Ed Cline*
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SOME GROUPS WHOSE S_3-SUBGROUPS HAVE MAXIMAL CLASS

ED CLINE

1. Introduction

In this paper, we investigate several classes of groups, among which the most general is defined as follows:

DEFINITION 1.1. A finite group G is a *SR-group* if it contains a subgroup P_1 of order 3 satisfying:

(a) A/S_3-subgroup P_2 of $N_G(P_1)$ is elementary of order 9;

(b) $N_G(P_2)/P_2$ acts semi-regularly by conjugation on the conjugates of P_1 contained in P_2.

To emphasize the role of P_1, we sometimes say G is a SR-group with respect to P_1.

The main result of this paper is

THEOREM 1.2. *If G is a SR-group, then $0^3(G)$ is a proper subgroup of G.*

It is clear that the definition of SR-groups can be easily generalized to primes other than 3, but the conclusion of Theorem 1.2 does not carry over to these primes.

The class of SR-groups contains several interesting subclasses, e.g., let X be a finite group, P_1 a subgroup of Aut(X) such that $|P_1| = |C_X(P_1)| = 3$. Then the semidirect product $G = P_1X$ is a SR-group. If $X = PSL(3, q)$, where q is congruent to 1 mod 3 but not mod 9, let α be the automorphism of X induced by the matrix $\begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, where λ is a primitive cube root of unity in the field with q elements. If P_1 is the cyclic group generated by α, the semidirect product $G = P_1X$ is a SR-group.

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The proof of Theorem 1.2 is given in section 2. In section 3, we consider a smaller class of groups called *SRTI* groups (cf. Def. 3.1), and characterize the 3-solvable groups in this class. If we let G be a *SRTI* group which is minimal subject to non-3-solvability, the results of this section yield an analogue to the maximal subgroup theorem (Theorem 8.6.3 of [5]) for G. However, we do not include this result here.

In section 4, we apply Theorem 1.2 to the theory of Frobenius Regular groups as defined by Keller in [7]. We recall G is a Frobenius Regular group if it contains a subgroup M such that $N_G(M) = MQ$ is Frobenius with kernel M, and M and Q are *TI* sets in G.

Frobenius Regular groups may be viewed as a two parameter family of groups if we specify the number α of transitive constituents of MQ which have length $|M|$, and the number β of constituents of length $|MQ|$. If this is done, we call a Frobenius Regular group G an (α, β) group.

THEOREM 1.3. *Let G be a $(2, \beta)$ group resrepented on the cosets of MQ. Let $\pi = \{2, 3\}$, and consider the characteristic series,*

$$G \geq G_1 = 0^{\pi'}(G) \geq G_2 = 0^3(G_1) > 1.$$

If $\tau = [G_1 : G_2]$, then

(i) *G_2 is simple;*

(ii) *$\tau \leq 3$;*

(iii) *G_2 is a $(2, \sigma)$ group on $M(Q \cap G_2)$, where $\sigma = [G : G_2]\beta$;*

(iv) *$N_{G_2}(Q \cap G_2)$ is a Frobenius group;*

(v) *$G = QG_2$.*

The notation in this paper is consistent with that of [4], with the exception that we use $V_G(X; Y)$ to denote the weak closure of X in Y with respect to G.

2. Proof of Theoerm 1.2.

It is clear from the definition of *SR*-groups that the S_3-subgroups of an *SR*-group have maximal class. We use the following properties of such groups:

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LEMMA 2.1. (*Blackburn* [1]) *Let P be a p-group, p a prime.*

(i) *If π is an element of order p in P such that $C_P(\pi)$ has order p^2, then P has maximal class.*

(ii) *If P has maximal class, the normal subgroups of P contained in P' form a chain.*

(iiii) *If $p = 3$, P' is abelian with at most 2 generators.*

The key step in the proof of Theorem 1.2 is the following description of the Sylow 3-structure of a SR-group.

THEOREM 2.2. *Let G be a SR-group with respect to P_1. Suppose P is a S_3-subgroup of G, has order 3^n, and contains P_1, then*

(i) *for each $i = 1, \cdots, n$ there is a unique subgroup P_i of P such that $|P_i| = 3^i$, and P_i contains P_1;*

(ii) *if $1 < i < n$, $P_{i+1} = N_G(P_i)$;*

(iii) *P has index at most 2 in its normalizer;*

(iv) *If ω is an involution in G which normalizes P, then ω acts regularly on $P/D(P)$;*

(v) *$N_G(P_1)$ has a normal 3-complement B. If $B \neq 1$, then $N_G(P_1)/P_1$ is Frobenius with kernel BP_1/P_1.*

Proof. Part (b) of the definition of SR-groups says that if $P_2 = N_P(P_1)$, then P_2 is a self-normalizing S_3-subgroup of $N_G(P_1)$, so (v) is an immediate consequence of Burnside's theorem.

For the proof of (i) and (ii), we use induction on i, noting that (i) is obvious for $i = 1$, while (ii) is vacuous. We may assume for any conjugate P_1^μ of P_1 contained in P, that (i) holds for all $k \leq i$, and (ii) holds for all $j \leq i-1$. Of course, we assume $1 < i < n$. Note first that

(2.1) $N_G(P_i)/P_i$ is semi-regular on the conjugates of P_{i-1} (under G) which are contained in P_i.

Suppose $\tau \in N_G(P_i)$ normalizes some conjugate P_{i-1}^μ of P_{i-1}, and that P_{i-1}^μ is contained in P_i. By (i), P_{i-1}^μ and P_i are the unique subgroups of orders 3^{i-1} and 3^i respectively of P which contain P_1^μ. By part (ii), $P_i = N_G(P_{i-1}^\mu)$ if $i > 2$, so $\tau \in P_i$ and (2.1) follows if $i > 2$. When $i = 2$, (2.1) is part of the definition of SR-groups, so (2.1) holds in all cases.

Since $P_1 \leq P_i$, Lemma 2.1 implies P_i has maximal class, hence $P_i/D(P_i)$ is elementary of order 9, and it follows that P_i contains at most four conjugates of P_{i-1}. Since $i < n$, the index $[N_G(P_i) : P_i]$ is divisible by 3. Since complete reducibility implies any involution in $N_G(P_i)$ must normalize at least two maximal sugroups of P_i, (2.1) implies $[N_G(P_i) : P_i] = 3$.

Let H be any subgroup of order 3^{i+1} of which contains P_1. The uniqueness of P_i implies P_i is a maximal subgroup of H, hence is normal in H, so $H = N_G(P_i) = P_{i+1}$ is unique. This completes the proof of parts (i) and (ii). We note that for $i = n$, (2.1) is a consequence of (i) and (ii). If ω is an involution in $N_G(P)$, ω normalizes at least two maximal subgroups of P. Since it cannot normalize P_{n-1}, (iii) follows at once from (2.1), and so does (iv).

COROLLARY 2.3. *Assume the hypothesis of Theorem* 2.2. *If P contains an elementary subgroup E of type* (3,3,3,), *then $P = P_1E$ has order* 81, *and is self-normalizing in G.*

Proof. By Lemma 8.4 of [4], P has an abelian normal subgroup A with 3 generators. By Lemma 2.1, it follows that A is maximal in P, and we can assume $E = \Omega_1(A)$. Then P_1 fixes exactly three elements of E, so P_1E is isomorphic to the wreath product of a cyclic group of order 3 with itself. Here $P_1(P_1E)'$ is the unique subgroup of P_1E which is non-abelian of order 27 and exponent 3, hence is characteristic in P_1E. By Theorem 2.2, it follows that $N_G(P) \leq P_1E = P$, and the proof is complete.

As a second application of Theorem 2.2, we obtain more information about the 3-local subgroups of G in

LEMMA 2.4. *Let G satisfy the hypothesis of Theorem* 2.2. *Let A be a maximal abelian normal subgroup of P, and $C = C_P(Z_2(P))$. If P has class at least* 3, *and P' is not weakly closed in P, either*

(i) $A = C = V_G(P'; P)$, *or*

(ii) $|P| = 81$, *and $C = A$ has type* (9,3).

In case (i), *either $N_G(A)/C_G(A)$ is isomorphic to $SL(2,3)$, or P has order* 81, *A is elementary, and $N_G(A)/C_G(A)$ is isomorphic to A_4 or the non-abelian group of order* 39.

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Proof. Since P' is not weakly closed in P, Lemma 2 of [3] implies $A = C$. If $|P| > 81$, the proof of Lemma 2 of [3] applies in this situation and yields

$$A = V_G(P'; P)$$

as well as

$$SL(2,3) \leq N_G(A)/C_G(A). \tag{2.2}$$

By Corollary 2.3, A has two generators, so Theorem 2.2 implies equality in (2.2).

Suppose $|P| = 81$, and $C = A$ does not have type $(9,3)$. Then A is elementary, and Corollary 2.3 implies P is self-normalizing in G. It follows that $N_G(A)/C_G(A)$ is a Frobenius group, and is isomorphic to a subgroup of $GL(3,3)$. The last statement of the lemma follows easily from this.

Remark. If $V = V_G(P'; P)$, $|P| = 81$, and A has type $(9,3)$, it is not difficult to see V must be non-abelian of order 27 and exponent 3. Here $N_G(V)/C_G(V/D(V))$ is isomorphic to $SL(2,3)$.

The proof of Theorem 1.1 is now easy. Grün's theorem, and part (iv) of Theorem 2.2 imply $N_G(P')/P'$ has a proper 3-factor group. If P has class 2, or if case (ii) of Lemma 2.4 occurs, then $N_G(P')$ contains $N_G(C)$, and Theorem 1 of [3] implies

$$0^3(N_G(P')) \cap N_G(C)) < N_G(C)).$$

By Lemma 2.4, $0^3(N_G(C)$ in all cases, so Theorem 1.1 follows from Theorem 1 of [3].

3. SRTI groups.

If G is a 3-solvable SR-group, Lemma 1.2.3 of [6] and Lemma 2.1 show that if P is a S_3-subgroup of G, then P' is contained in $0_{3'3}(G)$. By the Frattini argument,

$$G = N_G(V_G(P'; P))0_{3'}(G). \tag{3.1}$$

It is not hard to show (3.1) is the best possible result for 3-solvable SR-groups, so we consider a slightly stronger set of conditions which yield an improvement of (3.1).

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DEFINIITON 3.1. *A group G is a SRTI-group* if it contains a subgroup P_1 of order 3 which satisfies:

(a) If P_2 is a S_3-subgroup of $N_G(P_1)$, P_2 is elementary of order 9;

(b) P_2 is self-centralizing in $N_G(P_1)$;

(c) if $B = 0_{3'}(N_G(P_1))$, then $Q = P_1 \times B$ is a TI set in G.

Remark: Throughout this section, we use the notation introduced in Definition 3.1. If G is a SRTI-group, and π is the set of primes dividing the order of B, Frobenius' theorem on normal complements shows G satisfies D_π.

Our first lemmas provide some basic properties of SRTI-groups.

LEMMA 3.2. *If G satisfies (a) and (c) of Definition 3.1, either G has a normal 3-complement, or G is a SRTI-group.*

Proof. Since $Q = P_1 \times B$ is a TI set in G, $C_G(P_2) = P_2 \times C_B(P_2)$. If for some μ in G, $Q^\mu \cap P_2 > 1$, then $C_G(P_2) \leq N_G(Q^\mu)$, hence

$$C_B(P_2) \leq B^\mu \cap B.$$

If $C_B(P_2)$ is non-trivial, it follows that μ normalizes Q, hence exactly one conjugate of Q intersects P_2 non-trivially. Since $N_G(P_2)$ permutes these conjugates among themselves, $N_G(P_2) \leq N_G(Q)$, and Burnside's theorem implies G has a normal 3-complement.

LEMMA 3.3. *If G is a SRTI-group, it is also a SR-group.*

Proof. Suppose $P_1^\mu \leq P_2$, and an element σ of $N_G(P_2)$ normalizes P_1^μ. Then P_1^μ is contained in $Q^\mu \cap Q^{\mu\sigma}$, so σ normalizes Q^μ. Thus σ lies in $N_G(Q^\mu) \cap N_G(P_2)$. By part (b) of Definition 3.1, P_2 is self-normalizing in $N_G(P_2)$ so σ lies in P_2, and G is a SR-group.

For the remainder of this section, we let G be a *SRTI*-group with respect to P_1, and let P be a S_3-subgroup of G which contains P_1. Since G is a *SR*-group, we let P_i be the unique subgroup of P which contains P_1 and has order 3^i. We are interested in the properties of the subgroups of G which are normalized by P_i for various choices of i.

LEMMA 3.4. *Let X be a group whose S_3-subgroup P^* has maximal class. If every abelian subgroup of P^* has two generators, and if P^* is self-normalizing in X, then X has a normal 3-complement.*

Proof. Let $C = C_{P^*}(Z_2(P^*))$. By Theorem 1 of [3], and induction, we may assume C is normal in X, and if C is non-abelian, we may assume $P^{*\prime}$ is normal in X. Furthermore, if $Y \leq \{C, P^{*\prime}\}$, we know that X/Y has a normal 3-complement H/Y. Since Y is generated by two elements, $H/C_H(Y)$ is isomorphic to a 2-subgroup of $SL(2,3)$ which is normalized by a S_3-subgroup of $SL(2,3)$. Since P^* is self-normalizing in X, it follows that $H = C_H(Y)$ has a normal 3-complement, and the proof is complete.

LEMMA 3.5. (a) Let μ be an element of G, and X a subgroup of G such that $P_1^\mu \leq X \leq G$. One of the following occurs:

(i) $|X|_3 = 3$; X has a normal 3-complement;

(ii) $3 < |X|_3 < |G|_3$; X has a normal 3-complement; X is a $SRTI$-group;

(iii) $|X|_3 = |G|_3$; X is a $SRTI$-group.

(b) If N is normal in G, and 9 divides the index $[G : N]$, then G/N is a $SRTI$-group with respect to P_1N/N.

Proof. (a) Clearly, we can assume $P_1^\mu = P_1$. Also (i) is obvious, so suppose P^* is a S_3-subgroup of X, and $\tilde{P}$ a S_3-subgroup of G satisfying $P_1 < P^* < \tilde{P}$. Since G is a SR-group, Theorem 2.2 implies P^* has index 3 in its normalizer in G, thus P^* is self-normalizing in X. By Lemma 2.1, P^* has maximal calss, and by Corollary 2.3, P^* satisfies the hypotheses of Lemma 3.4. It follows that X has a normal 3-complement.

Now $B' = 0_{3'}(N_X(P_1)) = B \cap X$ is the normal 3-complement of $N_X(P_1)$. If $\mu \in X$ is chosen so that $Q' = P_1 \times B'$ satisfies $Q' \cap (Q')^\mu > 1$, then $\mu \in N_X(P_1) \leq N_X(Q')$, so Q' is a TI set in X. All other parts of the definition are clearly satisfied, so (ii) follows. Since this paragraph applies equally well to part (iii), (a) holds.

For the proof of (b), we denote homomorphic images in G/N by barring the appropriate letter, e.g., $\bar{G} = G/N$. Assume first 3 divides the order of N. Theorem 2.2 implies $N_{\bar{G}}(\bar{P}_1)$ is elementary of order 9, since 9 divides the order of $\bar{G}$, so in this case Definition 3.1 holds.

If N is a 3'-group, we can use the Frattini argument to show $N_G(P_1N) = N_G(P_1)N$, so (a) and (b) of the definition are clear. Also $\bar{B}$ is the normal 3-complement of $N_{\bar{G}}(\bar{P}_1)$, and we need only show $\bar{Q}$ is a TI set in $\bar{G}$.

Suppose for $\bar{\mu} = N\mu$, $\bar{Q} \cap \bar{Q}^{\bar{\mu}} > \bar{1}$. Then $QN \cap Q^\mu > 1$. Since QN is a Frobenius group with Q as the subgroup fixing a letter, if π is the set of primes dividing the order of Q, QN satisfies D_π, so there is an η in N such that $Q^\mu \geq QN \cap Q^\mu$. It follows that $Q^\eta = Q^\mu$, and this completes the proof, since $\bar{Q}^\mu = \bar{Q}^\eta = \bar{Q}$.

We have already noted G is a SR-group, and introduced the subgroups P_i. We now state a uniqueness property for the P_i which is useful for the investigation of the subgroups normalized by the P_i:

(3.2) If $i \geq 2$, and $P^\tau \geq P_i$, then $P^\tau = P$.

To see this, note that for some $j \geq i$, $P^\tau \cap P = P_j = (P^\tau)_j$ by Theorem 2.2. If $j < n$, Theorem 2.2 implies $P_{j+1} = (P^\tau)_{j+1}$ which is impossible, so (3.2) follows.

LEMMA 3.6. *Suppose* $i \geq 2$, *and* P_i *normalizes the subgroup* U *of* G. *Suppose* $U \cap P_i = 1$. *Then* U *is a* $3'$-*group, and* P_i' *centralizes* U. *If* $i > 2$, $|U|$ *is prime to* $|N_G(P_1)|$, *and* U *is nilpotent of class at most* 2.

Proof. Let S be a S_3-subgroup of U chosen so that P_i normalizes S. If P^* is a S_3-subgroup of G containing P_iS, (3.2) implies $P^* = P$, and if $S > 1$, $S \cap P_i \geq Z(P)$, a contradiction. Thus U is a $3'$-group.

Since P_2 is abelian, we may assume $i > 2$, and let π be the set of primes dividing the order of B. We show U is a π'-group. If not, let $T > 1$ be a S_t-subgroup of U for some prime t in π. Since U is a $3'$-group, we may assume T is normalized by P_i. Since G is a $SRTI$-group, G satisfies D_π, so for some σ in G, $T \leq B^\sigma$. Since B is a TI set in G, it follows that P_i normalizes B^σ, which contradicts the fact that a S_3-subgroup of $N_G(B^\sigma)$ has order 9. If P_1^μ is any conjuagte of P_1 contained in P_i, then P_1^μ must act Frobeniusly on U, hence the main theorem of [8] implies U is nilpotent of class at most 2.

Let W be the kernel of the representation of P_i on U, and suppose $W < P_1'$. Consider the action of $\bar{E} = \bar{P}_1 \times Z(\bar{P}_i)$ on U, where barring a letter denotes taking homomorphic images in P/W. Since $\bar{E}$ is properly contained in $\bar{P}_i$, Theorem 2.2 implies $\bar{E}$ contains exactly three conjugates of $\bar{P}_1$. The last sentence of the preceeding paragraph shows that these conjugates all act Frobeniusly on U. The remaining cyclic subgroup of $\bar{E}$ is $Z(\bar{P}_i)$, and since $\bar{E}$ acts cyclicly on every irreducible submodule of $U/D(U)$, it follows that $Z(\bar{P}_i)$ centralizes $U/D(U)$, hence also U. This contradicts the faithfulness of $\bar{P}_i$ on U, hence W contains P_i'.

We obtain a corollary,

LEMMA 3.7. (i) *Suppose* $2 < i < n$, P_i *normalizes* U, *and* $U \cap P_i = 1$, *then* $P_i \cap P'$ *centralizes* U.

(ii) *If* 27 *divides the order of* G, *and* P_2 *normalizes* U, $U \cap P_2 = 1$, *then either* $U \leq N_G(P_1)$, *or* $U \leq C_G(Z(P))$. *In either case* U *is nilpotent.*

Proof. (i) The fact that $i < n$ insures P_i/P_i' contains three conjugates of P_1P_i'/P_i' under action by P_{i+1}, so the same argument applies.

(ii) We know U is a $3'$-group. Let $Q^* = Q \cap U$, so Q^* is a self-normalizing Hall subgroup of U which is a TI set. Either $U = Q^*$, $Q^* = 1$, or $U = Q^*K$ is Frobenius with kernel K. If U is Frobenius, it follows that all conjugates of P_1 contained in P_2 act Frobeniusly on K, hence, as above, we obtain $K \leq C_G(Z(P))$. However, the definition of $SRTI$-groups shows that $Z(P)Q^*$ is a Frobenius group which normalizes K. Since Q^* is non-trivial on K, $Z(P)$ does not centralize K, and we have a contradiction.

If $Q^* = 1$, $Z(P)$ centralizes U, so the proof of (ii) is complete. In both cases, U is nilpotent of class at most 2.

We can now apply these results, which hold in general for $SRTI$-groups, to characterize 3-solvable $SRTI$-groups.

THEOREM 3.8. *Let* G *be a* 3-*solvable SRTI-group with* S_3-*subgroup* P. *Then* $V_G(P'; P)$ *is normal in* G. *If* $U = 0_{3'}(G)$, *and* 27 *divides the order of* G, U *is nilpotent of class at most* 2, *and* $B = 1$. *If* $|G|_3 = 9$, *either* G *has a normal* 3-*complement* U, *and there is a nilpotent normal subgroup* K *of* G *contained in* U *such that* $P_1U = QK$, *and* $Q \cap K = 1$, *or* U *is nilpotent of class at most* 2, *and* $B = 1$, *or* U. *The possibilities for* G/U *are given in Theorem* 2.2, *and Lemma* 2.4.

Proof. By (3.1), to show $V_G(P'; P)$ is normal in G, it suffices to show P' centralizes U, but this follows from Lemma 3.6. If 27 divides the order of G, it is clear that U is nilpotent of class at most 2. Now B is a Hall subgroup of G, and it follows from (3.1) and Lemma 2.4 that B is contained in U, hence $B = 1$.

Suppose $|G|_3 = 9$. If P_2 is self-normalizing in G, G has a normal 3-complement U. If $U > B > 1$, it is a nilpotent self-normalizing TI set in U, so U is a Frobenius group with kernel K, and the second statement of Theorem 3.8 is obvious. This completes the proof since Theorem 2.2 and Lemma 2.4 are essentially an analysis of the possibilities for G/U.

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4. Proof of Theorem 1.3

Throughout this section, let G be a $(2, \beta)$ group on MQ. The proof of Theorem 1.3 consists of the following lemmas.

LEMMA 4.1. *Let X be a (α, β) group on $\tilde{M}\tilde{Q}$. If α is even, $T = S_2(Q)$ is cyclic. If T is non-trivial, then $X = F(X)\tilde{M}\tilde{Q}$.*

Proof. Suppose T is non-trivial. By Lemma 2.2 of [7], the index $[N_X(Q) : Q] = \alpha + 1$ is odd. Since $N_X(T)$ is contained in $N_X(Q)$, it follows that T is a S_2-subgroup of X. It is well known that T is either quaternion or cyclic (cf. Theorem 10.3.1 of [5]). If T is quaternion, a result of Brauer and Suzuki [2] implies $Z(X/0_{2'}(X))$ has order 2, hence $\tilde{M} \leq K = 0_{2'}(X)$. The definition of (α, β) groups implies $\tilde{M}$ is not normal in G, and since $\tilde{M}$ is a Hall subgroup of G, $\tilde{M}$ is not normal in K.

By the famous theorem of Feit and Thompson [4], there is a prime p such that $0_p(K) = K_1$ is not the identity. Clearly $\tilde{M} \cap K_1 = 1$, so $\tilde{M}K_1$ is a Frobenius group. However, $\tilde{M}$ is nilpotent of odd order, so it follows that $\tilde{M}$ is cyclic. Since this contradicts ths fact that $T\tilde{M}$ is Frobenius and T is quaternion, it follows that T must be cyclic.

Now K is a normal 2-complement for X. Suppose T is non-trivial, and $F(X)\tilde{M}\tilde{Q} < X$. Since $\tilde{M}$ is not normal in X, $F(X) \cap \tilde{M}\tilde{Q} = 1$, hence $F(X) = F(K)$ has order prime to $|\tilde{M}\tilde{Q}|$. Let K_1 be a normal subgroup of K minimal with respect to the containments $K \geq K_1 > F(X)$. If $M_1 = \tilde{M} \cap K_1 > 1$, then M_1 is a Hall subgroup of K_1, and the Frattini argument implies $X = F(X)N_X(M_1)$, a contradiction since $N_X(M_1)$ is contained in $\tilde{M}\tilde{Q}$. Thus it follows that $\tilde{M}K_1$ is a Frobenius group, hence by [9], K_1 is nilpotent, a contradiction to $K_1 > F(X) = F(K)$.

COROLLARY 4.2. *The order of Q is odd.*

Proof. If $T = S_2(Q)$, then $G = F(G)MQ$ by Lemma 4.1. Since M is a Hall subgroup of G, and is non-normal in G, we obtain $F(G) \cap MQ = 1$. If Y is any MQ-invariant section of $F(G)$, then $C_Y(Q) > 1$, since $MF(G)$ is a Frobenius group, and $|F(G)|$ is prime to $|MQ|$. Since $[N_G(Q) : Q] = 3$, it follows that $F(G)$ is an elementary 3-group, and MQ acts irreducibly on $F(G)$. In particular, Q acts as a multiple of the regular $Z_3(Q)$-module (here Z_3 is the field with three elements). Since $[N_G(Q) : Q] = 3$, it follows that

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$|F(G)| = 3|Q|$. Since the smallest prime q dividing M is at least 5, the Frobenius group MQ does not have a faithful irreducible representation of degree $|Q|$ over Z_3, so the corollary follows.

This corollary shows the Sylow subgroups of Q are cyclic, hence in particular, Q has a normal 3-complement B by Burnside's theorem. Let U be a S_3-subgroup of $N_G(Q)$, and let $V = U \cap Q$ be the corresponding S_3-subgroup of Q.

LEMMA 4.3. *Either G has a normal 3-complement, or U is elementary of order at most 9.*

Proof. Suppose U is not elementary of order at most 9. Then it follows that V contains a characteristic subgroup K of U such that $K > 1$. Since Q is a TI set in G, $N_G(U) \leq N_G(Q)$ and it follows that U is a S_3-subgroup of G. Since $N_G(Q)$ has a normal 3-complement, if $C > 1$ is any characteristic subgroup of U contained in V, our previous argument shows $N_G(C)$ has a normal 3-complement. If U is abelian, G has a normal 3-complement by Burnside's Theorem.

Suppose U is non-abelian, then U contains an abelian subgroup of type $(3, 3)$ so there is an element σ of order 3 in U such that $U = \langle\sigma\rangle V$. Since σ is an automorphism of V of order 3, a simple computation shows

$$U' \leq Z(U) = D(V) = D(U) < U.$$

This implies U has class 2, hence $\Omega_1(U)$ has exponent 3. Since U does not have exponent 3, $\Omega_1(U) \leq \langle\sigma\rangle \times D(U)$. If A is an abelian subgroup of U for which the minimum number of generators is maximal, then $\Omega_1(A) = \Omega_1(U)$ and since U is non-abelian, we have $U > AD(U) \geq \langle\sigma\rangle \times D(U)$. The maximality of $\langle\sigma\rangle \times D(U)$ in U implies A is contained in $\langle\sigma\rangle \times D(U)$. If $J(U)$ is the subgroup of U generated by all abelian subgroups of U for which the minimum number of generators is maximal, then

$$J(U) = \langle\sigma\rangle \times D(U) \tag{4.1}$$

Since $Z(T)$ is contained in V, our remarks in the first paragraph show $C_G(Z(T)) \leq N_G(Q)$ has a normal 3-complement. If $|(U)| > 3$, then the subgroup $\Omega_1(D(U))$ is characteristic in $J(U)$, so $N_G(J(U))$ is contained in $N_G(Q)$, and it has a normal 3-complement. By Thompson's theorem [9], G has a normal 3-complement.

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Suppose $J(U)$ is elementary. Then U has order 27 by (4.1). $C_G(J(U))$ is contained in $N_G(Q)$, hence has odd order. If R is the normal 3-complement in Q, it follows that $C_G(J(U)) = J(U) \times R'$, where $R' = C_R(J(U))$. Since R' is a normal 3-complement in $C_G(J(U))$, if $R' > 1$, $N_G(UC_G(J(U)) \leq N_G(Q)$, and if $R' = 1$, $N_G(UC_G(J(U))) = N_G(U) \leq N_G(Q)$, so in any case it has odd order. Since $\bar{N} = N_G(J(U))/C_G(J(U))$ is isomorphic to a subgroup of $GL(2,3)$, our preceeding statement shows $\bar{U} = UC_G(J(U))/C_G(J(U))$ is a self-normalizing S_3-subgroup of $\bar{N}$. This implies $\bar{N} = \bar{U}$, hence $N_G(J(U))$ has a normal 3-complement and so does G by Thompson's Theorem.

LEMMA 4.4. *G does not have a normal 3-complement.*

Proof. Let H be a normal 3-complement for G. Since $N_G(M) = MQ$ is Frobenius, it follows that M is contained in H. Since M is a nilpotent Hall subgroup of G, the results of [10] allow us to use the Frattini argument to obtain $G = QH$. In particular, Q contains a S_3-subgroup of G which is not the case.

Consider the structure of $N_G(Q)$. Corollary 4.2 implies Q is metacyclic, so if we let Q_1 be the maximal normal cyclic Hall-subgroup of Q, $Q = RQ_1$ where R is a cyclic Hall-subgroup of Q. We note that Lemmas 4.3 and 4.4 imply V is central in Q, hence R is a $3'$-Hall subgroup of G. Choose R so that U normalizes R, then UR is represented on Q_1. Since Q_1 is cyclic, and R is non-trivial on Q_1, the group UR must be abelian. If Φ is the set of primes dividing the order of R, Burnside's theorem implies G has a normal Φ-complement G^*, and clearly $G = QG^*$. Let $Q^* = Q \cap G^*$, and let $B^* = 0_{3'}(Q)$. If $C = C_{B^*}(U)$, then C is a Hall subgroup of G^*, and for the set of primes $\tilde{\Phi}$ dividing $|C|$, G^* has a normal $\tilde{\Phi}$-complement G_1. Clearly $QG_1 = G$, and since G/G_1 has order prime to 6, G_1 contains $0^{\pi'}(G)$. The proof of Theorem 1.3 will be complete if we can show G_1 satisfies parts (i)-(iv) of Theorem 1.3 since this also implies $G_1 = 0^{\pi'}(G)$. Let $Q_1 = V \times \tilde{B} = Q \cap G_1$.

LEMMA 4.5. *If $V > 1$, G_1 is a SRTI-group.*

Proof. Consider $V = P_1$ by Lemmas 4.3, and 4.4, V is cyclic of order 3, $Q_1 = V \times \tilde{B}$ is a *TI* set in G_1, and an S_3-subgroup P_2 of $N_{G_1}(P_1)$ is elementary of order 9. If $\tilde{B} > 1$, the discussion above shows P_2B/P_1 is a Frobenius group, hence P_2 is self-centralizing in $N_{G_1}(P_1)$, and the lemma follows.

By Theorem 1.2, either $V = 1$, or there is a subgroup G_2 of index 3 in G_1 which is normal in G_1, and which satisfies $VG_2 = G_1$.

LEMMA 4.6. $\tilde{B} > 1$.

Proof. If $\tilde{B} = 1$, $Q \cap G_2 = 1$. Since M is not normal in G, it is not normal in G_2, hence $G_2 = KM$ is a Frobenius group, and $G = KMQ$.

As in Corollary 4.3, K is an elementary abelian group of order $3^{|Q|}$, and MQ operates faithfully and irreducibly on K. Thus M is cyclic and $|M|$ is prime to 6. The same contradiction obtained in Corollary 4.3 applies here, so the lemma holds.

The next lemma completes our proof.

LEMMA 4.7. *G_2 is simple.*

Proof. Let N be a non-identity normal subgroup of G_2. Since $\tilde{B} > 1$, $N_{G_2}(\tilde{B})$ is a Frobenius group. From this it follows that $M \cap N > 1$. By the Frattini Argument, and the fact that M is a TI set, $G = NMQ$. If $N < G$, G/N is isomorphic to a factor group of MQ, and this contradicts the fact that $N_{G_2}(B)$ is Frobenius. Thus $N = G$ is simple.

The simplicity of G_2 implies $G_2 = 0^3(G_1)$, and $G_1 = 0^{\pi'}(G)$. The fact that G_1 and G_2 are $(2, \gamma)$ groups for the appropriate choices of γ is trivial. The fact that $[G_1 : G_2] \leq 3$ follows immediately from the statement $G_1 = VG_2$. This completes the proof.

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University of Minnesota