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EXTENDING A RESULT OF RYAN ON WEAKLY COMPACT OPERATORS

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Abstract An elegant result of Ryan gives a characterization of weakly compact operators from a Banach space A into $c_0(X)$, the space of null sequences in a Banach space X. It would be a useful tool if the analogue of Ryan's result were valid when $c_0(X)$ is replaced by $c(X)$, the space of convergent sequences in X. This seems plausible and has been assumed to be true by some authors. Unfortunately, it is false in general; Ylinen has produced a counterexample. But when A is a C^*-algebra, or, more generally, when the dual of A is weakly sequentially complete, we show that the desired extension of Ryan's result does hold. The latter result turns out to be 'best possible'.

Keywords: weakly compact; weakly sequentially complete; C^*-algebras

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1. Introduction

The origin of this paper stems from observing that some results on non-commutative, finitely additive vector measures (i.e. weakly compact operators from a C^*-algebra to a Banach space) do not depend on the domain being a C^*-algebra but are essentially Banach space results.

Let A and X be Banach spaces and let (T_n) $(n = 1, 2, \dots)$ be a sequence of weakly compact operators mapping A into X. For each $z \in A^{**}$ let $(T_n^{**}z)$ $(n = 1, 2, \dots)$ be a Cauchy sequence. Since, for each n, T_n is weakly compact, the range of T_n^{**} is in X. By the uniform boundedness theorem there is a bounded operator $T^\# : A^{**} \mapsto X$ such that $\lim_{n\to\infty} T_n^{**}z = T^\# z$ for each z in A^{**}. It would be natural to expect $T^\#$ to be weakly compact but, in general, this is false. This follows from the following example constructed by Ylinen [**6**].

In [**6**, Proposition 2.1], $A = l^1 = X$. For each n, $T_n : l^1 \mapsto l^1$ is defined by

$$T_n(x_1, x_2, \dots, x_k, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots).$$

Then each T_n is weakly compact (because its range is finite dimensional). Ylinen proves that $(T_n^{**}z)$ $(n = 1, 2, \dots)$ converges for each z in the dual of l^∞ but the pointwise limit of the sequence of operators (T_n) $(n = 1, 2, \dots)$ is not weakly compact.

However, if A is a C^*-algebra, then there does exist a weakly compact operator $T : A \mapsto X$ such that $\lim_{n\to\infty} T_n^{**}z = T^{**}z$ for each z in A^{**}. This is an immediate consequence of [**1**, Corollary 3.3]. In this note we show that a positive result is also obtained if A^* is weakly complete. (We recall that the dual of a C^*-algebra is always weakly complete.) We shall also see that, in a sense made precise here, the latter result is 'best possible'.

Ryan [**4**] characterized weakly compact operators from a Banach space A into $c_0(X)$, the space of null sequences in a Banach space X (see Proposition 3.4 below). When $c_0(X)$ is replaced by $c(X)$, the space of convergent sequences in X, the natural extension of Ryan's characterization does not hold, in general. But when X^* is weakly (sequentially) complete, then we show, in § 3, that Ryan's characterization can be generalized successfully by applying the results we obtain in § 2. This can then be applied to underpin some fundamental work on weak compactness and multilinear operators on Banach spaces [**3**].

2. Convergent sequences of weakly compact operators

Let us recall that a Banach space Z is said to be *weakly complete* if, whenever (z_n) $(n = 1, 2, \dots)$ is a sequence in Z such that (ϕz_n) $(n = 1, 2, \dots)$ is a Cauchy sequence for every ϕ in Z^*, then there exists z in Z such that $\phi z_n \to \phi z$ for every ϕ in Z^*. Some authors use the term *weakly sequentially complete* for the same property.

Theorem 2.1. *Let A be a Banach space such that A^* is weakly complete. Let X be a Banach space and let (T_n) $(n = 1, 2, \dots)$ be a sequence of weakly compact operators from A into X. Let $(T_n^{**}z)$ $(n = 1, 2, \dots)$ be a Cauchy sequence for each z in A^{**}. Then there exists a weakly compact operator T such that $\|(T^{**} - T_n^{**})z\| \to 0$ for each z in A^{**}.*

Proof. Since T_n is weakly compact, T_n^{**} maps A^{**} into X. Let $T^{\#}z = \lim T_n^{**}z$ for each z in A^{**}. Then, by the uniform boundedness theorem, $T^{\#}$ is a bounded linear operator from A^{**} into X. Let T be the restriction of $T^{\#}$ to A.

Fix $\phi \in X^*$. Then, for each $z \in A^{**}$,

$$\lim_{n\to\infty} \langle T_n^{**}z, \phi\rangle = \langle T^{\#}z, \phi\rangle.$$

So

$$\lim_{n\to\infty} \langle z, T_n^{*}\phi\rangle = \langle T^{\#}z, \phi\rangle.$$

So $(T_n^*\phi)$ $(n = 1, 2, \dots)$ is a weakly Cauchy sequence in A^*. By the hypothesis that A^* is weakly complete, it follows that there exists a unique $\alpha \in A^*$ such that $\langle z, \alpha\rangle = \langle T^{\#}z, \phi\rangle$ for all z in A^{**}.

All that is now needed is to show that $T^{**} = T^{\#}$. Since this has been a source of error in the past we wish to avoid being too glib and so give a detailed elementary argument.

Let (z_t) be a net in A^{**} which converges to 0 in the $\sigma(A^{**}, A^*)$-topology. So $\langle z_t, \alpha\rangle \to 0$. Thus $\langle T^{\#}z_t, \phi\rangle \to 0$ for each ϕ in X^*. So $T^{\#}$ is a continuous map of A^{**}, equipped with the weak*-topology, to X equipped with the weak topology. Since the norm closed unit ball of A^{**} is weak* compact, the image of the unit ball of A^{**} under the map $T^{\#}$ is weakly compact. Hence $T^{\#}$, and its restriction to A, T, is weakly compact. Thus, by Lemma VI.2.3 and Theorem VI.4.2 of [**2**], T^{**} is weak* to weak continuous from A^{**} to X. By Goldstine's theorem (see [**2**, Theorem V.4.5]), the norm closed unit ball of A is weak*-dense in the norm closed unit ball of A^{**}. Hence $T^{\#} = T^{**}$. □

Remark. Let A be a C^*-algebra. Its dual is then the predual of a von Neumann algebra and so, by [**5**, Corollary III.5.2], the dual of A is weakly complete. Hence Theorem 2.1 applies whenever A is a C^*-algebra.

It turns out that Theorem 2.1 is 'best possible'. To make this claim precise it is convenient to introduce the following definition.

Definition 2.2. Let X be a Banach space. A Banach space A is said to have the *weak compactness stability property with respect to* X if, given any sequence of weakly compact operators (T_n) $(n = 1, 2, \dots)$, each mapping A into X, and with $(T_n^{**}z)$ $(n = 1, 2, \dots)$ a Cauchy sequence for each z in A^{**}, there exists a weakly compact operator T such that $lim_{n\to\infty} T_n^{**}z = T^{**}z$ for each z in A^{**}.

Proposition 2.3. *Let A be a Banach space with the weak compactness stability property with respect to some non-zero Banach space X. Then A^* is weakly complete.*

Proof. Let (ϕ_n) $(n = 1, 2, \dots)$ be a weakly Cauchy sequence in A^*. Then, for each z in A^{**}, $\lim_{n\to\infty}\langle z, \phi_n\rangle$ exists. By the uniform boundedness theorem, there exists a bounded linear functional $\psi^{\#}$ on A^{**} such that $\psi^{\#}(z) = \lim_{n\to\infty}\langle z, \phi_n\rangle$ for each z in A^{**}.

Since X is a non-zero Banach space it contains a non-zero element x_0. For each n, let $T_n : A \mapsto X$ be defined by

$$T_n(a) = \langle a, \phi_n\rangle x_0.$$

Then T_n has a one-dimensional range and so is (weakly) compact. Furthermore, $T_n^{**}(z) = \langle z, \phi_n\rangle x_0$ for each z in A^{**}. It now follows from the weak compactness stability property for X that there exists a weakly compact operator T mapping A into X, such that

$$T^{**}(z) = \lim_{n\to\infty} T_n^{**}(z) = \lim_{n\to\infty}\langle z, \phi_n\rangle x_0 = \psi^{\#}(z)x_0 \quad \text{for each } z \text{ in } A^{**}.$$

Since T is weakly compact, then, as remarked in the proof of Theorem 2.1, T^{**} is weak* to weak continuous as a map from A^{**} to X. Thus $\psi^{\#}$ is a weak* continuous linear functional on A^{**}. So, by [**2**, Theorem V.3.9], $\psi^{\#}$ may be identified with an element of A^*. Hence (ϕ_n) $(n = 1, 2, \dots)$ is weakly convergent. Thus A^* is weakly complete. □

Corollary 2.4. *Let A be a Banach space. Then the following conditions are equivalent:*

(i) *A^* is weakly complete;*

(ii) *A has the weak compactness stability property with respect to some Banach space of non-zero dimension;*

(iii) *A has the weak compactness stability property with respect to every Banach space X.*

Proof. By Theorem 2.1, (i) implies (iii). Trivially (iii) implies (ii). By Proposition 2.3, (ii) implies (i). □

3. Extending Ryan's lemma

For any Banach space X, let $c(X)$ be the Banach space of all (norm) convergent sequences in X, equipped with the supremum norm. Those elements of $c(X)$ which are sequences in X converging (in norm) to 0 form a closed subspace which is denoted by $c_0(X)$.

For each positive integer n, let T_n be a bounded linear operator from a Banach space A into a Banach space X. Let $\lim T_n a$ exist for each a in A. Then $(T_n a)$ $(n = 1, 2, \dots)$ is a vector in $c(X)$. Let T_∞ be the linear map from A into X defined by $T_\infty a = \lim T_n a$ for each a in A. We use $\boldsymbol{T}$ to denote the operator from A to $c(X)$ associated with the sequence (T_n) $(n = 1, 2, \dots)$ and defined by $\boldsymbol{T}(a) = (T_n a)$ $(n = 1, 2, \dots)$. By applying the uniform boundedness theorem we see that T_∞ and $\boldsymbol{T}$ are both bounded linear operators. Conversely, every bounded operator from A into $c(X)$ arises in this way from a sequence of operators from A into X.

Let us recall [**4**] that, for $1 \leqslant p < \infty$ and X an arbitrary Banach space, $l^p(X)$ is the Banach space whose points are the sequences $\boldsymbol{x} = (x_n)$ $(n = 1, 2, \dots)$ in X for which $\sum_1^\infty \|x_n\|^p < \infty$. The norm of $\boldsymbol{x}$ is defined to be $(\sum_1^\infty \|x_n\|^p)^{1/p}$. Also, $l^\infty(X)$ is defined to be the Banach space whose points are all bounded sequences in X and where the norm of $\boldsymbol{x} = (x_n)$ $(n = 1, 2, \dots)$ is defined to be $\sup\{\|x_n\| : 1 \leqslant n\}$.

Given $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots)$ in $l^1(X^*)$ and $\boldsymbol{x} = (x_n)$ $(n = 1, 2, \dots)$ in $c(X)$, let

$$L_{\boldsymbol{\phi}}(\boldsymbol{x}) = \phi_0(\lim x_n) + \sum_{n=1}^{\infty} \langle x_n, \phi_n \rangle.$$

Straightforward calculations then show that $L_{\boldsymbol{\phi}}$ is a bounded linear functional on $c(X)$ and its norm is $\sum_{n=0}^\infty \|\phi_n\|$. Furthermore, the map $\boldsymbol{\phi} \mapsto L_{\boldsymbol{\phi}}$ can be shown to be a surjective isometry of $l^1(X^*)$ onto $c(X)^*$.

Then the canonical bilinear form $\langle \cdot, \cdot \rangle$ arising from the dual pair $(c(X), c(X)^*)$, where $l^1(X^*)$ and $c(X)^*$ are isometrically isomorphic by the map $\boldsymbol{\phi} \mapsto L_{\boldsymbol{\phi}}$ described above, is given by

$$\langle \boldsymbol{x}, L_{\boldsymbol{\phi}} \rangle = \Big\langle \lim_{n\to\infty} x_n, \phi_0 \Big\rangle + \sum_{n=1}^{\infty} \langle x_n, \phi_n \rangle$$

for each $\boldsymbol{x} = (x_1, x_2, \dots) \in c(X)$ and $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots) \in l^1(X^*)$.

It follows from the remarks in [**4**] that the dual of $l^1(X^*)$ can be identified in a natural way with $l^\infty(X^{**})$. Thus $c(X)^{**}$ can be identified with $l^\infty(X^{**})$. Let $\natural$ be the canonical embedding of X into X^{**}. Then a sequence (x_n) $(n = 1, 2, \dots)$ in $c(X)$ is mapped to $(\lim \natural x_n, \natural x_1, \natural x_2, \dots)$ in $l^\infty(X^{**})$.

Lemma 3.1. *Let $\boldsymbol{T}$ be a bounded operator from a Banach space A into $c(X)$ and let T_n $(n = 1, 2, \dots)$ and T_∞ be the operators from A into X associated with $\boldsymbol{T}$ as above.*

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Fix L in $c(X)^*$. Then let $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots)$ be the corresponding element of $l^1(X^*)$. Then, for each $z \in A^{**}$,

$$\langle \boldsymbol{T}^{**}z, L\rangle = \langle T_\infty^{**}z, \phi_0\rangle + \sum_{n=1}^{\infty}\langle T_n^{**}z, \phi_n\rangle.$$

Proof. For each $a \in A$,

$$\langle \boldsymbol{T}a, L\rangle = \langle T_\infty a, \phi_0\rangle + \sum_{n=1}^{\infty}\langle T_n a, \phi_n\rangle.$$

Now let z be in the unit ball of A^{**}. Then, by Goldstine's theorem (see above) there is a net (a_t) in the unit ball of A which converges weak* to z. Then $\boldsymbol{T}a_t \to \boldsymbol{T}^{**}z$ in the weak* topology of $c(X)^{**}$. So $\langle \boldsymbol{T}a_t, L\rangle \to \langle \boldsymbol{T}^{**}z, L\rangle$. Similarly, for each N,

$$\langle T_\infty a_t, \phi_0\rangle + \sum_{n=1}^{N}\langle T_n a_t, \phi_n\rangle \to \langle T_\infty^{**}z, \phi_0\rangle + \sum_{n=1}^{N}\langle T_n^{**}z, \phi_n\rangle.$$

Choose $\varepsilon > 0$. Choose N large enough to ensure that

$$\|\boldsymbol{T}\| \sum_{n=N+1}^{\infty} \|\phi_n\| \leqslant \varepsilon.$$

Then for any w in the unit ball of A^{**},

$$\left| \sum_{n=N+1}^{\infty} \langle T_n^{**}w, \phi_n\rangle \right| \leqslant \|\boldsymbol{T}\| \sum_{n=N+1}^{\infty} \|\phi_n\| \leqslant \varepsilon.$$

From this it follows by routine arguments that

$$\langle \boldsymbol{T}^{**}z, L\rangle = \langle T_\infty^{**}z, \phi_0\rangle + \sum_{n=1}^{\infty}\langle T_n^{**}z, \phi_n\rangle.$$

□

We have seen that $c(X)^{**}$ can be identified with $l^\infty(X^{**})$. When this identification is made appropriately, we have the following corollary.

Corollary 3.2. *For each z in A^{**} we have*

$$\boldsymbol{T}^{**}(z) = (T_\infty^{**}z, T_1^{**}z, T_2^{**}z, \dots, T_n^{**}z, \dots).$$

The following lemma was, in essence, proved by Ylinen [**6**]. For the convenience of the reader, we give a brief proof here as an application of Corollary 3.2.

Lemma 3.3. *Let A and X be Banach spaces and let $\boldsymbol{T}$ be a weakly compact operator from A into $c(X)$. Let (T_n) $(n = 1, 2, \dots)$ be the sequence of operators from A into X such that $\boldsymbol{T}(a) = (T_n a)$ $(n = 1, 2, \dots)$ for each a in A. Then each T_n is weakly compact. Also, T_∞, the pointwise limit of (T_n) $(n = 1, 2, \dots)$, is weakly compact. Furthermore, $\lim T_n^{**}(z) = T_\infty^{**}z$ for each z in A^{**}.*

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Proof. We recall that the product of a bounded operator and a weakly compact operator is weakly compact. Let π_n be the canonical projection of $c(X)$ onto the nth coordinate. Then $T_n = \pi_n \boldsymbol{T}$. Hence T_n is weakly compact. Let π_∞ be the operator which maps $(a_1, a_2, \dots)$ in $c(X)$ to $\lim a_n$. Then $T_\infty = \pi_\infty \boldsymbol{T}$ and so is also weakly compact.

Since $\boldsymbol{T}$ is a weakly compact operator from A into $c(X)$, $\boldsymbol{T}^{**}$ maps A^{**} into the canonical image of $c(X)$ in the second dual $c(X)^{**}$. Hence, for every $z \in A^{**}$, there exists $\boldsymbol{x} = (x_1, x_2, \dots)$ in $c(X)$ such that

$$\langle (x_1, x_2, \dots), L_{\boldsymbol{\phi}} \rangle = \langle (\phi_0, \phi_1, \phi_2, \dots), (T_\infty^{**} z, T_1^{**} z, T_2^{**} z, \dots) \rangle$$

for all $\boldsymbol{\phi} = (\phi_0, \phi_1, \phi_2, \dots)$.

Now take any $\phi \in X^*$ and consider

$$\boldsymbol{\phi}^{(k)} = (\delta_{k,n} \phi)_{n=0}^{\infty} \in l^1(X^*) \quad \text{for } k = 0, 1, 2, \dots.$$

Then we have

$$\langle x_k, \phi \rangle = \langle \phi, T_k^{**} z \rangle$$

for all $k \geqslant 1$ and

$$\Big\langle \lim_{k \to \infty} x_k, \phi \Big\rangle = \langle \phi, T_\infty^{**} z \rangle,$$

that is, $\natural x_n = T_n^{**} z$ for all n and $\lim_{n \to \infty} \natural x_n = T_\infty^{**} z$. Hence it follows that

$$\| T_n^{**} z - T_\infty^{**} z \| \to 0, \quad n \to \infty.$$

□

Proposition 3.4 (Ryan [4]). *Let A and X be Banach spaces. Let (T_n) $(n = 1, 2, \dots)$ be a sequence of bounded operators from A into X. Let $\|T_n z\| \to 0$ for each z in A. Then $\boldsymbol{T}$ is a weakly compact operator from A into $c_0(X)$ if, and only if, each T_n is weakly compact and $\|T_n^{**} z\| \to 0$ for each z in A^{**}. When $\boldsymbol{T}$ is weakly compact, $\boldsymbol{T}^{**}(z) = (T_n^{**}(z))$ $(n = 1, 2, \dots)$ for each z in A^{**}.*

Proposition 3.4 is a special case of the following result, which is essentially due to Ylinen [**6**].

Proposition 3.5. *Let A and X be Banach spaces and let $\boldsymbol{T}$ be a bounded operator from A into $c(X)$. Let (T_n) $(n = 1, 2, \dots)$ be the sequence of operators from A into X such that $\boldsymbol{T}(a) = (T_n a)$ $(n = 1, 2, \dots)$ for each a in A and let T_∞ be the operator from A into X which is defined by $T_\infty a = \lim T_n a$ for each $a \in A$. Then $\boldsymbol{T}$ is weakly compact if and only if the following conditions are satisfied.*

(i) *For each n, T_n is weakly compact.*

(ii) *For each z in A^{**}, $\lim T_n^{**}(z) = T_\infty^{**}(z)$.*

(iii) *The operator T_∞ is weakly compact.*

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Proof. By Lemma 3.3, when $\boldsymbol{T}$ is weakly compact the three conditions are satisfied. Now suppose that the conditions are satisfied. So, for each z in A^{**}, condition (iii) implies that $T_\infty^{**}z$ is in X and condition (i) implies that $T_n^{**}z$ is in X for each n. Hence, by condition (ii), $(T_\infty^{**}z, T_1^{**}z, T_2^{**}z, \dots, T_n^{**}z, \dots)$ is in the canonical image of $c(X)$ in $c(X)^{**}$. Hence, by Corollary 3.2, $\boldsymbol{T}^{**}$ maps A^{**} into $c(X)$. So $\boldsymbol{T}$ is weakly compact. □

Theorem 3.6. *Let X be any Banach space. Let A be a Banach space whose dual space, A^*, is weakly complete. Let (T_n) $(n = 1, 2, \dots)$ be a sequence of weakly compact operators from A into X such that $(T_n^{**}(z))$ $(n = 1, 2, \dots)$ is a Cauchy sequence for each z in A^{**}. Then $\boldsymbol{T}$ is a weakly compact operator from A into $c(X)$.*

Proof. Because the dual of A is weakly complete, Theorem 2.1 implies the existence of a weakly compact operator $T_\infty : A \mapsto X$ such that $T_n^{**}(z) \to T_\infty^{**}(z)$ for each z in A^{**}. So conditions (i)–(iii) of Proposition 3.5 are satisfied. □

Remark. If A^* is not weakly complete, then it follows from Proposition 2.3 that we can find a sequence of weakly compact operators, (T_n) $(n = 1, 2, \dots)$, each mapping A into c, such that $(T_n^{**}(z))$ $(n = 1, 2, \dots)$ is a convergent sequence for each z in A^{**} but T_∞ is not weakly compact (where $T_\infty(a) = \lim T_n(a)$ for each a in A). It then follows from Proposition 3.5 that $\boldsymbol{T}$ is not weakly compact. So the hypothesis that A^* is weakly complete is essential for the validity of Theorem 3.6.

References

1. J. K. Brooks, K. Saitô and J. D. M. Wright, Operator algebras and a theorem of Dieudonné, *Rend. Circ. Mat. Palermo* **52** (2003), 5–14.
2. N. Dunford and J. T. Schwartz, *Linear operators*, Volume I (Interscience, New York, 1958).
3. J. M. Gutiérrez and I. Villanueva, Extensions of multilinear operators and Banach space properties, *Proc. R. Soc. Edinb.* A **133** (2003), 549–566.
4. R. A. Ryan, Dunford–Pettis properties, *Bull. Polish Acad. Sci. Math.* **27** (1979), 373–379.
5. M. Takesaki, *Theory of operator algebras*, Volume I (Springer, 1979).
6. K. Ylinen, Weakly compact operators into sequence spaces: a counterexample, *Proc. Am. Math. Soc.* **133** (2005), 1423–1425.

https://doi.org/10.1017/S0013091504001403 Published online by Cambridge University Press