REMARKS ON INVARIANT SUBSPACE LATTICES

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If A is a bounded linear operator on an infinite-dimensional complex Hilbert space H, let lat A denote the collection of all subspaces of H that are invariant under A; i.e., all closed linear subspaces M such that $x \in M$ implies $(Ax) \in M$. There is very little known about the question: which families F of subspaces are invariant subspace lattices in the sense that they satisfy $F = \text{lat } A$ for some A? (See [5] for a summary of most of what is known in answer to this question.) Clearly, if F is an invariant subspace lattice, then $\{0\} \in F$, $H \in F$ and F is closed under arbitrary intersections and spans. Thus, every invariant subspace lattice is a complete lattice.

Suppose that $F = \text{lat } A$ and that N and M are in F with N contained in M. Suppose also that the dimension of $M \ominus N$ is finite. Then the quotient transformation induced by A on $M \ominus N$ is an operator on a finite-dimensional space. Therefore $\{L \in F : N \subset L \subset M\}$ must have at least $[1 + \dim(M \ominus N)]$ elements. Also the sublattice $\{L \in F : N \subset L \subset M\}$ of F must be self-dual, since the lattice of invariant subspaces of an operator on a finite-dimensional space is self-dual [2].

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These restrictions are the only general restrictions on invariant subspace lattices that we know. The well-known invariant subspace problem is the question: is $\{\{0\},H\}$ an invariant subspace lattice?

In this note we make several remarks that add a little more information related to the general question mentioned above. An _atom_ in a lattice with 0 is an element a of the lattice such that the only member of the lattice strictly less than a is 0. An operator A is _polynomially compact_ if there exists a non-zero polynomial p such that p(A) = 0.

THEOREM 1. _If_ A _is a polynomially compact operator such that_ lat A _has a spanning set of atoms_, _then the dual of_ lat A _has at least one atom_.

Proof. The invariant subspace theorem for polynomially compact operators [1] implies that all the atoms in lat A are one-dimensional as subspaces of H, for if M has dimension greater than 1 then $A|M$ must have a non-trivial invariant subspace. Thus the eigenvectors of A span H.

Let p be a non-zero polynomial such that p(A) is compact. Since every eigenvector of A is an eigenvector of p(A), the eigenvectors of p(A) span H. We shall show that A* has an eigenvalue.

Let K denote the nullspace of p(A). If $K = H$ then A is algebraic. This implies that A* is algebraic too, and thus that A* has an eigenvalue. If $K \neq H$ then p(A) has a non-zero eigenvalue λ. Since p(A) is compact $\bar{\lambda}$ is an eigenvalue of $[p(A)]^*$. The spectral

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mapping theorem implies that A^* has an eigenvalue.

Thus A^* has an eigenvector, and hence lat A^* has an atom. But lat A^* is the dual of lat A, since lat $A^* = \{M : M^{\perp} \in \text{lat } A\}$.

COROLLARY 1. _If_ A _is polynomially compact, and if the dual of_ lat A _has a spanning set of atoms, then_ A _has an eigenvector_.

Proof. Corollary 1 follows immediately from Theorem 1 by interchanging A and A*; (the adjoint of a polynomially compact operator is obviously polynomially compact).

We recall that the _unilateral shift_ is the operator U defined, on a Hilbert space with o.n. basis $\{e_n\}_{n=0}^{\infty}$, by $Ue_n = e_{n+1}$. The invariant subspace lattice of U has been intensively studied [4].

COROLLARY 2. _There is no polynomially compact operator_ A _such that_ lat A _is order-isomorphic to_ lat U.

Proof. It is easily seen that lat U has no atoms and that lat U* (i.e., the dual of lat U) has a spanning set of atoms. Thus Corollary 1 applies.

Theorem 1 considers lattice-theoretic properties of lat A and thus is a partial answer to the question: which abstract lattices are order-isomorphic to an invariant subspace lattice of a polynomially compact operator? In the following we do not consider properties of abstract lattices, but merely consider a class of subspace lattices i.e., lattices given as collections of subspaces of H.

THEOREM 2. _Let_ M _be a proper subspace of_ H _of dimension at_

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least 2, *and let* $F = \{N : N \subset M$ *or* $N \supset M\}$. *If* $F \subset \text{lat } A$ *then* lat A *contains a two-dimensional subspace* K *such that* $K \notin F$.

Proof. Choose any $x \notin M$. If N is the smallest subspace of H which contains x and M, then N is in F and hence also in lat A. Therefore there is a complex number α such that $Ax = \alpha x + y$ for some $y \in M$. Since $y \in M$, y is an eigenvector of A, and thus the two-dimensional subspace spanned by $\{x,y\}$ is a suitable K.

Thus the F's that are considered in Theorem 2 are not invariant subspace lattices. This is true for $\dim M = 1$ too. If M is any proper subspace of H other then $\{0\}$ and if $F = \{N : N \subset M$ or $N \supset M\}$ then F is not an invariant subspace lattice. One way of showing this is by using the observations made preceeding Theorem 1. If $M_1 \subset M$ and $\dim(M \ominus M_1) = 1$, and if $M_2 \supset M$ and $\dim(M_2 \ominus M_1)$ is finite, then $\{N \in F : M_1 \subset N \subset M_2\}$ is not self-dual.

An interesting example of a subspace lattice that satisfies all the general restrictions mentioned above Theorem 1 but nonetheless is not an invariant subspace lattice has been found by J. E. McLaughlin [3].

REFERENCES

1. A.R. Bernstein and A. Robinson, Solution to an invariant subspace problem of K.T. Smith and P.R. Halmos. Pacific J. Math. 16 (1966) 421-431.

2. L. Brickman and P.A. Fillmore, The invariant subspace lattice of a linear transformation. Canad. J. Math. 19 (1967) 810-822.

3. P.R. Halmos, Invariant subspaces. Proc. 1968 Oberwolfoch Conference on Approximation Theory (to appear).

4. K. Hoffman, Banach spaces of analytic functions. (Prentice-Hall, Englewood Cliffs, 1962).

5. Peter Rosenthal, Examples of invariant subspace lattices. Duke Math. J. (to appear).

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