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STABILITY OF DHOMBRES' EQUATION

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One of the most important examples of conditional Cauchy equations with the condition dependent on the unknown function is Dhombres' equation

$$f(x) + f(y) \neq 0 \implies f(x+y) = f(x) + f(y).$$

This paper is devoted to the stability of the above equation.

1. INTRODUCTION

Our investigations are at the intersection of two research areas. One of them concerns conditional Cauchy equations, where the validity of Cauchy equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

is postulated for pairs (x, y) that satisfy some additional condition (see for example, [**2**, **3**]). We shall combine this research direction with the stability question, which, following Ulam (see [**8**]) and Hyers (see [**4**]), has been widely investigated (see for example [**5**]).

Our main results concern the stability of Dhombres' equation (see for example [**3**])

$$f(x) + f(y) \neq 0 \implies f(x+y) = f(x) + f(y), \tag{2}$$

as a fundamental example of conditional Cauchy equations with the condition dependent on the unknown function. Dhombres' equation is a symmetric analogue of Mikusiński's equation, stability of which was investigated in [**1**]. These results motivated us to solve a similar problem posed with respect to equation (2).

Throught the paper $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{C}$ are used to denote the sets of all positive integers, nonnegative integers and complex numbers, respectively.

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2. Stability results

In this section we prove the stability of Dhombres' equation (2).

Let a function f mapping a semigroup $(G,+)$ into a normed space $(X,\|\cdot\|)$ satisfy

$$\|f(x)+f(y)\|>\delta \implies \|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon \tag{3}$$

for all $x,y \in G$, with given $\delta,\varepsilon \geqslant 0$. For an arbitrary $x \in G$ exactly one of the following conditions holds:

(i) $\|f(2^n x)\| > \delta/2$ for $n \in \mathbb{N}_0$;

(ii) there exists an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $\|f(2^{n_k}x)\| \leqslant \delta/2$ for $k \in \mathbb{N}$;

(iii) there exists $k \in \mathbb{N}$ such that $\|f(2^{k-1}x)\| \leqslant \delta/2$ and $\|f(2^n x)\| > \delta/2$ for $n \geqslant k$.

Firstly we shall prove two auxiliary lemmas.

Lemma 1. *Let $(G,+)$ be a semigroup and let $(X,\|\cdot\|)$ be a normed space. If $f: G \to X$ satisfies* (3) *with some $\delta,\varepsilon \geqslant 0$ and all $x,y \in G$, then for an arbitrary $x \in G$ satisfying* (ii) *or* (iii) *we have*

$$\|f(x)\| \leqslant \max\left\{\varepsilon, \frac{1}{2}\delta\right\}. \tag{4}$$

Proof: Let us observe that for an arbitrary $x \in G$ satisfying (ii) or (iii) there exists a smallest nonnegative integer k with $\|f(2^k x)\| \leqslant \delta/2$. For the proof it is enough to use (3) with x and y replaced sequentially by x, $2x$, $\dots$, $2^{k-1}x$, and combine the inequalities obtained (the case $k=0$ is obvious). ∎

Lemma 2. *Let $(G,+)$ be a semigroup and let $(X,\|\cdot\|)$ be a normed space. If $f: G \to X$ satisfies* (3) *with some $\delta,\varepsilon \geqslant 0$ and all $x,y \in G$, then for an arbitrary $x \in G$ satisfying* (iii) *we have*

$$\|f(2^k x)\| \leqslant 3\varepsilon + 2\delta, \tag{5}$$

with k defined by (iii).

Proof: Let $x \in G$ satisfy (iii) and suppose that

$$\|f(2^k x) + f(2^{k-1}x)\| > \delta$$

(the opposite case is trivial). Substituting x by $2^k x$ and y by $2^{k-1}x$ in (3) we obtain

$$\|f(3\cdot 2^{k-1}x) - f(2^k x) - f(2^{k-1}x)\| \leqslant \varepsilon, \tag{6}$$

which, according to the definition of k, implies (5) in the case where

$$\|f(3\cdot 2^{k-1}x) + f(2^{k-1}x)\| \leqslant \delta.$$

Considering the opposite case, that is,

$$\|f(3\cdot 2^{k-1}x)+f(2^{k-1}x)\|>\delta.$$

we replace x with $3\cdot 2^{k-1}x$ and y with $2^{k-1}x$ in (3) and obtain

$$\|f(2^{k+1}x)-f(3\cdot 2^{k-1}x)-f(2^{k-1}x)\|\leqslant\varepsilon.$$

Adding this inequality and (6) side by side we get

$$\|f(2^{k+1}x)-f(2^{k}x)-2f(2^{k-1}x)\|\leqslant 2\varepsilon. \tag{7}$$

On the other hand, using (3) with x and y replaced by 2^kx, we obtain

$$\|f(2^{k+1}x)-2f(2^{k}x)\|\leqslant\varepsilon.$$

Now, according to the definition of k, condition (5) results easily from the inequality above and inequality (7). □

Our main stability result concerning Dhombres' equation (2) reads as follows:

THEOREM 1. *Let $(G,+)$ be an Abelian group and let $(X,\|\cdot\|)$ be a Banach space. If, for some $\varepsilon,\delta\geqslant 0$ and all $x,y\in G$, a function $f:G\to X$ satisfies condition (3) then there exists a unique additive function $a:G\to X$ such that*

$$\|f(x)-a(x)\|\leqslant\max\Big\{\varepsilon,\frac{1}{2}\delta\Big\} \tag{8}$$

for all $x\in G$.

PROOF: The proof proceeds in three steps.

STEP 1. We shall prove that for all $x\in G$ and sufficiently large $n\in\mathbb{N}$ we have

$$\Big\|\frac{f(2^nx)}{2^n}-f(x)\Big\|\leqslant 3\varepsilon+\frac{3}{2}\delta. \tag{9}$$

Fix an arbitrary $x\in G$. If x satisfies (i) then it is easy to observe that

$$\Big\|\frac{f(2^nx)}{2^n}-f(x)\Big\|\leqslant\Big(1-\frac{1}{2^n}\Big)\varepsilon\quad\text{for all } n\in\mathbb{N}. \tag{10}$$

If x satisfies (ii) then 2^nx (with an arbitrary $n\in\mathbb{N}$) also satisfies (ii). Thus, by Lemma 1 and the triangle inequality, we have

$$\Big\|\frac{f(2^nx)}{2^n}-f(x)\Big\|\leqslant\Big(1+\frac{1}{2^n}\Big)\max\Big\{\varepsilon,\frac{1}{2}\delta\Big\},$$

which implies (9). If, finally, x satisfies (iii), then making use of (10), we have

$$\Big\|\frac{f(2^{n-k}2^kx)}{2^{n-k}}-f(2^kx)\Big\|\leqslant\Big(1-\frac{1}{2^{n-k}}\Big)\varepsilon\quad\text{for } n>k, \tag{11}$$

where $k \in \mathbb{N}$ is chosen so that $2^k x$ satisfies (i). Using the triangle inequality this gives

$$\begin{aligned}\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| &\leqslant \frac{1}{2^k}\left\|\frac{f(2^{n-k}2^k x)}{2^{n-k}} - f(2^k x)\right\| + \left\|\frac{f(2^k x)}{2^k}\right\| + \|f(x)\| \\ &\leqslant 3\varepsilon + \frac{3}{2}\delta,\end{aligned}$$

where the last inequality results from (11), Lemma 2 and Lemma 1. This makes the proof of (9) complete.

Let us fix an arbitrary $x \in G$ and $n, m \in \mathbb{N}$, $n > m$ and observe that

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right\| = \frac{1}{2^m}\left\|\frac{f(2^{n-m}2^m x)}{2^{n-m}} - f(2^m x)\right\|.$$

Now apply (9) with x replaced by $2^m x$: for sufficiently large n, (depending on m), we then have

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\right\| \leqslant \frac{1}{2^m}\left(3\varepsilon + \frac{3}{2}\delta\right),$$

which means that $\left(f(2^n x)/2^n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence for an arbitrary $x \in G$. Thus, the map $a : G \to X$ given by

$$a(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \quad \text{for } x \in G$$

is well defined.

STEP 2. We shall show the additivity of $a : G \to X$.

Consider $x, y \in G$ with $a(x) + a(y) \neq 0$ and observe that $\|f(2^n x) + f(2^n y)\| > \delta$ for a sufficiently large $n \in \mathbb{N}$. Thus, using the definition of a and (3) we obtain $a(x + y) = a(x) + a(y)$. Consequently a is additive, as a solution of Dhombres' equation (2) (see [**2**, Proposition 5.6]).

STEP 3. We shall show inequality (8).

Letting $n \longrightarrow \infty$ in (9) we have

$$\|f(x) - a(x)\| \leqslant 3\varepsilon + \frac{3}{2}\delta \quad \text{for } x \in G \tag{12}$$

according to the definition of a.

Let us consider $x \in G$ satisfying (ii) or (iii) (if x satisfies (i) then (8) follows easily from (10)). By (12) and the additivity of a we obtain

$$a(x) = \lim_{n\to\infty} \frac{f(nx)}{n}. \tag{13}$$

Without loss of generality we may assume, that $a(x) \neq 0$. Then, for a sufficiently large $n \in \mathbb{N}$ and all $p \in \{1, 2, \ldots, 2^{n-1}\}$, we have

$$\left\|f\left((2^n - p)x\right) + f(x)\right\| \geqslant \delta.$$

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Consequently, replacing x sequentially by $(2^n-1)x, (2^n-2)x, \dots, 2^{n-1}x$ and y by x in (3), we obtain

$$\left\|f(2^n x) - f\big((2^n-1)x\big) - f(x)\right\| \leqslant \varepsilon,$$
$$\left\|f\big((2^n-1)x\big) - f\big((2^n-2)x\big) - f(x)\right\| \leqslant \varepsilon,$$
$$\vdots$$
$$\left\|f\big((2^{n-1}+1)x\big) - f(2^{n-1}x) - f(x)\right\| \leqslant \varepsilon,$$

respectively. Adding the above inequalities up, side by side, we have

$$\|f(2^n x) - f(2^{n-1}x) - 2^{n-1}f(x)\| \leqslant 2^{n-1}\varepsilon. \tag{14}$$

Moreover, for a sufficiently large $n \in \mathbb{N}$ one has $\|f(2^n x)\| > \delta$, hence applying (3) with x and y replaced by $2^{n-1}x$ we get

$$\|f(2^n x) - 2f(2^{n-1}x)\| \leqslant \varepsilon.$$

Using the inequality above and (14), we obtain

$$\left\|\frac{f(2^{n-1}x)}{2^{n-1}} - f(x)\right\| \leqslant \left(1 + \frac{1}{2^{n-1}}\right)\varepsilon,$$

and then, letting $n \to \infty$, we have

$$\|f(x) - a(x)\| \leqslant \varepsilon,$$

which implies (8).

One can easily check that if an additive function a satisfies (8) then $a(x) = \lim_{n\to\infty}\big(f(2^n x)/2^n\big)$ for $x \in G$, so a is uniquely determined. □

REMARK 1. The assumption that $(G,+)$ is commutative may be weakened. It suffices to assume the power-associativity of G (see [**6**]). Then, in Step 2 of the proof, we use [**3**, Theorem 8] instead of [**2**, Proposition 5.6].

REMARK 2. It is easy to observe that the constant of approximation in Theorem 1 is the best possible one.

As a corollary we have the superstability of Dhombres' equation in the multiplicative form

$$\big(f(x)+f(y)\big)\big(f(x+y) - f(x) - f(y)\big) = 0. \tag{15}$$

THEOREM 2. *Let $(G,+)$ be an Abelian group. If for some $\varepsilon \geqslant 0$ a function $f : G \to \mathbb{C}$ satisfies*

$$\left|\big(f(x)+f(y)\big)\big(f(x+y) - f(x) - f(y)\big)\right| \leqslant \varepsilon \quad \text{for } x, y \in G, \tag{16}$$

then f is either additive, or bounded with $|f(x)| \leqslant \sqrt{\varepsilon/2}$.

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PROOF: Applying Theorem 1, with δ and ε replaced by $\sqrt{2\varepsilon}$ and $\sqrt{\varepsilon/2}$, respectively, we obtain the existence of an additive function $a : G \to \mathbb{C}$ such that

$$|f(x) - a(x)| \leqslant \sqrt{\varepsilon/2} \quad \text{for } x \in G. \tag{17}$$

If f is bounded, then $a = 0$ and $|f(x)| \leqslant \sqrt{\varepsilon/2}$. Thus, let us assume that f is unbounded. By (17) a is nontrivial and f is of the form $f = a + b$, with some bounded function b. Taking into account this representation and (16) one can easily see that the function

$$y \longmapsto a(y)\big(b(x+y) - b(x) - b(y)\big) \quad \text{for } y \in G$$

is bounded for an arbitrary $x \in G$. This implies that

$$b(x) = \lim_{n\to+\infty}\big(b(x+y_n) - b(y_n)\big) \quad \text{for } x \in G,$$

where $(y_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in G with $|a(y_n)| \to +\infty$ (such a sequence exists since a is nontrivial). Therefore

$$\begin{aligned} b(x+y) &= \lim_{n\to+\infty}\big(b(x+y+y_n) - b(y_n)\big) \\ &= \lim_{n\to+\infty}\big(b(x+y+y_n) - b(y+y_n) + b(y+y_n) - b(y_n)\big) \\ &= b(x) + b(y), \end{aligned}$$

where the last equality follows from the fact that

$$b(x) = \lim_{n\to+\infty}\big(b(x+y+y_n) - b(y+y_n)\big),$$

as $|a(y+y_n)| \to +\infty$, whenever $|a(y_n)| \to +\infty$. Thus $b = 0$ as a bounded and additive function, and consequently $f = a$. ∎

REMARK 3. The method of the above proof is motivated by Schwaiger (see [7]).

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