H. Fujimoto
Nagoya Math. J.
Vol. 37 (1970), 91–106

ON THE AUTOMORPHISM GROUP OF A HOLOMORPHIC FIBER BUNDLE OVER A COMPLEX SPACE

HIROTAKA FUJIMOTO

Dedicated to Prof. K. Ono on his 60th birthday

§1. Introduction.

In [8], A. Morimoto proved that the automorphism group of a holomorphic principal fiber bundle over a compact complex manifold has a structure of a complex Lie group with the compact-open topology. The purpose of this paper is to get similar results on the automorphism groups of more general types of locally trivial fiber spaces over complex spaces.[1) We study automorphisms of a holomorphic fiber bundle over a complex space which has a complex space Y as the fiber and a (not necessarily complex Lie) group G of holomorphic automorphisms of Y as the structure group (see Definition 3. 1).

The main result is the following

THEOREM. *For any holomorphic fiber bundle* $\boldsymbol{B}$ *over a complex space* X, *the automorphism group of* $\boldsymbol{B}$ *is a* (*real*) *Lie group if the structure group is a locally compact subgroup of the holomorphic automorphism group of the fiber and* X *is* $*$-*strongly pseudo-concave* (Theorem 4. 1).

As a special case of this, we see that the group of all fiber-preserving holomorphic automorphisms of a locally trivial fiber space over a $*$-strongly pseudo-concave complex space is a Lie group if the holomorphic automorphism group of the fiber is locally compact.

Let $\boldsymbol{B}$ be a holomorphic fiber bundle over a compact normal complex space X and M be an analytic set of codimension $\geqq 2$ in X. We can prove

Received February 5, 1969

1) In this paper, a complex space means a reduced complex analytic space which is always assumed to be σ-compact and irreducible.

that any automorphism of the portion $\boldsymbol{B}|X-M$ of $\boldsymbol{B}$ over $X-M$ sufficiently near to the identity is the restriction of an automorphism of $\boldsymbol{B}$ if the structure group G is a locally compact subgroup of the holomorphic automorphism group of the fiber. In this case, the automorphism group $F(\boldsymbol{B}|X-M)$ of $\boldsymbol{B}|X-M$ is a Lie group. Moreover, if G is a complex Lie transformation group of the fiber, $F(\boldsymbol{B}|X-M)$ is shown to be a complex Lie group.

§2. Holomorphic maps of complex spaces into mapping spaces.

Let X, Y and Z be complex spaces. We denote the space of all holomorphic maps of Y into Z with the compact-open topology by $\mathrm{Hol}(Y,Z)$ and the space of all holomorphic automorphisms of Y as a subspace of $\mathrm{Hol}(Y) := \mathrm{Hol}(Y,Y)$ by $\mathrm{Aut}(Y)$.

DEFINITION 2.1. Take an arbitrary subset H of $\mathrm{Hol}(Y,Z)$. A map g: $X\to H$ is called to be *holomorphic* if the induced map $\tilde{g}(x,y) := g(x)(y)$ $(x\in X, y\in Y)$ of $X\times Y$ into Y is holomorphic (c.f. W. Kaup [5], p. 75).

By $\mathrm{Hol}(X,H)$ we denote again the space of all holomorphic maps of X into H with the compact-open topology, where H is considered as a topological subspace of $\mathrm{Hol}(Y,Z)$. Obviously, we may consider $\mathrm{Hol}(X, \mathrm{Hol}(Y,Z)) = \mathrm{Hol}(X\times Y, Z)$.

Let h: $X'\to X$ be a holomorphic map for another complex space X'. For any $g\in \mathrm{Hol}(X,H)$ $(H\subset \mathrm{Hol}(Y,Z))$, the composite $g\cdot h$: $X'\to H$ is also holomorphic. Particularly, the normalization μ: $X^*\to X$ of X gives the map μ^*: $\mathrm{Hol}(X,H)\to \mathrm{Hol}(X^*,H)$ defined as $\mu^*(g) = g\cdot\mu$ for each $g\in\mathrm{Hol}(X,H)$.

(2.2) *The topological space* $\mathrm{Hol}(X,H)$ *can be canonically identified with a closed subspace of* $\mathrm{Hol}(X^*,H)$.

The injectivity of μ^* is evident. While, $\mathrm{Hol}(X\times Y,Z)$ can be identified with a closed subspace of $\mathrm{Hol}(X^*\times Y,Z)$ because $\mu\times 1_Y$: $X^*\times Y\to X\times Y$ is a proper, nowhere degenerate surjective map, where 1_Y: $Y\to Y$ is the identity map. Since we may consider $\mathrm{Hol}(X,H)\subset \mathrm{Hol}(X\times Y,Z)$ and $\mathrm{Hol}(X^*,H)\subset \mathrm{Hol}(X^*\times Y,Z)$, we conclude easily the assertion (2.2).

Each $g\in\mathrm{Hol}(X, \mathrm{Hol}(Y))$ gives a map $g^* := 1_X\times\tilde{g}\in\mathrm{Hol}(X\times Y)$ (i.e. $g^*(x,y) = (x, g(x)y)$ for any $x\in X$, $y\in Y$). By this correspondence, $\mathrm{Hol}(X, \mathrm{Hol}(Y))$ is homeomorphic with the subspace

$$\mathrm{Hol}_X(X\times Y) := \{g\in\mathrm{Hol}(X\times Y);\ \pi_X g = \pi_X\}$$

https://doi.org/10.1017/S0027763000013325 Published online by Cambridge University Press

of $\mathrm{Hol}(X\times Y)$, where π_X: $X\times Y\to X$ is the canonical projection. Moreover, we see easily

(2. 3) *Each* $g^*\in \mathrm{Aut}(X\times Y)$ *with the property* $\pi_X g^*=\pi_X$ *corresponds exactly to a map* $g\in \mathrm{Hol}(X, \mathrm{Aut}(Y))$ *whose inverse* g^{-1}: $X\to \mathrm{Aut}(Y)$ *is also holomorphic, where* $g^{-1}(x)=g(x)^{-1}(x\in X)$.

Let G be an effective complex Lie transformation group of Y. We may identify G with a subset of $\mathrm{Aut}(Y)$. Since the given transforming map $\psi(g, y)=g\cdot y(g\in G,\ y\in Y)$ of $G\times Y$ into Y is holomorphic, we have easily

(2. 4) *If a map* g: $X\to G$ *is holomorphic with respect to the given complex structure of* G, *it is also holomorphic in the sense of Definition* 2. 1.

Conversely, we can prove

PROPOSITION 2. 5. *For a normal* X, *if* g: $X\to G\,(\subset \mathrm{Aut}(Y))$ *is holomorphic in the sense of Definition* 2. 1, *then* g *is holomorphic with respect to the complex structure of* G.

To prove this, we give

LEMMA 2. 6. *Let* X, H, Y *and* Z *be complex spaces and* φ: $H\times Y\to Z$ *be a holomorphic map such that* X *is normal and* Z *is holomorphically separable and* $\varphi(h, y)=\varphi(h', y)$ *for any* $y\in Y$ *only if* $h=h'(h, h'\in H)$. *For a map* g: $X\to H$, *if the map* $\varphi(g(x), y)$ $(x\in X,\ y\in Y)$ *of* $X\times Y$ *into* Z *is holomorphic, then* g *is also a holomorphic map of* X *into* H.

Proof of Lemma 2. 6. It suffices to show that the graph $\Gamma_g:=\{(x, g(x));\ x\in X\}$ of g is analytic in $X\times H$. Indeed, in this case, $\pi_X|\Gamma_g$: $\Gamma_g\to X$ is a bijective holomorphic map. Then $(\pi_X|\Gamma_g)^{-1}$: $X\to\Gamma_g$ is also holomorphic by the normality of X(c.f. ([9]) and so $g=\pi_H(\pi_X|\Gamma_g)^{-1}$: $X\to H$ is holomorphic. Now, we consider the family $\mathscr{F}$ of all holomorphic functions $\varphi^{f,y}$ on $X\times H$ defined as

$$\varphi^{f,y}(x, h):=f(\varphi(h, y))-f(\varphi(g(x), y)) \qquad (x\in X,\ h\in H)$$

for any $y\in Y$ and holomorphic function f on Z. And we put

$$A:=\{(x, h)\in X\times Y;\ \psi(x, h)=0 \text{ for any } \psi\in\mathscr{F}\}.$$

Since A is the set of the common zeros of a family of holomorphic functions on $X\times H$, it is analytic in $X\times H$ by the well-known H. Cartan's theorem. We want to show $\Gamma_g = A$. Evidently, $\Gamma_g \subset A$. Conversely, if $(x, h) \in A$, i.e. $f(\varphi(h, y)) = f(\varphi(g(x), y))$ for any $y \in Y$ and holomorphic f on Z, $\varphi(h, y) = \varphi(g(x), y)$ for any $y \in Y$ because Z is holomorphically separable. By the assumption, we conclude $h = g(x)$, which shows $(x, h) \in \Gamma_g$. This completes the proof of Lemma 2. 6.

Proof of Proposition 2. 5. Let $\psi: G \times Y \to Y$ be the given transforming map. We can take an open neighborhood N of the identity in G such that

$$N = \{g \in G;\ \psi(g, y) \in Y'' \text{ for any } y \in \bar{Y}'\},$$

where Y'' is a holomorphically separable open set in Y and Y' $(\Subset Y'')$ is a non-empty open set. If $\psi(g, y) = \psi(g', y)$ for any $y \in Y'$ $(g, g' \in N)$, it remains valid for any $y \in Y$ by the theorem of identity and hence $g = g'$ by the assumption of the effectivity. This shows that the map $\varphi := \psi | N \times Y' : N \times Y' \to Y''$ satisfies the conditions in Lemma 2. 6 for the spaces $H := N, Y := Y'$ and $Z := Y''$.

Now, take a map $g: X \to G$ which is holomorphic in the sense of Definition 2. 1. Each $x_0 \in X$ has obviously a neighborhood U such that $h(x) := g(x)g(x_0)^{-1} \in N$ for any $x \in U$. By Lemma 2. 6, since $h: U \to N$ is holomorphic in the sense of Definition 2. 1, h is holomorphic with respect to the complex structure of G. So, g is also holomorphic on U in the same sense.

For later uses, we give the following proposition on the continuability of holomorphic maps.

Proposition 2. 7. *Let G be a locally compact subgroup of* Aut(Y). *Then we can find a neighborhood N of the identity in G satisfying the following conditions:*

For any connected open $V \subset X$ and analytically thin set M in V a holomorphic map $g: V - M \to G$ is holomorphically continuable to V if there is an open set $D (\subset V - M)$ such that $g(D) \subset N$ and every holomorphic function on D is continuable to V.

Proof. By the assumption, G has a relatively compact neighborhood N of the identity in G which can be written

$$N = \{g \in G;\ g(Y'_j) \subset Y''_j \text{ for any } j (1 \leqq j \leqq s)\},$$

where Y'_j, Y''_j are Stein open sets in Y with $Y'_j \Subset Y''_j$. We shall show that N satisfies the desired conditions. Let $g: V - M \to G$ be a holomorphic map

https://doi.org/10.1017/S0027763000013325 Published online by Cambridge University Press

with $g(D)\subset N$. Since $g(x)(Y_j')\subset Y_j''$ for any $x\in D$, we see $\tilde{g}(D\times Y_j')\subset Y_j''$ for each $j(1\leqq j\leqq s)$, where $\tilde{g}(x,y)=g(x)(y)$ $(x\in V-M,\ y\in Y)$. Then $\varphi_j:=\tilde{g}|D\times Y_j'$ is continuable to a holomorphic map $\psi_j\colon V\times Y_j'\to Y_j''$ by H. Kerner's theorem, because every holomorphic function on $D\times Y_j'$ is continuable to $V\times Y_j'$ (e.g. [1], Corollary 1 to Theorem 10, p. 63) and Y_j'' is Stein. This shows also $g(x)(Y_j')\subset Y_j''(1\leqq j\leqq s)$ i.e. $g(x)\in N$ for any $x\in V-M$. Take an arbitrary $x\in M$. For any sequence $\{x_n\}$ in $V-M$ with $\lim_{n\to\infty}x_n=x$, $\{g(x_n)\}$ has a convergent subsequence in $\bar{N}$ whose limit $g_0(x)$ satisfies the condition $g_0(x)(y)=\psi_j(x,y)$ for any $y\in Y_j'$. And any convergent subsequence of $\{g(x_n)\}$ has the same limit $g_0(x)$ by the theorem of identity. So, we obtain $g_0(x)=\lim_{x'\to x}g(x')\in\bar{N}\subset G$ for any $x\in M$. Obviously, g_0 is continuous on V and so $\tilde{g}_0$: $V\times Y\to Y$ is also continuous. Since $\tilde{g}_0|(V-M)\times Y=\tilde{g}$ is holomorphic, $\tilde{g}_0$: $V\times Y\to Y$ is also holomorphic according to Riemann's theorem on removable singularities. This shows that g has a holomorphic continuation $g_0\colon V\to\bar{N}\subset G$.

§3. Holomorphic fiber bundles over complex spaces.

Let B, X and Y be complex spaces, $\pi\colon B\to X$ be a holomorphic map and G be a subgroup of $\mathrm{Aut}(Y)$.

DEFINITION 3.1. The space B is said to have a structure of a holomorphic fiber bundle over X with fiber Y and structure group G if X has an open covering $\{U_i;\ i\in I\}$ such that each $\pi^{-1}(U_i)$ $(i\in I)$ is mapped onto $U_i\times Y$ by a biholomorphic map γ_i with the property $\pi_{U_i}\cdot\gamma_i=\pi$ on $\pi^{-1}(U_i)$, where π_{U_i}: $U_i\times Y\to U_i$ is the canonical projection, and each $\gamma_i\gamma_j^{-1}\in\mathrm{Hol}_{U_i\cap U_j}((U_i\cap U_j)\times Y)$ $(i,j\in I)$ can be written $\gamma_i\gamma_j^{-1}(x,y)=(x,g_{ij}(x)y)$ $(x\in U_i\cap U_j,\ y\in Y)$ with a holomorphic map $g_{ij}\colon U_i\cap U_j\to G$ which we say a transition function. Another structure on B given by an open covering $\{V_k\colon k\in K\}$ and the biholomorphic maps $\gamma_k'\colon \pi^{-1}(V_k)\to V_k\times Y$ with the property as the above is said to be equivalent to the above structure if there is a holomorphic map $\bar{g}_{ki}\colon V_k\cup U_i\to G$ such that $\gamma_k'\gamma_i^{-1}(x,y)=(x,\bar{g}_{ki}(x)y)$ $(x\in V_k\cap U_i,\ y\in Y)$ for each $i\in I$, $k\in K$. As usual, *a holomorphic fiber bundle* is defined to be an equivalence class of structures of holomorphic fiber bundles attached to a fixed space B. For brevity, we denote a holomorphic fiber bundle over X with fiber Y and structure group G by $\boldsymbol{B}=B(X,Y,G,\pi)$, or simply $\boldsymbol{B}$.

Let $\boldsymbol{B}=B(X,Y,G,\pi)$ and $\boldsymbol{B}'=B'(X',Y,G,\pi')$ be two holomorphic fiber bundles with the same fiber Y and the same structure group G.

https://doi.org/10.1017/S0027763000013325 Published online by Cambridge University Press

DEFINITION 3. 2. By a *homomorphism* φ of $\boldsymbol{B}$ into $\boldsymbol{B}'$ we mean a holomorphic map $\varphi: B \to B'$ with the following properties;

(i) $\pi\varphi = \bar{\varphi}\pi$ for a suitable $\bar{\varphi} \in \mathrm{Hol}(X, X')$,

(ii) taking structures $\{(U_i, \gamma_i);\ i \in I\}$ on B and $\{(V_k, \gamma'_k);\ k \in K\}$ on B' as in Definition 3. 1, we can write $\gamma'_k \gamma_i^{-1} = \bar{\varphi} \times \tilde{g}_{ki}$ on $(U_i \cap \bar{\varphi}^{-1}(V_k))XY$ with holomorphic maps $g_{ki}: U_i \cap \bar{\varphi}^{-1}(V_k) \to G(i \in I,\ k \in K)$, where $\tilde{g}_{ki}(x, y) = g_{ki}(x)(y)$.

If a homomorphism $\varphi: \boldsymbol{B} \to \boldsymbol{B}$ has the inverse homomorphism, it is said to be an automorphism of $\boldsymbol{B}$. By $F(\boldsymbol{B})$ we denote the set of all automorphisms of $\boldsymbol{B}$.

By definition an automorphism of $\boldsymbol{B} = B(X, Y, G, \pi)$ is a holomorphic automorphism of the space B. So, $F(\boldsymbol{B})$ is considered as a subspace of $\mathrm{Aut}(B)$ with the compact-open topology. According to the well-known Bochner-Montgomery's theorem (c.f. W. Kaup [5], Satz 4, p. 83 and Satz 6, p. 85), we see

(3. 3) *If $F(\boldsymbol{B})$ is locally compact, it has a structure of a Lie transformation group of B.*

EXAMPLE 3. 4. (i) A holomorphic principal fiber bundle over a complex space X in the usual sense defines canonically a holomorphic fiber bundle over X in the sense of Definition 3. 1 by (2. 4). If X is normal, an automorphism of this bundle is nothing but an automorphism of this as a holomorphic principal fiber bundle in the usual sense (c.f. [8], p. 158) according to Proposition 2. 5.

(ii) Let $\boldsymbol{B} = B(X, Y, G, \pi)$ be assumed that X is normal and G is an effective complex Lie transformation group of Y with the topology induced from $\mathrm{Aut}(Y)$. In view of Proposition 2. 5, $\boldsymbol{B}$ is regarded as an associated fiber bundle $\boldsymbol{P}$ over X with structure group G which is canonically defined by the same transition functions as $\boldsymbol{B}$. Moreover, as is easily seen, $F(\boldsymbol{B})$ is topologically isomorphic with the group of all automorphisms of $\boldsymbol{P}$.

(iii) Let $\pi: B \to X$ be a locally trivial fiber space over a complex space X with fiber Y, i.e. π be a holomorphic map such that, for a suitable open covering $\{U_i;\ i \in I\}$ of X, each $\pi^{-1}(U_i)$ is biholomorphic with $U_i \times Y$ by a map γ_i with $\pi_{U_i} \cdot \gamma_i = \pi$. Then we can define canonically a holomorphic

https://doi.org/10.1017/S0027763000013325 Published online by Cambridge University Press

fiber bundle $\boldsymbol{B} = B(X, Y, \text{Aut}(Y), \pi)$, where the transition functions $g_{ij}: U_i \cap U_j \to \text{Aut}(Y)$ are given so as to satisfy $\gamma_i \gamma_j^{-1} = 1_{U_i \cap U_j} \times \tilde{g}_{ij}$. In this case, an automorphism of $\boldsymbol{B}$ means exactly a fiber-preserving holomorphic automorphism of B, i.e. an element $\varphi \in \text{Aut}(B)$ with the property $\pi \varphi = \bar{\varphi} \pi$ for some $\bar{\varphi} \in \text{Aut}(X)$.

Let $h: X' \to X$ be a holomorphic map and $\boldsymbol{B}$ be a holomorphic fiber bundle over X which is defined by the structure with the transition functions g_{ij}. As usual, the induced bundle $h^{-1}(\boldsymbol{B})$ over X' can be defined by the structure with the transition functions $g_{ij} \cdot h$. If X' is an open set in X and $h: X' \to X$ is the inclusion map, we call $\boldsymbol{B}|X' := h^{-1}(\boldsymbol{B})$ the portion of $\boldsymbol{B}$ over X'.

Take the normalization $\mu: X^* \to X$ of X. A holomorphic fiber bundle $\boldsymbol{B} = B(X, Y, G, \pi)$ induces the bundle $\boldsymbol{B}^* := \pi^{-1}(\boldsymbol{B}) = B^*(X^*, Y, G, \pi^*)$.

PROPOSITION 3. 5. *The automorphism group $F(\boldsymbol{B})$ is topologically isomorphic with a closed subgroup of $F(\boldsymbol{B}^*)$.*

Proof. By the definition of the induced bundle, a holomorphic map $\tilde{\mu}: B^* \to B$ with $\pi \cdot \tilde{\mu} = \mu \cdot \pi^*$ is defined canonically. And each $\varphi \in F(\boldsymbol{B})$ gives exactly one $\mu^*(\varphi) := \varphi^* \in F(\boldsymbol{B}^*)$ with $\tilde{\mu} \varphi^* = \varphi \tilde{\mu}$. The map $\mu^*: F(\boldsymbol{B}) \to F(\boldsymbol{B}^*)$ is obviously a continuous injective group homomorphism. It suffices to show the closedness of $\mu^*(F(\boldsymbol{B}))$ in $F(\boldsymbol{B}^*)$. Take a sequence $\{\varphi_n\}$ in $F(\boldsymbol{B})$ such that $\{\mu^*(\varphi_n)\}$ converges to φ^* in $F(\boldsymbol{B}^*)$. Since $\tilde{\mu}$ is a proper nowhere degenerate, surjective map, we can find easily some $\varphi \in \text{Hol}(B)$ with $\tilde{\mu} \varphi^* = \varphi \tilde{\mu}$ and $\bar{\varphi} \in \text{Hol}(X)$ with $\bar{\varphi} \pi = \pi \varphi$. Then, it can be easily proved by (2. 2) that φ satisfies the condition (ii) in Definition 3. 2 in its local representation. On the other hand, $\{\mu^*(\varphi_n)^{-1}\}$ converges also to φ^{*-1} in $F(\boldsymbol{B}^*)$. By the same argument as the above, we have the inverse homomorphism of φ and so $\varphi \in F(\boldsymbol{B})$. Thus $\mu^*(F(\boldsymbol{B}))$ is closed in $F(\boldsymbol{B}^*)$.

§4. Holomorphic fiber bundles over a $*$-strongly pseudo-concave space.

In this section, we prove the following main theorem.

THEOREM 4. 1. *For any $\boldsymbol{B} = B(X, Y, G, \pi)$, if G is a locally compact subgroup of* Aut (Y) *and X is $*$-strongly pseudo-concave (see Definition* 8. 1 *in* [2], p. 104), *then $F(\boldsymbol{B})$ has a structure of a Lie transformation group of B.*

We need some preparations.

LEMMA 4. 2. *Let N be a compact subset of* Aut (Y) *such that* $N \subset \{g \in \mathrm{Aut}(Y);\ g(\bar{Y}') \subset Y''\}$, *where Y' is a non-empty open set and Y'' is relatively compact in some K-complete open subset of Y. Then the set*

$$\mathfrak{N} := \{g \in \mathrm{Hol}(X, \mathrm{Aut}(Y));\ g(x) \in N \text{ for any } x \in X\}$$

is also compact in Hol $(X, \mathrm{Aut}(Y))$.

Proof. By Arzelà-Ascoli's theorem it suffices to show that $\mathfrak{N}$ is equicontinuous on X with the canonical uniform structure of Aut (Y) because $\{g(x);\ g \in \mathfrak{N}\}$ is included in a compact N for any $x \in X$. By the assumption, the restriction map $r: N \to \mathrm{Hol}(Y', Y'')$ $(r(g) := g | Y'$ for each $g \in N)$ is well-defined. With each $g \in \mathfrak{N}$ we associate the map $g' = r \cdot g: X \to \mathrm{Hol}(Y', Y'')$ and $\tilde{g}': X \times Y' \to Y''$ with $\tilde{g}'(x, y) = g'(x)(y)$ $(x \in X,\ y \in Y')$. In view of the assumption of Y'', $\{\tilde{g}';\ g \in \mathfrak{N}\}$ is relatively compact in Hol $(X \times Y', Y)$ ([2], Theorem 2. 1, p. 86) and so equicontinuous on $X \times Y'$, where Y is considered as a metric space with a suitable metric. Then $\{g';\ g \in \mathfrak{N}\}$ $(\subset \mathrm{Hol}(X, \mathrm{Hol}(Y', Y'')))$ is also equicontinuous on X. On the other hand, since r is injective by the theorem of identity, r is a topological map of N onto a compact subset of Hol (Y', Y''). Therefore, $\mathfrak{N}$ itself is equicontinuous.

For the proof of Tehrorem 4. 1, we have only to show that $F(\boldsymbol{B})$ is locally compact by (3. 3). Moreover, X may be assumed to be normal. Indeed, in Theorem 4. 1, the normalization X^* of X is also $*$-strongly pseudo-concave and hence the induced bundle $\boldsymbol{B}^*$ of $\boldsymbol{B}$ over X^* satisfies all conditions in Theorem 4. 1. If $F(\boldsymbol{B}^*)$ is shown to be locally compact, $F(\boldsymbol{B})$ is also locally compact according to Proposition 3. 5. In the following, $\boldsymbol{B} = B(X, Y, G, \pi)$ denotes a holomorphic fiber bundle over a normal $*$-strongly pseudo-concave space X with a locally compact $G (\subset \mathrm{Aut}(Y))$.

By definition, there is a positive real-valued continuous function v on X such that v is $*$-strongly $(\dim X - 1)$-convex on $X - K$ (see [2], p. 101) for a suitable compact $K \subset X$ and $\{x;\ v(x) > c\} \Subset X$ for any $c > 0$. We put $c_0 := \min\{v(x);\ x \in K\}$, $X_c = \{x \in X;\ v(x) > c\}$ and $B_c := \pi^{-1}(X_c)$ for any $c (0 < c < c_0)$.

LEMMA 4. 3. *In the above situation, we can find a neighborhood $\boldsymbol{N}$ of the identity in $F(\boldsymbol{B})$ such that, for a suitable $c (< c_0)$, any sequence $\{\varphi_n\}$ in $\boldsymbol{N}$ has a subsequence $\{\varphi_{n_k}\}$ having the following properties;*

(i) $\{\varphi_{n_k}|B_c\}$ *converges to an injective map* φ *in* $\mathrm{Hol}(B_c, B)$,

(ii) *the limit* φ *is a homomorphism of* $\boldsymbol{B}|X_c$ *into* $\boldsymbol{B}$,

(iii) $\{\bar{\varphi}_{n_k}\}$ *converges in* $\mathrm{Aut}(X)$, *where* $\bar{\varphi}$ *denotes an automorphism of* X *with* $\pi\varphi = \bar{\varphi}\pi$ *for each* $\varphi \in F(\boldsymbol{B})$.

Proof. The compact set $\bar{X}_{c_0}$ has an open covering $\{V_i: 1 \leqq i \leqq k\}$ such that each $\boldsymbol{B}|V_i$ has a locally trivial bundle structure, i.e. there are biholomorphic maps $\gamma_i: \pi^{-1}(V_i) \to V_i \times Y$ with $\pi_{V_i} \cdot \gamma_i = \pi$ on $\pi^{-1}(V_i)$ and holomorphic maps $g_{ij}: V_i \cap V_j \to G$ with $\gamma_i\gamma_j^{-1} = 1_{V_i \cap V_j} \times \tilde{g}_{ij}$ on $(V_i \cap V_j) \times Y\,(1 \leqq i, j \leqq k)$. Moreover we take open coverings $\{U_i; 1 \leqq i \leqq k\}$ and $\{U'_i; 1 \leqq i \leqq k\}$ of X with the property $U_i \Subset U'_i \Subset V_i$ for each i. By Theorem 8. 3 in [2], p. 105, $\mathrm{Aut}(X)$ is locally compact. There is a compact neighborhood N of 1_X in $\mathrm{Aut}(X)$ such that

$$N \subset \{g \in \mathrm{Aut}(X);\ g(U'_i) \subset V_i \text{ for any } i\}.$$

We consider the set

$$\boldsymbol{N}^* = \{\varphi \in F(\boldsymbol{B});\ \text{the corresponding } \bar{\varphi} \in N\}.$$

For each $\varphi \in \boldsymbol{N}^*$, since $\bar{g}(U'_i) \subset V_i$, the holomorphic maps $\varphi_i := \gamma_i \cdot \varphi \cdot \gamma_i^{-1}$: $U'_i \times Y \to V_i \times Y$ are well-defined $(1 \leqq i \leqq k)$. Then we have holomorphic maps $g_i(\varphi): U'_i \to G$ with the property $\varphi_i = \bar{\varphi} \times g_i(\tilde{\varphi})$ on $U'_i \times Y$. On the other hand, there is a compact neighborhood N' of the identity in G such that $N' \subset \{g \in \mathrm{Aut}(Y);\ g(\bar{Y}') \subset Y''\}$ for open sets Y', Y'' with the same properties as in Lemma 4. 2. We put

$$\boldsymbol{N} := \{\varphi \in \boldsymbol{N}^*;\ g_i(\varphi)(x) \in N' \text{ for any } x \in \bar{U}_i (1 \leqq i \leqq k)\},$$

which is obviously a neighborhood of 1_B in $F(\boldsymbol{B})$. We shall prove that $\boldsymbol{N}$ has the desired properties in Lemma 4. 3 for $c := \sup\{v(x);\ x \notin \cup_{i=1}^{k} U_i\}\ (< c_0)$.

Take an arbitrary $\{\varphi_n\}$ in $\boldsymbol{N}$. Since $\{\bar{\varphi}_n\} \subset N$, a suitable subsequence $\{\bar{\varphi}_{n_k}\}$ converges to some $\bar{\varphi}$ in $\mathrm{Aut}(X)$. On the other hand, $\{g_i(\varphi_n)\}$ is relatively compact in $\mathrm{Hol}(U_i, G)$ by Lemma 4. 2. Relabeling indices suitably, we may assume that $\{g_i(\varphi_{n_k})\}$ converges to some g_i^0 in $\mathrm{Hol}(U_i, G)$ for each i. Since $\mathrm{Hol}(U_i \times Y, Y) = \mathrm{Hol}(U_i, \mathrm{Hol}(Y))$, $\{\gamma_i \cdot \varphi_{n_k} \cdot \gamma_i^{-1}\}$ converges to $\varphi_i^0 := \bar{\varphi} \times \tilde{g}_i^0$ in $\mathrm{Hol}(U_i \times Y, V_i \times Y)$. So, $\{\varphi_{n_k}|\pi^{-1}(U_i)\}$ converges to $\gamma_i^{-1} \cdot \varphi_i^0 \cdot \gamma_i$ in $\mathrm{Hol}(\pi^{-1}(U_i), B)$. The subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ converges obviously on $B_c \subset \cup_i \pi^{-1}(U_i)$ and satisfies all desired conditions.

https://doi.org/10.1017/S0027763000013325 Published online by Cambridge University Press

REMARK. If X is compact, we may take $X_c = X$. In this case, Theorem 4. 1 is an immediate consequence of Lemma 4. 3.

LEMMA 4. 4. *Let* $\{\varphi_n\}$ *be a sequence in* $F(\boldsymbol{B})$ *such that* $\{\varphi_n | B_c\}$ *converges to an injective map* φ *in* $\mathrm{Hol}(B_c, B)$ *for some* $c < c_0$ *and the corresponding* $\{\bar{\varphi}_n\}$ *converges to some* $\bar{\varphi}$ *in* $\mathrm{Aut}(X)$. *Then any* x *with* $v(x) = c$ *has a neighborhood* U *such that* $\{\varphi_n | \pi^{-1}(U)\}$ *converges to an injective map* φ^* *in* $\mathrm{Hol}(\pi^{-1}(U), B)$.

Proof. Take a Stein neighborhood V of x such that $\pi^{-1}(V)$ is biholomorphic with $V \times Y$ by a map γ. By Lemma 7. 2 in [2], p. 101, there is a connected Stein neighborhood U of $x (U \Subset V)$ such that, for a suitable open $D \Subset X_c$, every holomorphic function on D is uniquely continuable to U and any 1-codimensional analytic subset of U intersects D. Moreover, we choose Stein open sets U' and U^* in X such that $x \in U \Subset U' \Subset V$, $\varphi(\bar{U}') \subset U^*$, $\bar{\varphi}_n(\bar{U}') \subset U^*$ for almost all n and $\pi^{-1}(U^*)$ is biholomorphic with $U^* \times Y$ by a map γ'. Then we can define the maps $\varphi' = \gamma' \cdot \varphi \cdot \gamma^{-1}: U'_c \times Y \to U^* \times Y$ and $\varphi'_n := \gamma' \cdot \varphi_n \cdot \gamma^{-1}: U' \times Y \to U^* \times Y$ for almost all n, where $U'_c = U' \cap \{v > c\}$. Take an open set D' with $D \Subset D' \Subset U'_c$ and Stein neighborhoods W, W' of y with $W \Subset W' \Subset Y$ for each $y \in Y$. Since $D \times W \Subset D' \times W' \subset U'_c \times W'$ and φ' is injective on $U'_c \times Y$, there is a suitable n_0 such that

$$\varphi'_n(D \times W) \Subset \varphi_{n_0}(D' \times W') \subset \varphi'_{n_0}(U' \times W') \subset U^* \times Y$$

for any $n \geqq n_0$ by Lemma 3. 2 in [2], p. 89. Now, every holomorphic function on $D \times W$ is uniquely continuable to $U \times W$ (e.g. Corollary 1 in [1], p. 63) and $\varphi'_{n_0}(U' \times W')$ is Stein. As in the proof of Lemma 7. 3 in [2], p. 102, we have $\varphi'_n(U \times W) \subset \varphi'_{n_0}(U' \times W')$ for any $n \geqq n_0$ by H. Kerner's theorem ([6]). Since $\varphi'_{n_0}(U' \times W')$ may be assumed to be relatively compact in some K-complete subset of $U^* \times Y$, $\{\varphi'_n;\, n \geqq n_0\}$ may be considered as a normal family in $\mathrm{Hol}(U \times W, U^* \times Y)$ by Theorem 2. 1 in [2], p. 86. A suitable subsequence of $\{\varphi'_n\}$ converges on $U \times W$. Covering Y by countably many open sets W's with the above property, we can choose a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi'_n\}$ which converges to a map ψ in $\mathrm{Hol}(U \times Y, U^* \times Y)$. Then $\{\varphi_{n_k} | \pi^{-1}(U)\}$ converges to the map $\varphi^* := \gamma'^{-1} \psi \gamma$ in $\mathrm{Hol}(\pi^{-1}(U), B)$. Moreover, any subsequence of $\{\varphi_n\}$ has a subsequence which converges to the same limit φ^*. So, $\{\varphi_n\}$ itself converges to φ^* in $\mathrm{Hol}(\pi^{-1}(U), B)$. It remains to prove the injectivity of φ^* on $\pi^{-1}(U)$, or equivalently, ψ on $U \times Y$. Let E be the set of

https://doi.org/10.1017/S0027763000013325 Published online by Cambridge University Press

degeneracy of ψ, which is analytic in $U\times Y$ by R. Remmert's theorem ([9]). Then E is of codimension $\geqq 2$. In fact, if $\operatorname{codim} E=1$, $E\cap\{y=y_0\}$ can be identified with an analytic set of codimension $\geqq 1$ at x_0 in U for any $(x_0, y_0)\in E$. By the assumption of U, E intersects $U_c\times Y$. This contradicts with the injectivity of φ. Now, we apply Lemma 7. 3 in [2], p. 102, whence ψ is injective on $U\times Y$. This completes the proof.

Proof of Theorem 4. 1. Take a neighborhood $\boldsymbol{N}$ of 1_B in $F(\boldsymbol{B})$ satisfying the conditions in Lemma 4. 3. We shall prove that $\boldsymbol{N}' := \{\varphi\in F(\boldsymbol{B});\ \varphi$ and $\varphi^{-1}\in\boldsymbol{N}\}$ is a relatively compact neighborhood of 1_B in $F(\boldsymbol{B})$. An arbitrary $\{\varphi_n\}$ in $\boldsymbol{N}'$ has a subsequence $\{\varphi_{n_k}\}$ which satisfies the conditions (i), (ii), (iii) in Lemma 4. 3. As in the proof of Theorem 7. 5, p. 103, putting

$$\Gamma := \{c;\ \{\varphi_{n_k}\} \text{ converges to an injective map in } \operatorname{Hol}(B_c, B)\},$$

we conclude that $\inf\Gamma=0$ according to Lemma 4. 4. This shows that $\{\varphi_{n_k}\}$ converges to some φ in $\operatorname{Hol}(B, B)$, which satisfies obviously $\bar{\varphi}\pi=\pi\varphi$ for $\bar{\varphi}=\lim_{k\to\infty}\bar{\varphi}_{n_k}\in\operatorname{Aut}(X)$. Then, applying the same argument to the sequence $\{\varphi_n^{-1}\}$, we see easily $\varphi\in\operatorname{Aut}(B)$. It remains to show $\varphi\in F(\boldsymbol{B})$. The condition (ii) in Definition 3. 2 is of local nature. For each $x\in X$ taking sufficiently small neighborhood U of x and V of $\bar{\varphi}(x)$, we may assume that $\pi^{-1}(U)=U\times Y$, $\pi^{-1}(V)=V\times Y$, $\varphi(U\times Y)\subset V\times Y$ and $\varphi_{n_k}(U\times Y)\subset V\times Y$ for almost all k. Moreover, it may be assumed that $\varphi_{n_k}=\bar{\varphi}_{n_k}\times\tilde{g}_{n_k}$ for suitable $g_{n_k}\in\operatorname{Hol}(U, G)$ by Definition 3. 2 and $\varphi=\bar{\varphi}\times\tilde{g}$ on $U\times Y$ for a suitable $g\in\operatorname{Hol}(U, \operatorname{Aut}(Y))$ by Example 3. 4 (iii). Then we have $g=\lim_{k\to\infty} g_{n_k}\in\operatorname{Hol}(U, G)$ because G is closed in $\operatorname{Aut}(Y)$. This concludes $\varphi\in F(\boldsymbol{B})$. Theorem 4. 1 is completely proved.

According to Example 3. 4 (iii), Theorem 4. 1 implies

COROLLARY 4. 5. *Let B be a locally trivial fiber space over a $*$-strongly pseudo-concave complex space with fiber Y. If* $\operatorname{Aut}(Y)$ *is locally compact, the group of all fiber-preserving holomorphic automorphisms of B is a Lie group.*

We have also in view of Example 8. 2 in [2], p. 104,

COROLLARY 4. 6. *Let X be a compact complex manifold and M be an analytic set with* $\operatorname{emdim} M\leqq\dim X-2$. *For any $\boldsymbol{B}=B(X-M, Y, G, \pi)$, if G is a locally compact subgroup of* $\operatorname{Aut}(Y)$, *$F(\boldsymbol{B})$ is a Lie group.*

In connection with this, for a holomorphic fiber bundle $\boldsymbol{B}$ defined over the total X we give more precise informations on automorphisms of $\boldsymbol{B}|X-M$ in the next section.

§5. Holomorphic fiber bundles over a compact complex space.

Let $\boldsymbol{B}=B(X, Y, G, \pi)$ be a holomorphic fiber bundle over a compact complex space X. We want to study $F(\boldsymbol{B}|X-M)$ for a thin analytic set M in X. By $\boldsymbol{G}^0$ we denote the connected component of the identity in a topological group $\boldsymbol{G}$.

THEOREM 5.1. *If X is a compact normal complex space, G is a locally compact subgroup of* Aut(Y) *and M is of codimension $\geqq 2$, then any element in $F(\boldsymbol{B}|X-M)^0$ is the restriction of an automorphism of $\boldsymbol{B}$ over the total X.*

Proof. It suffices to show that there is a neighborhood $\boldsymbol{N}$ of 1_B in $F(\boldsymbol{B}|X-M)$ such that any $\varphi \in \boldsymbol{N}$ is the restriction of some $\psi \in F(\boldsymbol{B})$, because any element in $F(\boldsymbol{B}|X-M)$ is represented as the product of finitely many elements in $\boldsymbol{N}$. The compact set M has an open covering $\{V_i;\ 1\leq i\leq k\}$ such that, for a suitable connected open subset D_i of V_i with $D_i \Subset X-M$, any holomorphic function on D_i is uniquely continuable to V_i for each i. If we choose sufficiently small $V_i (1\leqq i\leqq k)$, it may be assumed that there is another open covering $\{U_i;\ 1\leqq i\leqq k\}$ of M such that each U_i is Stein, $V_i \Subset U_i$ and $\boldsymbol{B}|U_i$ has a locally trivial bundle structure, and so $\pi^{-1}(U_i)$ is biholomorphic with $U_i\times Y$ by a map γ_i. We consider the set

$$N:=\{g\in \mathrm{Aut}(X-M);\ g(\bar{D}_i')\subset U_i \text{ and } g^{-1}(\bar{D}_i')\subset U_i \text{ for any } i(1\leqq i\leqq k)\}$$

where D_i' is an open set with $D_i \Subset D_i' \Subset U_i - M$. And, as in the proof of Lemma 4.3, we put

$$\boldsymbol{N}^*:=\{\varphi\in F(\boldsymbol{B}|X-M);\ \text{the corresponding } \bar{\varphi}\in N\}.$$

Since $\bar{\varphi}(\bar{D}_i')\subset U_i$ and so $\varphi(\pi^{-1}(D_i'))\subset \pi^{-1}(U_i)$, each $\varphi\in \boldsymbol{N}^*$ defines a holomorphic map $g_i(\varphi)\colon D_i'\to G$ such that $\varphi_i := \gamma_i\varphi\gamma_i^{-1} = \bar{\varphi}\times \widetilde{g_i(\varphi)}$ on $D_i'\times Y$. Now, taking a neighborhood N' with the property in Proposition 2.7, we put

$$\boldsymbol{N}'=\{\varphi\in \boldsymbol{N}^*;\ g_i(\varphi)(x)\in N' \text{ for any } x\in \bar{D}_i,\ 1\leqq i\leqq k\}.$$

We shall prove that $\boldsymbol{N}:=\{\varphi\in F(\boldsymbol{B}|X-M);\ \varphi\in \boldsymbol{N}' \text{ and } \varphi^{-1}\in \boldsymbol{N}'\}$ is a desired neighborhood of 1_B in $F(\boldsymbol{B})$.

Take an arbitrary $\varphi\in \boldsymbol{N}$. The corresponding $\bar{\varphi}\in \mathrm{Aut}(X-M)$ with $\pi\varphi=\bar{\varphi}\pi$ defines the map $\bar{\varphi}|D_i\colon D_i\to U_i$ for each i. Since U_i is Stein, $\bar{\varphi}|D_i$

is uniquely continuable to a map $\bar{\varphi}_i: V_i \to U_i$ by H. Kerner's theorem ([6]). Obviously, $\bar{\varphi}_i = \bar{\varphi}_j$ on $V_i \cap V_j$ if $V_i \cap V_j \neq \phi$. Therefore $\bar{\varphi}$ has a continuation to X, which we denote by the same notation $\bar{\varphi}$. Since N is symmetric, $\bar{\varphi}^{-1}$ has also a continuation to X. Thus we conclude $\bar{\varphi} \in \mathrm{Aut}(X)$.

Now, since $\bar{\varphi}(V_i) \subset U_i$ by the above argument, $\varphi(\pi^{-1}(V_i - M)) \subset \pi^{-1}(U_i)$ and so the map $\varphi_i: \gamma_i \varphi \gamma_i^{-1}: (V_i - M) \times Y \to U_i \times Y$ is well-defined for each i. Then we get a holomorphic map $g_i: V_i - M \to G$ with $\varphi_i = \bar{\varphi} \times \tilde{g}_i$, which satisfies $g_i(D_i) \subset N'$. By the assumption of N', each g_i is continuable to a holomorphic map $g_i^0: V_i \to G$ in view of Proposition 2. 7. We consider the map $\psi_i: = \bar{\varphi} \times \tilde{g}_i^0: V_i \times Y \to U_i \times Y$. The map $\psi \in \mathrm{Hol}(B)$ with $\psi = \gamma_i^{-1} \cdot \psi_i \cdot \gamma_i$ on $\pi^{-1}(V_i)$ is obviously well-defined and a continuation of φ to the total B. The above proof shows also $\psi \in F(\boldsymbol{B})$. This completes the proof.

As a special case of Theorem 5. 1, we have

COROLLARY 5. 2. *Let M be an analytic subset of codimension $\geqq 2$ in a normal compact complex space X. Then any element in* $\mathrm{Aut}(X - M)^0$ *is the restriction of an automorphism of X.*

In Corollary 5. 2, an automorphism of $X - M$ not belonging to $\mathrm{Aut}(X - M)^0$ is not necessarily the restriction of an automorphism of X. In fact, for any integer k, we can construct a normal compact complex space X and an analytic set M in X with $\mathrm{codim}\, M \geqq k$ such that there is an automorphism of $X - M$ which cannot be continuable to the total X. To construct such a space X and an analytic set M, take a continuous proper modification $\pi: Y_1 \to Y_2$ with normal compact complex spaces Y_1 and Y_2 such that, for a suitable $M_2: = \{x_0\} \subset Y_2$, $M_1: = \pi^{-1}(M_2)$ is biholomorphic with the Riemann sphere $\boldsymbol{P}^1$ and $\pi | Y_1 - M_1: Y_1 - M_1 \to Y_2 - M_2$ is a biholomorphic map (Grauert-Remmert [3], §4, p. 292), where it can be assumed that $\dim Y_1 = \dim Y_2 \geqq k$ for any given k and Y_1 is not biholomorphic with Y_2. We consider the complex space $X: = Y_1 \times Y_2$ and the analytic set $M: = (Y_1 \times M_2) \cup (M_1 \times Y_2)$ in X, which is of codimension $\geqq k$. Let $g: = \pi | Y_1 - M_1$ and $h: = g^{-1}: Y_2 - M_2 \to Y_1 - M_1$. The map $\bar{g}(x, y): = (h(y), g(x))$ $(x \in Y_1 - M_1, y \in Y_2 - M_2)$ is an automorphism of $(Y_1 - M_1) \times (Y_2 - M_2) = X - M$. And it cannot be the restriction of any automorphism of X. For, if there is some $g' \in \mathrm{Aut}(X)$ with $\bar{g} = g'$ on $X - M$, g is necessarily continuable to a biholomorphic map of Y_1 onto Y_2. This is a contradiction.

COROLLARY 5. 3. *Let* $\boldsymbol{B}=B(X, Y, G, \pi)$ *be a holomorphic fiber bundle over a compact complex space and* M *be an analytic set with* codim $M \geqq 2$. *If* G *is a locally compact subgroup of* Aut(Y), $F(\boldsymbol{B}|X-M)$ *has a structure of a Lie transformation group of* $B-\pi^{-1}(M)$.

Proof. Without loss of generality, we may assume that X is normal by Proposition 3. 5. As a special case of Theorem 4. 1, $F(\boldsymbol{B})$ has a structure of a Lie transformation group of B (c.f. Remark to the proof of Lemma 4. 3). Then the closed subgroup $\boldsymbol{G}:=\{\varphi \in F(\boldsymbol{B});\ \varphi(\pi^{-1}(M))=\pi^{-1}(M)\}$ of $F(\boldsymbol{B})$ is also a Lie group and can be identified with a topological subgroup of $F(\boldsymbol{B}|X-M)$ by the restriction map. On the other hand, in view of the proof of Theorem 5. 1, there is a neighborhood $\boldsymbol{N}$ of the identity in $F(\boldsymbol{B}|X-M)$ such that any $\varphi \in \boldsymbol{N}$ is the restriction of some $\psi \in \boldsymbol{G}$. This shows that $F(\boldsymbol{B}|X-M)$ and $\boldsymbol{G}$ have a common neighborhood of the identity. Thus $F(\boldsymbol{B}|X-M)$ is also a Lie transformation group of $B-\pi^{-1}(M)$.

THEOREM 5. 4. *Assume that* $\boldsymbol{B}$, X *and* M *satisfy the conditions in Theorem* 5. 1 *and, furthermore,* G *is a complex Lie transformation group of* Y. *Then* $F(\boldsymbol{B}|X-M)$ *has a structure of a complex Lie transformation group of* $B-\pi^{-1}(M)$.

Proof. According to Example 3. 4 (ii), $\boldsymbol{B}$ is an associated fiber bundle of the canonically defined holomorphic principal fiber bundle $\boldsymbol{P}$ over X and $F(\boldsymbol{B}|X-M)$ is isomorphic with the automorphism group of $\boldsymbol{P}|X-M$. As is easily seen, if the automorphism group of $\boldsymbol{P}$ is shown to be a complex Lie transformation group, $F(\boldsymbol{B}|X-M)$ is also a complex Lie transformation group of $B-\pi^{-1}(M)$. From the beginning, we may assume that $\boldsymbol{B}$ itself is a holomorphic principal fiber bundle, i.e. G acts on the fiber $Y=G$ as the left translations. In this case, $F(\boldsymbol{B})$ is a complex Lie transformation group of B. For, we know that an arbitrary infinitesimal transformation group on a complex space is locally integrable (W. Kaup [5], Satz 3, p. 82) and so it can be proved that the Lie algebra of $F(\boldsymbol{B})$ is canonically isomorphic with the Lie algebra of infinitesimal transformations D on B with $R_g' \cdot D=D$ for any $g \in G$ by the same argument as in the proof of Proposition 1 in [8], p. 163, where R_g denotes the right translation by g acting on B. On the other hand, the closed subgroup $\boldsymbol{G}:=\{\varphi \in F(\boldsymbol{B});\ \varphi(\pi^{-1}(M))=\pi^{-1}(M)\}$ of $F(\boldsymbol{B})$ and $F(\boldsymbol{B}|X-M)$ have a common neighborhood of the identity. Theorem 5. 4 is a direct result of W. Kaup [5], Korollar to Satz 2, p. 80.

COROLLARY 5. 5. *Let X be a compact complex space and M be an analytic subset of X. If* $\operatorname{codim} M \geqq 2$, $\operatorname{Aut}(X-M)$ *has a structure of a complex Lie transformation group of $X-M$.*

Proof. Let $\mu: X^* \to X$ be the normalization of X and put $M^* = \mu^{-1}(M)$. The space $X^* - M^*$ is the normalization of $X - M$. Then $\operatorname{Aut}(X^* - M^*)$ is a complex Lie transformation group of $X^* - M^*$ as a special case of Theorem 5. 4. The rest of the proof of Corollary 5. 5 is due to the following Lemma.

LEMMA 5. 6. *Let X be a complex space and $\mu: X^* \to X$ be the normalization of X. If* $\operatorname{Aut}(X^*)$ *is a complex Lie transformation group of X^*,* $\operatorname{Aut}(X)$ *is also a complex Lie transformation group of X as a closed subgroup of* $\operatorname{Aut}(X^*)$.

Proof. To each $g \in \operatorname{Aut}(X)$ corresponds exactly one $g^* \in \operatorname{Aut}(X^*)$ with $g \cdot \mu = \mu \cdot g^*$. By this correspondence, $\operatorname{Aut}(X)$ is considered as a closed subgroup of $\operatorname{Aut}(X^*)$ (e.g. [2], Proposition 4. 2) and so a real Lie group. For our purpose, it suffices to show that any real one-parameter subgroup $\{g_t\}$ of $\operatorname{Aut}(X)$ can be extended to a complex one-parameter group of transformations of X. The given $\{g_t\}$ in $\operatorname{Aut}(X)$ gives a real one-parameter group $\{g_t^*\}$ in $\operatorname{Aut}(X^*)$. By the assumption, $\{g_t^*\}$ is extended to a complex one-parameter group of transformations of X^*, which we denote by the same notation $\{g_t^*\}$. For each g_t^*, we can define a map $g_t': X \to X$ with $\mu \cdot g_t^* = g_t' \cdot \mu$. In fact, if $\mu(x_1^*) = \mu(x_2^*)$ for any fixed x_1^*, x_2^*, the holomorphic maps $\mu \cdot g_t^*(x_1^*)$ and $\mu \cdot g_t^*(x_2^*)$ of the complex number space $\boldsymbol{C}$ into X coincide with each other for any real t. By the theorem of identity, it remains valid for any $t \in \boldsymbol{C}$. This shows that the single-valued map $g_t' := \mu g_t^* \mu^{-1}: X \to X$ is well-defined. Since μ is proper nowhere degenerate surjective, $g_t'(x)$ ($t \in \boldsymbol{C}$, $x \in X$) is obviously continuous on $\boldsymbol{C} \times X$ and $g_t' g_s' = g_{t+s}'$ for any t, $s \in \boldsymbol{C}$. And $g_t'(x)$ is holomorphic in x for any fixed real t. For any $x_0 \in X$, take a neighborhood W of $(0, x_0)$ in $\boldsymbol{C} \times X$ such that $g_t'(x) \in V$ with a Stein open set V in X for any $(t, x) \in W$. Then $f(g_t'(x))$ is holomorphic on W for any holomorphic function f on V (H. Kerner [7], Hilfssatz 4, p. 285) because $f(g_t \cdot \mu(x))$ ($t \in \boldsymbol{C}$, $x \in X^*$) is holomorphic on $\boldsymbol{C} \times X^*$. From these fact, we conclude that $g_t'(x)$ ($t \in \boldsymbol{C}$, $x \in X$) is holomorphic and hence $\{g_t'\}$ defines a complex one-parameter transformation group of X with the property $g_t' = g_t$ for any real t. This completes the proof.

REFERENCES

[1] H. Fujimoto, Vector-valued holomorphic functions on a complex space, J. Math. Soc. Japan, **17** (1965), 52–66.
[2] F. Hujimoto, On the holomorphic automorphism groups of complex spaces, Nagoya Math. J., **33** (1968), 85–106.
[3] H. Grauert and R. Remmert, Zur Theorie der Modifikationen. I, Math. Ann., **129** (1955), 274–296.
[4] H. Grauert and R. Remmert, Komplexe Räume, Math. Ann., **136** (1958), 245–318.
[5] W. Kaup, Infinitesimale Transformationsgruppen komplexer Räume, Math. Ann., **160** (1965), 72–92.
[6] H. Kerner, Über die Fortsetzung holomorpher Abbildungen, Arch. Math., **11** (1960), 44–49.
[7] H. Kerner, Über die Automorphismengruppen kompakter komplexer Räume, Arch. Math., **11** (1960), 282–288.
[8] A. Morimoto, Sur le groupe d'automorphismes d'un espace fibré principal analytique complexe, Nagoya Math. J., **13** (1958), 157–168.
[9] R. Remmert, Holomorphe und meromorphe Abbildungen komplexer Räume, Math. Ann., **133** (1957), 328–370.

Mathem
Nagoya