)$. *Then*

$$\left\|k_{n+1}*f - k_n*f\right\|_r \leq 2K\lambda\left(V_{n+1}\right)^{\alpha} ,$$

where K *depends only on* f .

Proof. For each n ,

$$k_n * f - f = \int_G \left({}_y f-f\right)k_n\left(y^{-1}\right)d\lambda(y)$$

and, using Minkowski's inequality for integrals, for all n suitably large,

$$\begin{aligned} \left\|k_n*f - f\right\|_r &\leq \int_{V_n} \left\|{}_y f-f\right\|_r k_n\left(y^{-1}\right)d\lambda(y) \\ &\leq K \sup\left\{d(y, 0)^{\alpha} : y \in V_n\right\} \\ &= K\lambda\left(V_{n+1}\right)^{\alpha} . \end{aligned}$$

We can assume that this inequality holds for all $n \geq 1$, and hence

$$\left\|k_{n+1}*f - k_n*f\right\|_r \leq K\left[\lambda\left(V_{n+2}\right)^{\alpha}+\lambda\left(V_{n+1}\right)^{\alpha}\right] \leq 2K\lambda\left(V_{n+1}\right)^{\alpha} .$$

We now have our main result.

THEOREM 1. *Let* $1 \leq p \leq 2$. *There exists* $g \in L^p(G) \cap L^p_*(G)$ *such*

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that for all $f \in \text{Lip}(\alpha;\, q)$ *with* $\alpha > q^{-1}$ *, there exists* $h \in L^q(G)$ *satisfying* $f = g * h$ *.*

Proof. Choose $\beta \in \left(q^{-1},\, \alpha\right)$ and put

$$g = k_1 + \sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right)^{\beta}\left(k_{n+1}-k_n\right) .$$

Clearly $g \in L^p(G) \cap L^p_*(G)$ as $k_n^* = k_n$ and

$$\begin{aligned}\|g\|_p &\le \|k_1\|_p + \sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right)^{\beta}\left(\|k_{n+1}\|_p+\|k_n\|_p\right) \\ &\le \lambda\left(V_1\right)^{p^{-1}-1} + 2\sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right)^{\beta+p^{-1}-1} < \infty ,\end{aligned}$$

the last inequality following since $\beta + p^{-1} - 1 = \beta - q^{-1} > 0$ and $\lambda\left(V_{n+1}\right) \le 2^{-n}\lambda\left(V_1\right)$ (recall that V_{n+1} is a proper subgroup of V_n for each n). Now

$$h = k_1 * f + \sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right)^{-\beta}\left(k_{n+1}-k_n\right) * f$$

satisfies the conditions of the theorem. Indeed we have, from Lemma 1,

$$g * h = k_1 * f + \sum_{n=1}^{\infty} \left(k_{n+1}-k_n\right) * f = f ,$$

the second equality following from the property that $\left(k_n\right)$ is a bounded approximate unit for $L^1(G)$ (see the proof of Lemma 2, for example). Also Lemma 2 shows that

$$\begin{aligned}\|h\|_q &\le \|k_1 * f\|_q + \sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right)^{-\beta}\|\left(k_{n+1}-k_n\right) * f\|_q \\ &\le \|k_1 * f\|_q + 2K\sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right)^{\alpha-\beta} < \infty ,\end{aligned}$$

so that $h \in L^q(G)$.

For infinite groups the range of values of α in Theorem 1 cannot be

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extended. This we show using some of the properties of random Fourier series, but first we require a preliminary result, which is of interest in its own right. We introduce some notation.

Let G denote a compact group with dual object Σ , the set of equivalence classes of continuous irreducible unitary representations of G . For each $\sigma \in \Sigma$ fix a representative $U^{(\sigma)}$ and let H_σ be the Hilbert space in which $U^{(\sigma)}$ acts. The (finite) dimension of H_σ is denoted by d_σ . The Fourier series of $f \in L^1(G)$ is given by

$$\sum_{\sigma\in\Sigma} d_\sigma \operatorname{tr}\left[\hat{f}(\sigma)U^{(\sigma)}(x)\right] ,$$

where tr denotes the usual trace function and $\hat{f}(\sigma)$, the Fourier transform of f at σ , is given by

$$\hat{f}(\sigma) = \int_G f(x)U^{(\sigma)}\left(x^{-1}\right)d\lambda(x) .$$

Write $\mathrm{E}(\Sigma) = \prod_{\sigma\in\Sigma} B\left(H_\sigma\right)$, where $B\left(H_\sigma\right)$ denotes the space of linear operators on H_σ and, for each $E = \left(E_\sigma\right) \in \mathrm{E}(\Sigma)$, define the norm $\|E\|_p$ as in [6, (28.34)],

$$\|E\|_p = \left(\sum_{\sigma\in\Sigma} d_\sigma\|E_\sigma\|_{\varphi_p}^p\right)^{1/p}$$

where $\|\ \|_{\varphi_p}$ are the von Neumann norms of [6, (D.37), (D.36) (e)]. We put

$$\mathrm{E}_p(\Sigma) = \left\{E \in \mathrm{E}(\Sigma) : \|E\|_p < \infty\right\} .$$

Note that by the Peter-Weyl theorem, the Fourier transformation $f \to \hat{f}$ is an isomorphism of the Hilbert spaces $L^2(G)$ and $\mathrm{E}_2(\Sigma)$.

THEOREM 2. *Let G be compact and $1 \le p \le 2$. If $f \in \mathrm{Lip}(\alpha; q)$ with $\alpha > q^{-1}$ then $\hat{f} \in \mathrm{E}_r(\Sigma)$, where $r = 2p(3p-2)^{-1}$.*

Proof. From Theorem 1 we have

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$$\mathrm{Lip}(\alpha;\, q) \subset L^p * L^q(G) \subset L^p * L^2(G)$$

and, using [*6*, (28.36), (28.43) and (31.25)], we have that $\hat{f} \in E_q(\Sigma)E_2(\Sigma)$. The result now follows by appealing to [*6*, (28.33)].

Theorem 2, which is an extension of a classical theorem of Bernstein on the absolute convergence of Fourier series, has been obtained previously for G abelian (see [*2*, Theorem 2]), where it was also shown ([*2*, Theorem 4]) that the range of values of α cannot be extended. For G not necessarily abelian Benke ([*1*, Corollary, p. 323]) has a version of our Theorem 2, but only for $p = 2$ and for G satisfying a certain boundedness condition. Benke also shows that his result is sharp (see [*1*, Theorem 3]).

We can now prove that the results of Theorem 1 are sharp.

THEOREM 3. *Suppose G is infinite and let $1 < p \le 2$. There exists $f \in \mathrm{Lip}(q^{-1};\, q)$ with $f \notin L^p_* * L^q(G)$.*

Theorem 3 would also hold for $p = 1$, provided $\mathrm{Lip}(0;\, \infty)$ is defined to be $L^\infty(G)$. In this case the result would state that there existed bounded functions that are not continuous.

Our proof of the theorem is divided into two cases:

(i) G noncompact

Choose a sequence $\{x_n\}$ with $x_1 = 1$ and satisfying $V_1x_m \cap V_1x_n = \emptyset$ for $m \ne n$, and write $f = \sum_{n=1}^{\infty} \eta_n$, where η_n denotes the characteristic function of V_nx_n . We show that $f \in \mathrm{Lip}(q^{-1};\, q)$. First note that $f \in L^r(G)$ for all $r \ge 1$, since

$$\|f\|_r^r = \sum_{n=1}^{\infty} \lambda(V_nx_n) = \sum_{n=1}^{\infty} \lambda(V_n) \le \lambda(V_1) \sum_{n=1}^{\infty} 2^{1-n} = 2\lambda(V_1) .$$

Furthermore, for $a \in V_k \backslash V_{k+1}$ we see that ${}_a\eta_n = \eta_n$ if $n \le k$ and $\|{}_a\eta_n\|_q = \|\eta_n\|_q$ for all n ; for the latter equality just use the property that the Haar measure of the compact (hence unimodular) group V_1 is just

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the restriction of λ to V_1 , suitably normalised. It follows that for such a ,

$$\|{}_af-f\|_q \le 2 \sum_{n=k+1}^{\infty} \|\eta_n\|_q = 2 \sum_{n=k+1}^{\infty} \lambda(V_n)^{q^{-1}} = Kd(a,\, 0)^{q^{-1}}$$

for some constant K , and thus $f \in \mathrm{Lip}(q^{-1};\, q)$. However it is clear that $f \notin C_0(G)$ (the space of continuous functions on G vanishing at infinity) and hence $f \notin L_*^p * L^q(G)$ (see [5, (20.32) (e)]).

(ii) G compact

In view of the proof of Theorem 2 we need only exhibit $f \in \mathrm{Lip}(q^{-1};\, q)$ having $\hat{f} \notin E_r(\Sigma)$, where $r = 2p(3p-2)^{-1}$. We consider $k_n = \lambda(V_n)^{-1}\xi_n$, introduced in the introduction. It is clear that the restriction ω of $k_n d\lambda$ to V_n is just the normalised Haar measure on V_n and, by [6, (28.72) (g)] (note that since G is compact we have that V_n is normal),

$$\{\sigma \in \Sigma : \hat{\omega}(\sigma) \ne 0\} = \{\sigma \in \Sigma : \hat{\omega}(\sigma) = I\} = A_n ,$$

where $A_n = \left\{\sigma \in \Sigma : U^{(\sigma)}(x) = I \text{ for all } x \in V_n\right\}$ is the annihilator of V_n in Σ . In particular the Fourier series of k_n is given by

$$\sum_{\sigma \in A_n} d_\sigma \operatorname{tr}\left[U^{(\sigma)}(x)\right] .$$

Now by [3, Theorem 4], $W \in E(\Sigma)$ with each W_σ unitary can be chosen so that

$$\sum_{\sigma \in A_{n+1} \setminus A_n} d_\sigma \operatorname{tr}\left[W_\sigma U^{(\sigma)}(x)\right]$$

is the Fourier series of a function $l_n \in L^q(G)$, where $\|l_n\|_q \le K(q)\|l_n\|_2$ and $K(q)$ is a constant depending only on q . Define

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$$f = \sum_{n=1}^{\infty} \lambda(V_{n+1})^{q^{-1}+\frac{1}{2}} l_n .$$

Using the equalities

$$\|l_n\|_2 = \|k_{n+1}-k_n\|_2 = \left(\lambda(V_{n+1})^{-1}-\lambda(V_n)^{-1}\right)^{\frac{1}{2}}$$

we have

$$\|f\|_q \le \sum_{n=1}^{\infty} \lambda(V_{n+1})^{q^{-1}+\frac{1}{2}} K(q)\left(\lambda(V_{n+1})^{-1}-\lambda(V_n)^{-1}\right)^{\frac{1}{2}} < \infty ,$$

so that $f \in L^q(G)$. Furthermore, for $a \in V_k \backslash V_{k+1}$ and $n \le k-1$ we see that

$${}_a l_n - l_n = {}_a k_{n+1} * l_n - k_{n+1} * l_n = \left({}_a k_{n+1} - k_{n+1}\right) * l_n = 0$$

and, for some constant $C(q)$,

$$\begin{aligned}\|{}_a f - f\|_q &\le \sum_{n=k}^{\infty} \lambda(V_{n+1})^{q^{-1}+\frac{1}{2}} 2\|l_n\|_q \\ &\le 2K(q) \sum_{n=k}^{\infty} \lambda(V_{n+1})^{q^{-1}+\frac{1}{2}}\left(\lambda(V_{n+1})^{-1}-\lambda(V_n)^{-1}\right)^{\frac{1}{2}} \\ &\le C(q)\lambda(V_{k+1})^{q^{-1}} = C(q)d(a, 0)^{q^{-1}} ,\end{aligned}$$

which shows that $f \in \mathrm{Lip}\left(q^{-1}; q\right)$.

Now, using the property that the spectra of the l_n are pairwise disjoint,

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$$\begin{aligned}\|\hat{f}\|_r^r &= \sum_{\sigma\in\Sigma} d_\sigma \|\hat{f}(\sigma)\|_{\varphi_r}^r \\ &\geq \sum_{n=1}^{\infty} \sum_{\sigma\in A_{n+1}\setminus A_n} d_\sigma \lambda\left(V_{n+1}\right)^{r\left(q^{-1}+\frac{1}{2}\right)} \|\hat{l}_n(\sigma)\|_{\varphi_r}^r \\ &= \sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right) \sum_{\sigma\in A_{n+1}\setminus A_n} d_\sigma \|W_\sigma\|_{\varphi_r}^r \\ &= \sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right) \sum_{\sigma\in A_{n+1}\setminus A_n} d_\sigma^2 .\end{aligned}$$

As V_{n+1} is a proper subgroup of V_n we have $\lambda\left(V_n\right) - \lambda\left(V_{n+1}\right) \geq \frac{1}{2}\lambda\left(V_{n+1}\right)$. It follows that

$$\sum_{\sigma\in A_{n+1}\setminus A_n} d_\sigma^2 = \|k_{n+1}-k_n\|_2^2 \geq \tfrac{1}{2}\lambda\left(V_{n+1}\right)^{-1} ,$$

and so we deduce that $\hat{f} \notin E_r(\Sigma)$. This completes the proof of Theorem 3.

Theorem 1 admits the following generalisation. We suppose that $1 \leq p_1 \leq \ldots \leq p_m \leq \infty$ are given with $p_1^{-1} + \ldots + p_m^{-1} = m - 1$, where $m \geq 2$. If q_j denotes the index conjugate to p_j then each $\beta > p_m^{-1}$ can be written as $\beta = \beta_1 + \ldots + \beta_{m-1}$ with $\beta_j > q_j^{-1}$, $1 \leq j \leq m-1$, since $q_1^{-1} + \ldots + q_{m-1}^{-1} = p_m^{-1}$. For each $1 \leq j \leq m-1$ define

$$g_j = k_1 + \sum_{n=1}^{\infty} \lambda\left(V_{n+1}\right)^{\beta_j}\left(k_{n+1}-k_n\right) .$$

Then, using the notation of the proof of Theorem 1, we have

$$g = g_1 * \ldots * g_{m-1} ,$$

where $g_j \in L^{p_j}(G)$. Hence we obtain

THEOREM 4. *Let p_j, $1 \leq j \leq m$, be as above. Then, for $\alpha > p_m^{-1}$,*

$$\operatorname{Lip}\left(\alpha;\, p_m\right) \subset L^{p_1}(G) * \ldots * L^{p_m}(G) .$$

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The above details follow that of [9, Section 6], where the result is given for bounded Vilenkin groups.

References

[1] George Benke, "Smoothness and absolute convergence of Fourier series in compact totally disconnected groups", *J. Funct. Anal.* 29 (1978), 319-327.

[2] Walter R. Bloom, "Absolute convergence of Fourier series on totally disconnected groups", submitted.

[3] Alessandro Figà-Talamanca and Daniel Rider, "A theorem of Littlewood and lacunary series for compact groups", *Pacific J. Math.* 16 (1966), 505-514.

[4] Liang-Shin Hahn, "On multipliers of p-integrable functions", *Trans. Amer. Math. Soc.* 128 (1967), 321-335.

[5] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis* I. *Structure of topological groups, integration theory, group representations* (Die Grundlehren der mathematischen Wissenschaften, 115. Springer-Verlag, Berlin, Heidelberg, New York, 1963).

[6] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis* II. *Structure and analysis for compact groups. Analysis on locally compact abelian groups* (Die Grundlehren der mathematischen Wissenschaften, 152. Springer-Verlag, Berlin, Heidelberg, New York, 1970).

[7] Noël Lohoué, "Une condition d'appartenance à $A_p(T)$ ", *C.R. Acad. Sci. Paris Sér. A* 270 (1970), 736-738.

[8] Yoshikazu Uno, "Lipschitz functions and convolution", *Proc. Japan Acad.* 50 (1974), 785-788.

[9] Yoshikazu Uno, "Lipschitz functions and convolution on bounded Vilenkin groups", *Sci. Rep. Kanazawa Univ.* 23 (1978), 1-6.

[10] В.Я. Виленкин [N.Ja. Vilenkin], "Об одном классе полных ортонормальных систем" [On a class of complete orthonormal systems], *Izv. Akad. Nauk SSSR Ser. Mat.* **11** (1947), 363-400; English Transl: *Amer. Math. Soc. Transl.* (2) **28** (1963), 1-35.

[11] P.L. Walker, "Lipschitz classes on 0-dimensional groups", *Proc. Cambridge Philos. Soc.* **63** (1967), 923-928.

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