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PURE UNITAL LOCAL PRINCIPAL IDEAL DOMAINS IN LOCAL FIELDS

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The main purpose of this note is to give a characterization of p-pure unital subrings of the p-adic completion of the ring of integers R of an algebraic number field K localized at a maximal ideal p. This yields a characterization of the valued subfields of the p-adic field. In this context there turn up valuations of rational function fields in many indeterminates which seem to be new. The proof that the underlying function is indeed a valuation is quite easy here, however direct computations would involve a large amount of combinatorics. Our approach seems to fit well with Kronecker's, apparently forgotten, approach to ideal theory in rings of algebraic integers [**3**]. The concept of p-pure unital subrings arose from a study by the first author and A. Laroche of quasi-p-pure-injective (q.p.p.i.) abelian groups ([**1**], p. 582). It was shown in [**1**] that a q.p.p.i. torsion free abelian group is a module over some p-pure unital subring of the ring of p-adic integers. However, a complete characterization of such groups is still an open problem.

Let K be an algebraic number field and let R be the ring of integers of K. Let p be a maximal ideal of R. We denote by R_p the localisation of R at p and by $\hat{R}_p$ and $\hat{K}_p$ the p-adic completions.

Definition. A subring A of $\hat{R}_p$ containing R is said to be
(i) *p-pure* if $pA = A \cap p\hat{R}_p$;
(ii) *unital* if $\mathscr{U}(A) = A \cap \mathscr{U}(\hat{R}_p)$.

We note that every unital subring A of $\hat{R}_p$ contains R_p. In fact, if $r \notin p$ ($r \in R$) then $r \in \mathscr{U}(\hat{R}_p) \cap A = \mathscr{U}(A)$ therefore r is a unit in A and $R_p \subset A$.
Unital subrings of $\hat{R}_p$ can be characterized by the following:

PROPOSITION 1. *A is a unital subring of $\hat{R}_p$ if and only if $A \cap p\hat{R}_p$ is the Jacobson radical of A.*

Proof. The canonical map $\nu: A \rightarrow \hat{R}_p/p\hat{R}_p$ is surjective since restricted to R it is already surjective. Hence $A \cap p\hat{R}_p = \ker \nu$ is a maximal ideal in A since $\nu(A)$ is a field. Now, let $\alpha \in A \backslash \ker \nu$, then $\alpha \in \mathscr{U}(\hat{R}_p) \cap A = \mathscr{U}(A)$ since A is unital. Therefore A is a local ring whose maximal ideal is $A \cap p\hat{R}_p$. The converse is obvious.

We show next that the concept of p-pure unital subrings of $\hat{R}_p$ coincides with

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that of discrete valuation subrings of $\hat{R}_p$. We obtain thus a new characterization of these important rings.

PROPOSITION 2. *Let A be a subring of $\hat{R}_p$ containing R, and let L be the field of quotients of A. Then the following properties are equivalent:*

(i) *A is a p-pure unital subring of $\hat{R}_p$;*
(ii) *$A = L \cap \hat{R}_p$;*
(iii) *A is a local principal ideal domain;*
(iv) *A is a discrete valuation ring of L;*
(v) *A is a rank 1 valuation ring of L.*

Proof. We take the following circuit
(i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii) $\Rightarrow$ (i).
We observe that we need only prove (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) the other implications being well known results from valuation theory.

(i) $\Rightarrow$ (iii). A being unital, we see from the proof of Proposition 1 that A is a local ring whose maximal ideal is $A \cap p\hat{R}_p$. By the purity of A this ideal is principal, and so A is a local principal ideal domain.

(ii) $\Rightarrow$ (i). If $A = L \cap \hat{R}_p$ it follows that A is p-pure. Let $a \in A \cap \mathcal{U}(\hat{R}_p)$, then $a^{-1} \in L \cap \hat{R}_p = A$ so $a \in \mathcal{U}(A)$.

It should be noted that Proposition 2 gives a way of constructing all p-pure unital subrings of $\hat{R}_p$. In fact for every intermediate field F between K and $\hat{K}_p$, $F \cap \hat{R}_p$ is a p-pure unital subring and this correspondence is a bijection.

We examine now the circumstances under which a p-pure subring of $\hat{R}_p$ is also unital. But first we need the following characterization of p-pure subrings of $\hat{R}_p$ containing R_p.

LEMMA 3. *Let A be a subring of $\hat{R}_p$ containing R_p. Then A is p-pure if and only if $A = KA \cap \hat{R}_p$.*

Proof. Observe that A is p-pure if and only if $p^iA = A \cap p^i\hat{R}_p$ for every $i > 0$. Let π be a generator of p and let $A = KA \cap \hat{R}_p$ then $p\hat{R}_p = \pi\hat{R}_p$. Now let

$$a \in \pi\hat{R}_p \cap A = \pi\hat{R}_p \cap (\hat{R}_p \cap KA)$$

then $a = \pi\alpha$ where $\alpha \in KA \cap \hat{R}_p$, therefore $\alpha \in A$ and A is p-pure. Conversely, let $b \in KA \cap \hat{R}_p$, then $b = \pi^{-s}a$ for some $a \in A$ and

$$a = \pi^s b \in p^s\hat{R}_p \cap A = p^sA$$

therefore $b \in A$ and $A = KA \cap \hat{R}_p$.

In general KA will not be the field of quotients of A. In fact we have:

PROPOSITION 4. *Let A be a p-pure subring of $\hat{R}_p$ containing R_p. Then A is unital if and only if $KA = L$, where L is the field of quotients of A.*

Proof. If $KA = L$ then the result follows from Proposition 2 and Lemma 3. Conversely if A is p-pure and unital then $A = L \cap \hat{R}_p = KA \cap \hat{R}_p$, in view of these same results. Now, since $KA \subset L$ it suffices to show that KA is a field. Let $p = (\pi)$ where $\pi \in R$ then $(\pi \hat{R}_p) \cap A = \pi A$ is the unique maximal ideal in A, thus it suffices to show that π is invertible in KA but this is obvious.

Remark. Before we give a few illustrative examples, we would like at this point to acknowledge our indebtedness to the referee who pointed out to us a serious flaw in our original version of Proposition 4 and who led us to Example 5 (iii) below.

Example 5. Trivially R_p and $\hat{R}_p$ are p-pure unital subrings of $\hat{R}_p$. Here we construct a non-trivial example.

(i) Let $K \subset L \subset K_p$ with $|L:K| = m > 1$ finite and let S be the ring of algebraic numbers over R_p in L. Then write pS as a product of different prime ideals in S, say $pS = \prod_{i=1}^{n} p_i^{\alpha_i}$. Going to $\hat{R}_p$, we obtain

$$p\hat{R}_p = \prod_{i=1}^{n} (p_i \hat{R}_p)^{\alpha_i}.$$

Since $p\hat{R}_p$ is the unique maximal ideal in $\hat{R}_p$, there exists an i_0 which we may take to be 1, such that $\alpha_1 = 1$ and $p_i\hat{R}_p = \hat{R}_p$ for all $i > 1$. Moreover $S/p_1 \simeq R/p$ and from the fundamental equation $\sum \alpha_i f_i = [L:K]$ it follows that $n > 1$. Let A denote the localization of S at p_1. It is clear that $A \subset \hat{R}_p$, since all prime ideals in S except p_1 generate all of $\hat{R}_p$. Then $A/p_1 A \simeq R/p$ and $p_1 A = \text{rad}\,(A)$ so that A is unital and from this construction it follows that $pA = p_1 A = p\hat{R}_p \cap A$. Hence A is also p-pure.

Next, we present an example of a unital subring which is not p-pure.

(ii) Let $L = K(x) \subset \hat{K}_p$, where x is transcendental over K. Let $B = R_p[x]$. Then, assuming $x\hat{R}_p = p\hat{R}_p$ we have the canonical homomorphism

$$\varphi: B = R_p[x] \rightarrow F_q,$$

whose kernel is the maximal ideal $B\langle x, p\rangle$ of B. Let $m = B\langle x, p\rangle$. Localizing at m we have $mB_m = B_m \cap p\hat{R}_p$ and so by Proposition 1, B_m is unital. On the other hand B_m has krull dimension 2, therefore it is not a local principal ideal domain. Consequently B_m is not p-pure in view of Proposition 2.

Here is an example of a p-pure subring of $\hat{\mathbf{Z}}_p$ which contains $\mathbf{Z}_p$ but which is not unital.

(iii) Let $\alpha \in \mathscr{U}(\hat{\mathbf{Z}}_p)$ be transcendental over $\mathbf{Z}_p$ and let

$$A = \{x \in \hat{\mathbf{Z}}_p | \exists\, n \in \mathbf{Z},\, nx \in \mathbf{Z}_p[\alpha]\}.$$

A is a p-pure subring of $\hat{\mathbf{Z}}_p$ containing $\mathbf{Z}_p$. However, $\alpha \notin \mathscr{U}(A)$ as otherwise α would be algebraic over $\mathbf{Z}_p$.

Remark. Let v_p and $\hat{v}_p$ denote respectively the p-adic valuations on K and $\hat{K}_p$. Then the restriction of $\hat{v}_p$ to L is a valuation v_L on L for every subfield L of $\hat{K}_p$ containing K. Clearly $L \cap \hat{R}_p$ is the valuation ring of v_L. In the case of $L = K(x)$ where x is a transcendental element over K such that $x\hat{R}_p = p\hat{R}_p$ then

$$v_L(f(x)/g(x)) = v_p(f(x)) - v_p(g(x)).$$

Let π be a generator of p, then, in particular $x\pi^{-1}$ is a unit in $A = L \cap \hat{R}_p$ and if we write $x = \pi\alpha$ where α is a unit in $\hat{R}_p$ then $\alpha \equiv i(\pi)$ for some positive integer $i < \pi$ and therefore $(x - \pi i)/x$ lies in A, but is not a unit in A.

It seems that for the purely transcendental extensions, the valuation rings cannot be described easily. However, it is not difficult to see that all algebraic extensions have as valuation ring direct limits of the valuation rings described in Example 5 (i).

We conclude with a somewhat surprising result derived from Proposition 4.4., p. 583 in [**1**].

PROPOSITION 6. *Let L and M be subfields of $\hat{Q}_p$, v_L and v_M the induced valuation on L and M by the p-adic valuation on $\hat{Q}_p$. Then (L, v_L) is isomorphic to (M, v_M) if and only if $M = L$.*

REFERENCES

1. K. Benabdallah and A. Laroche, *Quasi-p-pure injective groups*, Can. J. Math. *29* (1977), 578–586.

2. O. Endler, *Valuation theory* (Springer-Verlag, New-York, 1972).

3. L. Kronecker, *Vorlesungen Über Zahlentheorie*, Leipzig, 1901, BD 2, 143–241.

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