Canad. Math. Bull. Vol. **25** (2), 1982

INTEGRAL REPRESENTATION BY BOUNDARY VECTOR MEASURES

BY
PAULETTE SAAB

ABSTRACT. In this paper we show that if X is a compact Hausdorff space, A is an arbitrary linear subspace of $C(X, C)$, and if E is a Banach space, then each element L of $(A \otimes E)^*$ can be represented by a boundary E^*-valued vector measure of the same norm as L.

Introduction. All the results obtained in this paper are valid for real and complex Banach spaces. However we shall deal only with complex Banach spaces.

Let X be a compact Hausdorff space and let E be a Banach space with dual E^*. Let $C(X, E)$ denote the Banach space of all continuous E-valued functions defined on X under the supremum norm. In [6] O. Hustad showed that if A is a linear subspace of $C(X, C)$ that separates the points of X and contains the constant functions, then each continuous linear function l on A can be represented by a "boundary measure" that has the same norm as l. Later Choquet [2] and Fuhr and Phelps [5] independently extended Hustad's theorem to the case in which the subspace A does not contain the constant functions. In [9], we showed that if the compact space X is metrizable and E is an arbitrary Banach space, then it is possible to extend Hustad's theorem to those subspaces of $C(X, E)$ that are of the form $A \otimes E$, where A is a linear subspace of $C(X, C)$ that separates points of X and $A \otimes E$ is the closed linear subspace of $C(X, E)$ generated by elements of the form $a \otimes v$, with a in A and v in E and where for all x in X we have;

$$a \otimes v(x) = a(x) \cdot v$$

In this paper we shall prove, using the technique we developed in [8], an extension of our result in [9]. Namely, we shall show that for any compact Hausdorff space X and any linear subspace A of $C(X, C)$, each continuous linear functional L on $A \otimes E$ can be represented by a boundary E^*-valued vector measure that has the same norm as L.

First let us collect some notations.

Received by the editors March 17, 1980.
A.M.S. (1970) subject classification: Primary 46A55, 46E40.

If V is a Banach space, we shall denote by V^* its topological dual.

It is known [7] that if Y is a compact Hausdorff space and E is a Banach space, the dual of $C(Y, E)$ is isometrically isomorphic to $M(Y, E^*)$, the space of w^*-regular E^*-valued vector measures defined on the σ-field of Borel subsets of Y and that are of bounded variation [3].

The space of all complex valued regular Borel measures on Y will simply be denoted by $M(Y)$. The subset of $M(Y)$ consisting of probability measures (resp., the positive measures) will be denoted $M_1^+(Y)$ (resp., $M^+(Y)$).

If μ is in $M(Y, E^*)$, and x is in E, we denote by $\langle x, \mu\rangle$ the element of $M(Y)$ defined as follows:

$$\langle x, \mu\rangle(B) = \mu(B)x \quad \text{for each Borel subset } B \text{ of } X.$$

1. **A representation theorem for point separating subspaces of** $C(X, C)$. Throughout this section X is a compact Hausdorff space, A is a linear subspace of $C(X, C)$ that *separates points* of X, and E is a Banach space. We shall denote by $A \otimes E$ the closed linear subspace of $C(X, E)$ generated by elements of the form $a \otimes x$, where a is in A and x is in E. Also we shall denote by U the unit ball of A^*, by $\phi: X \to U$ the canonical map, and by T the unit circle in the plane.

DEFINITION 1.1. A measure μ in $M(X, E^*)$ is called a *boundary vector measure* for A if its variation $|\mu|$ (when carried via ϕ) is maximal for the Choquet ordering on $M^+(U)$ [1].

The following Lemma can easily be obtained using the characterization of maximal measures on compact convex sets [1, 27.4].

LEMMA 1.2. *A vector measure μ in $M(X, E^*)$ is a boundary vector measure for A if and only if for each x in E the scalar measure $\langle x, \mu\rangle$ is a boundary measure for A.*

We are now ready to prove the main result of this section.

THEOREM 1.3. *Let X be a compact Hausdorff space, let A be a linear subspace of $C(X, C)$ that separates points of X, and let E be a Banach space. Then for each L in $(A \otimes E)^*$ there exists a vector measure μ in $M(X, E^*)$ such that*

(i) $\|\mu\| = \|L\|$,

(ii) $\int_X b\, d\mu = L(b)$ *for all b in $A \otimes E$, and*

(iii) *the measure μ is a boundary vector measure for A.*

Proof. If $A(U, E)$ denotes the Banach space of all continuous affine E-valued functions on U, then $A \otimes E$ embeds isometrically in $A(U, E)$ as follows: For $b=\sum_{i=1}^n a_i\otimes x_i$ we let $j(b)$ denote the element of $A(U,E)$ defined by

$$j(b)(a^*)=\sum_{i=1}^n a^*(a_i)\cdot x_i,\quad \text{for all } a^* \text{ in } U.$$

The mapping j is obviously linear. It is an isometry since the set of extreme points of U is included in $T\cdot\phi(X)$. Let L be an element of $(A\otimes E)^*$. With the help of the Hahn–Banach theorem, pick Φ in $A(U,E)^*$ such that, Φ when restricted to $j(A\otimes E)$, is equal to L and $\|\phi\|=\|L\|$.

By [8] the functional L can be represented by a measure λ in $M(U,E^*)$ such that

(i) $L(b)=\int_U j(b)\,d\lambda$ for all b in $A\otimes E$,

(ii) $\|\lambda\|=\|L\|$, and

(iii) the variation $|\lambda|$ of λ is maximal for the Choquet ordering on $M^+(U)$.

Since the measure $|\lambda|$ is maximal it is supported by $T\cdot\phi(X)$ [5]. Let $s:T\cdot\phi(X)\to T\times X$ be the Borel selection map defined by Fuhr and Phelps [5, Lemma 7.2]. Denote by $s(\lambda)$ the E^*-valued set function defined on Borel subsets of $T\times X$ as follows:

$$s(\lambda)(B)=\lambda(s^{-1}(B))\quad \text{for each Borel subset } B \text{ of } T\times X.$$

It is easily checked that $s(\lambda)$ is in $M(T\times X,E^*)$. Let μ be equal to $H_*s(\lambda)$, where for every ν in $M(T\times X,E^*)$ and every f in $C(X,E)$

$$H_*\nu(f)=\int_{T\times X} t\cdot f\,d\nu.$$

It can easily be checked that μ is in $M(X,E^*)$, and that for each x in E

$$\langle x, H_*s(\lambda)\rangle = Hs(\langle x,\lambda\rangle)$$

where H is Hustad's map see [6] or [5].

We claim that μ is our required element. For this, note that for each a in A and for each e in E

$$\mu(a\otimes e)=\langle e, H_*s(\lambda)\rangle(a)=Hs(\langle e,\lambda\rangle)(a)$$
$$=\int_{T\cdot\phi(X)} j(a\otimes e)\,d\lambda = L(a\otimes e).$$

This shows that $H_*s(\lambda)$ and L agree on $A\otimes E$. This proves (ii). To prove (i), it is easy to check that

$$\|H_*s(\lambda)\|\le\|\lambda\|=\|L\|,$$

hence $\|H_*s(\lambda)\|=\|L\|$.

Finally, since $|\lambda|$ is maximal for the Choquet ordering on $M^+(U)$, then for each x in E, the scalar measure $|\langle x,\lambda\rangle|$ is also maximal. This implies that

https://doi.org/10.4153/CMB-1982-022-4 Published online by Cambridge University Press

$Hs(\langle x, \lambda\rangle)$ is a boundary measure for A. An appeal to Lemma 1.2 shows that the vector measure $H_* s(\lambda)$ is a boundary vector measure for A since for each x in E $\langle x, H_* s(\lambda)\rangle = Hs(\langle x, \lambda\rangle)$. This completes the proof.

2. **A representation theorem for arbitrary subspaces of** $C(X, C)$. We shall now proceed to prove Theorem 1.3 for an arbitrary linear subspace A of $C(X, C)$.

For each a in A denote by $\tilde{a}$ the element of $C(\phi(X), C)$ defined by:

$$\tilde{a}(\phi(x)) = a(x) \quad \text{for all } x \text{ in } X.$$

It is clear the element $\tilde{a}$ is well defined. Denote by $\tilde{A}$ the set $\{\tilde{a} : a \in A\}$. If E is a Banach space, consider $\tilde{A} \otimes E$ the corresponding linear subspace of $C(\phi(X), E)$. The spaces $A \otimes E$ and $\tilde{A} \otimes E$ are isometrically isomorphic. We can now prove the main result of this paper.

THEOREM 2.1. *Let A be an arbitrary linear subspace of $C(X, C)$ and let E be a Banach space. Then for each L in $(A \otimes E)^*$ there exists a measure μ_L in $M(X, E^*)$ such that*

(i) $\|\mu_L\| = \|L\|$,

(ii) $\int_X b\, d\mu_L = L(b)$ *for all b in $A \otimes E$, and*

(iii) *the measure μ_L is a boundary vector measure for A.*

Proof. Let L be in $(A \otimes E)^*$ with $\|L\| = 1$. By virtue of the isometry of $A \otimes E$ and $\tilde{A} \otimes E$ we may and do assume that L is in $(\tilde{A} \otimes E)^*$. Apply Theorem 1.3 for $\tilde{A}$ and $\phi(X)$ to get a measure ν in $M(\phi(X), E^*)$ such that

(i) $\|\nu\| = \|L\|$,

(ii) $\int \tilde{b}\, d\nu = L(b)$ for all b in $A \otimes E$, and

(iii) the measure $|\nu|$ is maximal for the Choquet ordering on $M^+(U)$.

Since $|\nu|$ is in $M_1^+(\phi(X))$, there is a net of positive discrete measures $(\nu_i)_{i \in I}$ such that $\nu_i = \sum_{j=1}^{n_i} \alpha_j^i \varepsilon_{\phi(y_j^i)}$ and $\|\nu_i\| = 1$ for each i in I, and such that ν_i converges to $|\nu|$ in the weak* topology of $M_1^+(\phi(X))$.

For each $i \in I$, let

$$\mu_i = \sum_{j=1}^{n_i} \alpha_j^i \varepsilon_{y_j^i},$$

and note that the measure μ_i is in the weak* compact convex set $M_1^+(X)$. Let μ be a weak* cluster point of the net $(\mu_i)_{i \in I}$ in $M_1^+(X)$. It is a straightforward computation to show that $\phi(\mu) = |\nu|$. Since ν is in $M(\phi(X), E^*)$, it follows from [4, p. 389] that there exists a mapping $g : \phi(X) \to E^*$ that is $|\nu|$-essentially bounded by one, (scalarly) weak*-Borel measurable, and such that $\nu = g \cdot |\nu|$. Let $\mu_L = g \circ \phi \cdot \mu$ be the E^*-valued set function defined on Borel subset B of X by:

$$\mu_L(B)(x) = \int_B \langle g \circ \phi(\omega), x\rangle\, d\mu(\omega) \quad \text{for each } x \text{ in } E.$$

It is clear that μ_L is in $M(X, E^*)$ and that the variation $|\mu_L| \le \mu$. We claim that μ_L is our required element. To this end, note that for each a in A and for each x in E we have

$$\begin{aligned}\int_X a \otimes x \, d\mu_L &= \int_X \langle g \circ \phi(\omega), a(\omega) \cdot x \rangle \, d\mu(\omega) \\ &= \int_X \langle g \circ \phi(\omega), x \rangle \tilde{a}(\phi(\omega)) \, d\mu(\omega) \\ &= \int_{\phi(X)} \tilde{a} \otimes x \, d\nu = L(a \otimes x).\end{aligned}$$

This shows that $\mu_L = L$ on $A \otimes E$. Since $|\mu_L| \le \mu$, it follows that $\|\mu_L\| = \|\mu\| = 1$. Hence $|\mu_L| = \mu$. Finally, the measure μ_L is a boundary vector measure for A since $\phi(|\mu_L|) = |\nu|$ is maximal for the Choquet ordering on $M^+(U)$. This completes the proof.

ACKNOWLEDGEMENT. The author is grateful to Professor J. J. Uhl, Jr. for valuable discussions concerning this paper.

BIBLIOGRAPHY

1. G. Choquet, Lecture on Analysis, Volume II, Benjamin, New York, 1969. (MR40 #3253).

2. G. Choquet, Frontière-module et représentation intégrale [Résumé], Séminaire Choquet, Univ. de Paris (1971/1973), No. 8, 4 pp.

3. J. Diestel and J. J. Uhl, Jr., Vector measures. Math. Surveys, Volume 15, American Math. Soc., Providence, 1977.

4. N. Dinculeanu, Vector measures, Berlin, VEB Deutscher Verlag der Wissenschaften, 1966.

5. R. Fuhr and R. R. Phelps, Uniqueness of complex representing measures on the Choquet boundary, J. Functional Analysis 14 (1973), 1–27. (MR50 #14186).

6. O. Hustad, A norm preserving complex Choquet Theorem, Math. Scand. 29 (1971), 272–278. (MR48 #852).

7. I. Singer, Sur la meilleur approximation des fonctions abstraites continues à valeurs dans un espace de Banach. Rev. Roumaine Math. Pures Appl. 2 (1957), 245–262.

8. P. Saab, The Choquet integral representation in the affine vector-valued case. Aequationes Mathematicae 20 (1980), 252–262.

9. P. Saab, Representation intégrale dans des sous-espaces de fonctions à valeurs vectorielles. Séminaire Choquet, (1974–1975), No. 23, 10 p.

10. H. H. Schaefer, Banach Lattices and Positive Operators. Springer-Verlag, New York–Heidelberg–Berlin, 1974.

THE UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B.C., CANADA V6T 1Y4