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On Annelidan, Distributive, and Bézout Rings

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In memory of Vera Puninskaya

Abstract. A ring is called *right annelidan* if the right annihilator of any subset of the ring is comparable with every other right ideal. In this paper we develop the connections between this class of rings and the classes of right Bézout rings and rings whose right ideals form a distributive lattice. We obtain results on localization of right annelidan rings at prime ideals, chain conditions that entail left-right symmetry of the annelidan condition, and construction of completely prime ideals.

1 Introduction

In [25, 26], we introduced the class of right annelidan rings, obtained structural results, proved a "Principal Hopkins–Levitzki Theorem" for right annelidan rings, and showed that right annelidan rings are Armendariz. In [25] we promised a localization theory for these rings would be developed in a forthcoming paper. That theory is one of the principal aims of this paper.

A ring is said to be *right annelidan* if the right annihilator of every subset of the ring is a *right waist*, a right ideal that is comparable with every other right ideal of the ring. We will be studying right annelidan rings in conjunction with two conditions on right ideals, one internal, one external. The internal condition is that every finitely generated right ideal be principal; rings satisfying this condition are called *right Bézout rings*. The external condition is that the lattice of right ideals be distributive; rings satisfying this condition are called *right distributive rings*. Each of these three classes of rings—right annelidan, right distributive, right Bézout—broadly generalizes the class of *right uniserial rings* (also known as *right chain rings*, *i.e.,* those rings in which the lattice of right ideals is a chain). Right uniserial rings have long played a vital role in various branches of algebra: within the commutative setting, in algebraic number theory and algebraic geometry; within the noncommutative setting, in projective geometry, combinatorics, coding theory, etc.

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Motivation for the localization theory developed here comes from the elementary fact that the localization of a Dedekind domain at a prime ideal is a discrete valuation ring. Dedekind domains are precisely the commutative noetherian domains with a distributive ideal lattice. So one might expect that localizations of rings with distributive right ideal lattices should be right uniserial. Yet even the existence of a localization can be subtle. Quoting G. Sigurđsson [39]: "When a ring R is not commutative, it is rarely possible to localize R at a prime ideal P." Indeed, it is well known that a noncommutative domain (a special case of an annelidan ring) need not embed in any division ring, much less admit a localization at its zero ideal. We will show that under certain conditions, localizations of right annelidan rings at prime ideals exist and are right uniserial, which enables us to transfer information between right annelidan rings and right uniserial rings.

The nucleus of this paper occurs in Sections 4 and 5. In Section 4, we begin by proving (Theorem 4.1) that either one of two known properties of the set of right zero-divisors in a right annelidan ring actually characterizes right annelidan rings when one restricts attention to the class of right Bézout rings. We establish a parallel result (Theorem 4.3) relative to the class of right distributive rings. Right annelidan rings that are either right Bézout or right distributive are shown to be localizable at a prime ideal consisting of zero-divisors, with the resulting localization right uniserial (Theorems 4.6 and 4.7). Thus, these rings fulfill the expectations prompted by the example of Dedekind domains. When a two-sided version of the Bézout or distributive condition is assumed, we obtain a stronger result (Theorem 4.11) stating that the class of right annelidan rings is *characterized* by this pleasant localization property.

Section 5 concludes with a symmetry result (Theorem 5.13) stating that the annelidan condition is left-right symmetric for the classes of Bézout or distributive rings that satisfy any one of a number of mild chain conditions. This represents a significant expansion upon the symmetry results proved in [25]. The most intricate case is that of rings satisfying the descending chain condition on completely prime ideals. In order to deal with this case, we establish a number of general results on completely prime ideals, including criteria for containment of the left annihilator of a completely prime ideal in its right annihilator (Theorem 5.6), which in turn is based upon a construction of new completely prime ideals from old ones by taking double left annihilators (Proposition 5.1(ii)).

Throughout this paper, all rings are associative and unital. For any adjective A describing a class of modules, a ring R will be called *right A* if the module R_R satisfies A. We will call a ring *left A* if the opposite ring is right A, and we will call a ring A if it is both right A and left A. Whenever we say that two subsets of a ring are *comparable* or *incomparable*, we will always mean with respect to inclusion.

A module is called *uniserial* if its submodules are linearly ordered by inclusion. Given a set S of elements of a ring R, we will let (S) denote the ideal generated by S when it is clear from the context that the ring in question is R. Given a ring R, the right singular ideal of R will be denoted by $\mathcal{Z}(R_R)$, the Jacobson radical of R by $\operatorname{rad}(R)$, and the group of units of R by $\mathrm{U}(R)$. Given a subset A of the ring R, the right (resp. left) annihilator of A in R will be denoted by $\operatorname{ann}_r^R(A)$ (resp. $\operatorname{ann}_\ell^R(A)$). The right (resp. left) annihilator of an element $a \in R$ will be denoted by $\operatorname{ann}_r^R(a)$

(resp. $\mathrm{ann}_\ell^R(a)$). The set of right (resp. left) zero-divisors of the ring R will be denoted by $\mathrm{RZD}(R)$ (resp. $\mathrm{LZD}(R)$), *i.e.*,

$$\mathrm{RZD}(R) = \{a \in R : \mathrm{ann}_\ell^R(a) \neq 0\},$$
$$\mathrm{LZD}(R) = \{a \in R : \mathrm{ann}_r^R(a) \neq 0\}.$$

An ideal $\mathfrak{p}$ of R is *prime* if $\mathfrak{p} \subsetneqq R$ and $a, b \in R \smallsetminus \mathfrak{p} \Rightarrow aRb \notin \mathfrak{p}$. A one-sided ideal $\mathfrak{p}$ of R is *completely prime* if $\mathfrak{p} \subsetneqq R$ and $a, b \in R \smallsetminus \mathfrak{p} \Rightarrow ab \notin \mathfrak{p}$.

All other ring-theoretic terminology and notation will be standard, pursuant to the usage in [21, 22].

Our original impetus for the research herein was the paper [33] by Vera Puninskaya. We dedicate this work to her memory.

2 Background and Basic Results

Following [1], we say that a submodule K of a module M is a *waist* if for every submodule $N \subseteq M$ we have $K \subset N$ or $N \subseteq K$. A set of elements of a ring R will be called a *right waist* of R if that set is a right ideal that is a waist of the module R_R. *Left waists* are defined by the opposite property. We now recall the central concepts of [25, 26].

Definition A ring R is called *lineal* if its right annihilator lattice is linearly ordered, *i.e.*, for any subsets $A, B \subseteq R$ we have $\mathrm{ann}_r^R(A) \subseteq \mathrm{ann}_r^R(B)$ or $\mathrm{ann}_r^R(B) \subseteq \mathrm{ann}_r^R(A)$. A ring R is called *right annelidan* if for every subset $A \subseteq R$ the right annihilator $\mathrm{ann}_r^R(A)$ is a right waist.

We have the following obvious implications between conditions on a ring:

$$\text{right uniserial} \implies \text{right annelidan} \implies \text{lineal.}$$

Both implications are irreversible. The lineal condition is left-right symmetric [26, Theorem 2.1]; the annelidan condition is not [25, §7].

We will make frequent use of the following structural result on right annelidan rings, proved in [25].

Theorem 2.1 ([25, Theorem 3.1]) *Let R be a right annelidan ring.*
- (i) *The set $\mathrm{LZD}(R)$ is a completely prime ideal of R, and $\mathrm{LZD}(R) = \mathcal{Z}(R_R)$.*
- (ii) *The set $\mathrm{RZD}(R)$ is a completely prime ideal of R, $\mathrm{RZD}(R) \subseteq \mathrm{rad}(R)$, and $\mathrm{RZD}(R)$ is a right waist of R.*

Theorem 2.1 has the following corollary, which bears comparison with the localization results Theorems 4.6 and 4.7.

Corollary 2.2 *If a right annelidan ring R is a right order in a semisimple ring Q, then R is a right Ore domain and Q is its division ring of quotients.*

Several methods for constructing right annelidan rings are provided in [25, Sections 2 and 4]. Here we offer another. As noted in [25, Example 7.3], subrings and factor rings of right annelidan rings, in general, need not be right annelidan. We can ensure better behavior as follows.

Proposition 2.3 *Let R be a right annelidan ring.*

(i) *If T is a subring of R such that* $\mathrm{RZD}(R) \subseteq T$*, then T is right annelidan.*
(ii) *If I is an ideal of R that is a right annihilator* (i.e., $I = \mathrm{ann}_r^R(S)$ *for some* $S \subseteq R$*), then the factor ring R/I is right annelidan.*

Proof (i) It suffices to show that for any $a \in T$, the right annihilator $\mathrm{ann}_r^T(a)$ is a right waist of T, *i.e.*, for any $b \in T$, if $ab \neq 0$ then $\mathrm{ann}_r^T(a) \subseteq bT$. Take any $t \in \mathrm{ann}_r^T(a)$. Since R is right annelidan and $ab \neq 0$, it follows that $\mathrm{ann}_r^R(a) \subseteq bR$ and thus $t = br$ for some $r \in R$. Now $abr = 0$ and $ab \neq 0$ imply $r \in \mathrm{RZD}(R) \subseteq T$, and thus $t = br \in bT$. Hence, $\mathrm{ann}_r^T(a) \subseteq bT$.

(ii) Let $I = \mathrm{ann}_r^R(S)$. We will use bars to denote images under the canonical map $R \to R/I$. As in the proof of (i), it suffices to show that for any $\overline{a}, \overline{b}, \overline{t} \in \overline{R}$, if

$$\overline{a}\overline{b} \neq \overline{0} \quad \text{and} \quad \overline{a}\overline{t} = \overline{0}, \tag{2.1}$$

then $\overline{t} \in \overline{b}\,\overline{R}$. From (2.1) it follows that $ab \notin \mathrm{ann}_r^R(S)$ and $at \in \mathrm{ann}_r^R(S)$; thus, $b \notin \mathrm{ann}_r^R(Sa)$ and $t \in \mathrm{ann}_r^R(Sa)$. Since R is right annelidan, $\mathrm{ann}_r^R(Sa)$ is a right waist of R; therefore, $t \in bR$. Hence $\overline{t} \in \overline{b}\,\overline{R}$, as desired. ∎

For any ring R (not necessarily right annelidan), we present below a condition equivalent to the containment $\mathrm{RZD}(R) \subseteq T$, which occurred in Proposition 2.3(i).

Remark 2.4 Let R be any ring, and let T be a subring of R. Then

$$\mathrm{RZD}(R) \subseteq T \iff \mathrm{RZD}(R) = \mathrm{RZD}(T). \tag{2.2}$$

To prove (2.2), suppose $\mathrm{RZD}(R) \subseteq T$. Trivially, $\mathrm{RZD}(R) \supseteq \mathrm{RZD}(T)$. If $0 \neq r \in \mathrm{RZD}(R) \subseteq T$, then $ar = 0$ for some nonzero $a \in R$. Since $ara = 0$, we have $ra \in \mathrm{RZD}(R) \subseteq T$. If $ra \neq 0$, then $rar = 0$ implies $r \in \mathrm{RZD}(T)$. If $ra = 0$, then $a \in \mathrm{RZD}(R) \subseteq T$, whence $r \in \mathrm{RZD}(T)$. Therefore, $\mathrm{RZD}(R) \subseteq \mathrm{RZD}(T)$.

For a ring R, let $\mathrm{u.dim}(R_R)$ (resp. $\mathrm{u.dim}({}_RR)$) denote the right (resp. left) uniform dimension of R. Uniform dimension will play a significant role in our symmetry result Theorem 5.13 for Bézout and distributive rings (*cf.* Corollary 5.7 and Theorem 5.9). A right annelidan ring can have arbitrary right uniform dimension. Nevertheless, we have a simple criterion for the right uniform dimension to equal 1, given in the second part of the following result.

Proposition 2.5 *Let R be a right annelidan ring.*

(i) *If $a \in R \setminus \{0\}$ and the right ideal aR is not essential in R, then* $\mathrm{LZD}(R) = \mathrm{ann}_\ell^R(a)$.
(ii) *If* $\mathrm{ann}_r^R(\mathrm{LZD}(R)) = (0)$*, then* $\mathrm{u.dim}(R_R) = 1$.

Proof (i) Assume $a \in R \setminus \{0\}$ and the right ideal aR is not essential. Obviously $\mathrm{ann}_\ell^R(a) \subseteq \mathrm{LZD}(R)$. To prove the opposite inclusion, consider any element $c \in \mathrm{LZD}(R)$. By Theorem 2.1(i), $\mathrm{ann}_r^R(c)$ is an essential right ideal of R. Since aR is not an essential right ideal, $\mathrm{ann}_r^R(c) \nsubseteq aR$. As R is right annelidan, it follows that $aR \subseteq \mathrm{ann}_r^R(c)$, *i.e.*, $c \in \mathrm{ann}_\ell^R(a)$, as desired.

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(ii) If $\text{u.dim}(R_R) \neq 1$, then (i) implies that $\text{LZD}(R) = \text{ann}_\ell^R(a)$ for some $a \in R \setminus \{0\}$. Hence $0 \neq a \in \text{ann}_r^R(\text{LZD}(R))$. ∎

A well-known result due to A. W. Goldie states that the only possible values of the (right) uniform dimension of a domain are 1 and ∞. This theorem generalizes to right annelidan semiprime rings.

Corollary 2.6 *If R is a right annelidan semiprime ring, then*

$$\text{u.dim}(R_R) = 1 \quad \textit{or} \quad \text{u.dim}(R_R) = \infty.$$

Proof Recall from Theorem 2.1(i) that $\text{LZD}(R)$ is an ideal of R. By [26, Proposition 3.11(i)], a semiprime ring that is right annelidan (or, more generally, lineal) must be prime. If the ideal $\text{LZD}(R)$ of the prime ring R is nontrivial, then its right annihilator is trivial, and we are done by Proposition 2.5(ii). On the other hand, if the ideal $\text{LZD}(R)$ is trivial, then R is a domain, and we are done by Goldie's Theorem. ∎

Corollary 2.6 motivates a problem we have been unable to resolve.

Question What are the possible values of the right uniform dimension of a lineal semiprime ring?

As we remarked parenthetically in the proof of Corollary 2.6, semiprime lineal rings are actually prime. For right annelidan rings, more can be said.

Theorem 2.7 *If a right annelidan ring R is semiprime, then either R is a domain or else R is prime and $\text{RZD}(R)$ is the unique minimal ideal of R. In particular, a right annelidan semiprime ring that is not a domain must be subdirectly irreducible.*

Proof Assume R is right annelidan and semiprime, hence prime, but not a domain. By Theorem 2.1(ii), $\text{RZD}(R)$ is a (nontrivial) ideal of R. Let I be any nontrivial ideal of R. Since R is prime, $\text{ann}_\ell^R(I) = (0)$. Thus, for any nonzero element $a \in R$ we have $I \nsubseteq \text{ann}_r^R(a)$, hence $\text{ann}_r^R(a) \subseteq I$. This proves $\text{RZD}(R) \subseteq I$. The final assertion is clear. ∎

In the first sentence of Theorem 2.7, both cases can occur, even when R is right uniserial. It was, in fact, an open problem for many years whether a right uniserial semiprime ring need be a domain (see, *e.g.,* [14, p. 72], *cf.* also [37]); this problem was finally resolved, in the negative, in [9] (see also [7]).

Idempotent ideals play an important role in right uniserial rings. For example, it is well known that a nonzero idempotent ideal in such a ring is completely prime. (For a stronger result, see Lemma 3.1.) Using the structure theory developed in [25, §5], it is easy to show that if R is a right annelidan, left perfect ring, then its only idempotent ideals are (0) and R. We will now obtain the same conclusion for an independent class of right annelidan rings.

A *right Bézout ring* is a ring in which every finitely generated right ideal is principal. All von Neumann regular rings, for example, are right (and left) Bézout. Among domains, the right Bézout rings can be characterized as the right Ore domains that are semifirs: see [8, §2.3] for this and other interesting results.

Lemma 2.8 *Let R be a right Bézout ring; let A be a right ideal and B an ideal of R. Then*

$$AB = \{ab\colon a \in A,\ b \in B\}.$$

Proof If $r \in AB$, then $r = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ for some $n \in \mathbb{N}$, $a_i \in A$, and $b_i \in B$. Since R is right Bézout, $a_1R + a_2R + \cdots + a_nR = aR$ for some $a \in A$. Write $a_i = ar_i$ for some $r_i \in R$. Letting $b = r_1b_1 + r_2b_2 + \cdots + r_nb_n$ completes the proof. ∎

An obvious consequence of Lemma 2.8 is that a right Bézout ring cannot contain a nontrivial left or right T-nilpotent idempotent ideal; thus, a right uniserial, left perfect ring cannot contain a nontrivial proper idempotent ideal. One can strengthen this observation quite a bit, as we will see in Theorem 2.11.

In general, *ACC on principal right ideals* is a much weaker condition than *right noetherian,* which is, in turn, a much weaker condition than that all right ideals be principal. For right Bézout rings, however, these conditions become equivalent.

Proposition 2.9 *Let R be a ring. The following conditions are equivalent.*

(i) *R is a right Bézout ring with the ascending chain condition on principal right ideals.*
(ii) *R is a right Bézout and right noetherian ring.*
(iii) *R is a principal right ideal ring.*

Proof It is easy to see that a module is noetherian if and only if it has the ascending chain condition on finitely generated submodules. Therefore, a Bézout module is noetherian if and only if it has the ascending chain condition on cyclic submodules. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is clear. ∎

In a ring that satisfies the equivalent conditions of Proposition 2.9, one can rule out the possibility of an idempotent ideal if the Jacobson radical contains all left or right zero-divisors. Indeed, we have the following result.

Proposition 2.10 *Let R be a ring such that $\mathrm{LZD}(R) \subseteq \mathrm{rad}(R)$ or $\mathrm{RZD}(R) \subseteq \mathrm{rad}(R)$, and let A be a nonzero principal right ideal of R. Then for any proper ideal B of R, we have $AB \subsetneq A + B$.*

Proof Suppose $AB = A + B$. Then $A = B$, and $A = A^2$ is a nonzero proper ideal of R. By hypothesis, $A = aR$ for some nonzero $a \in R$, hence $aR = aA$. Write $a = ax$ for some $x \in A$. Since $a(1-x) = 0$, it follows that $a \in \mathrm{LZD}(R)$ and $1-x \in \mathrm{RZD}(R)$. By hypothesis, $a \in \mathrm{rad}(R)$ or $1 - x \in \mathrm{rad}(R)$. The former contradicts Nakayama's Lemma; the latter gives the contradiction $x \in \mathrm{U}(R) \cap A$. ∎

We now turn to further conditions that preclude the existence of idempotent ideals. Following Lemma 2.8, we observed that a right uniserial, left perfect ring—that is, a right uniserial ring with DCC on principal right ideals—cannot contain a nontrivial proper idempotent ideal. Of course, right uniserial rings are both right annelidan and right Bézout; moreover, according to the "Principal Hopkins-Levitzki Theorem" for right annelidan rings [25, Theorem 5.4], a right annelidan ring with DCC on principal right ideals must have ACC on principal right ideals. Thus, the following theorem generalizes our observation on right uniserial, left perfect rings.

Theorem 2.11 *A right annelidan, right Bézout ring with the ascending chain condition on principal right ideals cannot contain any nontrivial proper idempotent ideal.*

Proof Apply Theorem 2.1(ii) and Propositions 2.9 and 2.10. ∎

In Section 4, where *right distributive rings* are defined, a common theme will be that many results on right Bézout rings can also be proved for right distributive rings. For this reason, it is worthy of note that that is *not* the case for Theorem 2.11. To avoid excessive digression we will defer the definition of a right distributive ring until Section 4 and simply refer the reader to the implication table on page 10, which indicates that a commutative domain is right distributive if and only if it is a Prüfer domain. A. P. Grams gives a beautiful construction in [17, Example 2, p. 328] of a commutative one-dimensional Prüfer domain with ACC on principal ideals that contains precisely one idempotent maximal ideal. In particular, in contrast with Theorem 2.11, a right annelidan, right distributive ring with the ascending chain condition on principal right ideals can contain a nontrivial proper idempotent ideal. Grams's Prüfer domain is certainly not noetherian (confuting the "distributive" analogue of Proposition 2.9), since Dedekind domains cannot contain proper nontrivial idempotent ideals. Note that in Grams's example, every finitely generated ideal requires at most two generators [36, Corollary 4.3]; thus, in some sense, that ring is not too far from being Bézout.

3 An Application to Uniserial Rings

The main object of study in this brief section is the ideal

$$I^{\omega} = \bigcap_{n=1}^{\infty} I^n,$$

where I is an ideal of a ring R. This construction has a long history in ring theory; for example, according to the classical Krull Intersection Theorem, $I^{\omega} = (0)$ whenever I is a proper ideal of a commutative noetherian domain. This construction is important for a different reason in the ideal theory of right uniserial rings: as C. Bessenrodt, H. H. Brungs, and G. Törner proved in [4, Theorem 2.3(ii)], if I is a non-nilpotent ideal of a right uniserial ring, then I^{ω} is a completely prime ideal.

We begin with a lemma extending [4, Theorem 2.3(i)].

Lemma 3.1 *If $\mathfrak{p}$ is a nontrivial, proper, idempotent ideal, and a right waist of a ring R, then $\mathfrak{p}$ is a completely prime ideal of R.*

Proof Suppose the proper idempotent ideal $\mathfrak{p}$ is a waist in R_R but is not a completely prime ideal. Since $\mathfrak{p} \subseteq R_R$ is a waist, it is contained in every maximal right ideal and hence $\mathfrak{p} \subseteq \mathrm{rad}(R)$. For any $x \in R \setminus \mathfrak{p}$, we have $\mathfrak{p} \subset xR$, hence $\mathfrak{p} = \mathfrak{p}^2 \subseteq xR\mathfrak{p} \subseteq x\mathfrak{p} \subseteq \mathfrak{p}$, so $\mathfrak{p} = x\mathfrak{p}$.

Take $a, b \in R \setminus \mathfrak{p}$ such that $ab \in \mathfrak{p}$. Then $\mathfrak{p} = a\mathfrak{p} = a(b\mathfrak{p})$ implies $ab \in ab\mathfrak{p}$, so by Nakayama's Lemma, $ab = 0$. Therefore, $\mathfrak{p} = ab\mathfrak{p} = (0)$. ∎

A ring whose prime radical contains every nilpotent element of the ring is called *2-primal*. Basic examples of 2-primal rings include *reduced rings* (*i.e.*, rings without nonzero nilpotent elements) and commutative rings. More surprisingly, a right annelidan ring that is left or right noetherian must be 2-primal: this is a consequence of [25, Theorem 3.5].

Theorem 3.2 *Let I be a nontrivial, proper ideal of a ring R. Suppose that the following two "waist" properties hold:*

(W_1) *For every $n \in \mathbb{N}$ the ideal I^n is a waist of R in the category of (R, R)-bimodules (*i.e.*, I^n is comparable with each ideal of R).*

(W_2) *The ideal I^ω is a right waist of R.*

Then the following conditions are equivalent.

(i) *The ideal I^ω is a completely prime ideal of R.*
(ii) *The factor ring R/I^ω is reduced.*
(iii) *The factor ring R/I^ω is 2-primal and I is not a nilpotent ideal.*
(iv) *The factor ring R/I^ω is right annelidan and I is not a nilpotent ideal.*

Proof (i) ⇒ (ii) ⇒ (iii): This is trivial.

(iii) ⇒ (iv): First we show that I^ω is a prime ideal. If not, then there exist ideals J and K of R that properly contain I^ω such that $JK \subseteq I^\omega$. By property (W_1), there must exist positive integers j and k such that $I^j \subset J$ and $I^k \subset K$. Then $I^{j+k} \subseteq JK \subseteq I^\omega$, which implies that I^{j+k} is an idempotent ideal equal to I^ω. Applying property (W_2), we infer from Lemma 3.1 that the ideal I^ω is prime, a contradiction.

A fundamental characterization of 2-primal rings is that every minimal prime ideal is completely prime [38, Proposition 1.11]. Thus, assuming (iii), the factor ring R/I^ω is prime and 2-primal, so it must be a domain and hence right annelidan.

(iv) ⇒ (i): Assume that I is not nilpotent. We consider two cases.

Case 1: For some $n \in \mathbb{N}$ we have $I^n = I^{n+1}$. Then $I^\omega = I^n$ is a nontrivial, proper, idempotent ideal of R, and Lemma 3.1 implies that I^ω is completely prime.

Case 2: For every $n \in \mathbb{N}$ we have $I^n \neq I^{n+1}$. From (W_1) we infer that R/I^ω is a prime ring that is not subdirectly irreducible; now (iv) and Theorem 2.7 imply that I^ω is completely prime. ∎

A special case of Theorem 3.2 is the aforecited result of Bessenrodt–Brungs–Törner, used by V. Puninskaya in [33] to study the model theory of a module over a countable uniserial ring (this result was cited by Puninskaya in [33] as [34, Lemma 15.2] by G. E. Puninskiĭ and A. A. Tuganbaev).

Corollary 3.3 ([4, 34]) *Assume R is a right uniserial ring, and $I \subset R$ is any proper ideal. If I is not a nilpotent ideal, then I^ω is a completely prime ideal. In this case, there does not exist any prime ideal properly contained between I^ω and I.*

Proof The final assertion is clear; indeed, if I is an ideal and Q is a prime ideal of any ring whose ideals are linearly ordered by inclusion, and $I^\omega \subsetneqq Q \subseteq I$, then $Q = I$. The penultimate assertion follows from Theorem 3.2. ∎

Of course, given an ideal I even in a commutative noetherian ring, it is certainly possible to have $I^\omega \subsetneqq Q \subsetneqq I$ for infinitely many different prime ideals Q.

We will obtain another construction of completely prime ideals, under different hypotheses from those of Theorem 3.2, in Proposition 4.5.

4 Annelidan Right Bézout and Right Distributive Rings

As we will see in Theorems 4.1 and 4.3, the right annelidan condition admits an elegant characterization in right Bézout rings and in right distributive rings. Rather amazingly, it is the *same* elegant characterization for both classes of rings.

Right Bézout rings were defined and discussed in Section 2. Right distributive rings are characterized by "joins" and "meets" obeying the distributive laws in the lattice of right ideals. A lattice L is called *distributive* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all $a, b, c \in L$ (equivalently, if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all $a, b, c \in L$). A module M is *distributive* if its submodule lattice is distributive, *i.e.*, $(A + B) \cap C = (A \cap C) + (B \cap C)$ for all submodules A, B, and C of M. A ring R is *right* (resp. *left*) *distributive* if the module R_R (resp. ${}_RR$) is distributive. For a variety of important properties of distributive rings and modules, see [41].

A ring is called *right duo* if every right ideal is an ideal. The duo condition has important connections with distributivity of submodule lattices. For example, any right noetherian, right distributive ring must be right duo (see [41, p. 305, Corollary 3]).

Uniserial modules are obviously Bézout and distributive. There is a partial converse: a module over a local ring is uniserial if and only if it is Bézout or distributive. Among von Neumann regular rings, the right (and left) distributive rings are precisely the *abelian regular* rings (see [15, Chapter 3]).

In general, a right Bézout ring need not be right distributive, and a right distributive ring need not be right Bézout, even if the ring is annelidan. (This observation is especially germane in connection with Theorems 4.1 and 4.3 to follow.) For example, if R is a commutative Dedekind domain that is not a principal ideal domain, then R is distributive but not Bézout. On the other hand, by [16, Theorem 2.8], [24, Theorems 2 and 4], and [41, p. 307, Corollary 3], if R is an Ore polynomial ring $R = D[x; \sigma, \delta]$ where D is a division ring, σ is a ring automorphism of D, and δ is a σ-derivation of D, then R is a Bézout domain (indeed, every one-sided ideal of R is principal); however, R is not left or right distributive unless it is commutative (*i.e.*, D is a field, σ is the identity automorphism, and δ is the zero map). We see, then, that a noetherian domain can be Bézout without being distributive—a phenomenon exclusive to *noncommutative* ring theory (*cf.* the implication table below).

The Bézout and distributive conditions do, however, have the following relationship. If R is a *right quasi-duo* ring, *i.e.*, a ring in which every maximal right ideal is two-sided, and a right R-module M is Bézout, then M is distributive (see Theorem 4.4). In particular, any commutative Bézout ring is distributive. Indeed, *if we restrict our attention to commutative domains*, we have the following table of implications:

$$
\begin{array}{ccccc}
\text{PID} & \Rightarrow & \text{Dedekind domain} & & \\
\Downarrow & & \Downarrow & & \\
\text{Bézout} & \Rightarrow & \text{distributive} & & \\
 & & \Updownarrow & & \\
 & & \text{Prüfer domain} & \Rightarrow & \text{coherent, catenary,} \\
 & & & & \text{integrally closed, etc.}
\end{array}
$$

In addition to the characterization of Prüfer domains given in this table, numerous other characterizations can be found in [6, VI.2.Exercise 12].

We have remarked that for commutative rings, *Bézout* implies *distributive*. It is also worth pointing out that for semilocal rings, *right distributive* implies *right Bézout*. Indeed, if R is a semilocal right distributive ring with $J = \operatorname{rad}(R)$, and $\mathfrak{A} \subseteq R$ is a finitely generated right ideal, then $\mathfrak{A}/\mathfrak{A}J$ is a finitely generated, semisimple, distributive right (R/J)-module, hence cyclic by [41, p. 298, Corollary 2]. It follows from Nakayama's Lemma that $\mathfrak{A} \subseteq R$ is a principal right ideal.

Semiperfect and perfect right distributive rings have been completely described by W. Stephenson; see [41, Theorem 2.4].

By Theorem 2.1(ii), if R is a right annelidan ring then $\operatorname{RZD}(R) \subseteq \operatorname{rad}(R)$. Assuming every one-sided ideal of a ring R is principal, the structure theorem proved by I. Kaplansky in [20, Theorem 12.1] shows that, conversely, R is annelidan if $\operatorname{RZD}(R) \subseteq \operatorname{rad}(R)$. But a much stronger result is possible. For right Bézout rings, we can obtain the following converse to Theorem 2.1(ii).

Theorem 4.1 *For a right Bézout ring R, the following conditions are equivalent:*

(i) *R is right annelidan.*
(ii) $\operatorname{RZD}(R)$ *is a right waist of R.*
(iii) $\operatorname{RZD}(R) \subseteq \operatorname{rad}(R)$.

Proof (i) ⇒ (ii): This holds by Theorem 2.1(ii).

(ii) ⇒ (iii): Any proper right waist is contained in the Jacobson radical.

(iii) ⇒ (i): Let $a, b \in R$ be such that $aR \nsubseteq bR$. Since R is right Bézout, $aR + bR = cR$ for some $c \in R$. Write $a = c\alpha$, $b = c\beta$, and $c = a\gamma_1 + b\gamma_2$ for some $\alpha, \beta, \gamma_1, \gamma_2 \in R$. Then $a(1 - \gamma_1\alpha) = b\gamma_2\alpha$, so

$$aR \nsubseteq bR \quad \Rightarrow \quad 1 - \gamma_1\alpha \notin \mathrm{U}(R) \quad \Rightarrow \quad \alpha \notin \operatorname{rad}(R).$$

By (iii), $\alpha \notin \operatorname{RZD}(R)$. Therefore $\operatorname{ann}_\ell^R(a) = \operatorname{ann}_\ell^R(c) \subseteq \operatorname{ann}_\ell^R(b)$. By [25, Proposition 2.1(v)], R is right annelidan. ■

Theorem 4.1 bears comparison with [26, Proposition 3.3(i)], which states that a right Bézout ring is lineal if and only if $\operatorname{RZD}(R)$ is a right ideal of R. Combining Theorems 2.1(ii) and 4.1, we recover a result due to C. Lomp and A. Sant'Ana [23, Proposition 5.4]: if R is a right Bézout ring with $\operatorname{RZD}(R) \subseteq \operatorname{rad}(R)$, then $\operatorname{RZD}(R)$ is a completely prime ideal and a right waist of R.

In order to obtain an analogue of Theorem 4.1 for right distributive rings, let us recall a theorem due to Stephenson. Given a ring R, and given elements m and n in a right R-module M, we write

$$n^{-1}(mR) = \{r \in R\colon\ nr \in mR\} \subseteq R.$$

The following lemma is the right-hand version of part of [41, Theorem 1.6].

Lemma 4.2 (Stephenson) *Let R be a ring and M a right R-module. The following conditions are equivalent.*

(i) *M is a distributive module.*
(ii) *For all $a, b \in M$, we have $b^{-1}(aR) + a^{-1}(bR) = R$.*

We can now prove a parallel result to Theorem 4.1.

Theorem 4.3 *For a right distributive ring R the following conditions are equivalent.*

(i) *R is right annelidan.*
(ii) $\mathrm{RZD}(R)$ *is a right waist of R.*
(iii) $\mathrm{RZD}(R) \subseteq \mathrm{rad}(R)$.

Proof (i) ⇒ (ii) ⇒ (iii): This is clear.

(iii) ⇒ (i): Suppose $a, b \in R$ satisfy $\mathrm{ann}_\ell^R(a) \nsubseteq \mathrm{ann}_\ell^R(b)$. For some $r \in R$, we have $ra = 0 \neq rb$. By Lemma 4.2, we can write $1 = x + y$ where $x \in a^{-1}(bR)$ and $y \in b^{-1}(aR)$. Since $rby \in raR = \{0\}$, we have $y \in \mathrm{RZD}(R) \subseteq \mathrm{rad}(R)$, and thus $x = 1 - y \in \mathrm{U}(R)$, which implies $a^{-1}(bR) = R$, *i.e.*, $aR \subseteq bR$. By [25, Proposition 2.1(v)], R is right annelidan. ∎

Theorem 4.3 bears comparison with [26, Proposition 3.2(i)], which states, in part, that a right distributive ring is lineal if and only if $\mathrm{RZD}(R)$ is a right ideal of R.

The kindred Theorems 4.1 and 4.3 are remarkable inasmuch as *right Bézout* and *right distributive* are independent conditions even for domains. As mentioned above, the situation is different for commutative rings. The connections between the Bézout and distributive properties under appropriate commutativity conditions were first recognized by Stephenson, who proved in 1974 that if R is a right duo ring and M is a Bézout right R-module, then M is distributive. Six years later, A. A. Tuganbaev generalized this result to the case where R is right quasi-duo. We record a quick proof, by methods different from Stephenson's and Tuganbaev's.

Theorem 4.4 (Tuganbaev) *Let R be a right quasi-duo ring, and let M be a right R-module. If M_R is Bézout, then M_R is distributive.*

Proof Assume M_R is Bézout but not distributive. By Lemma 4.2, there exist $a, b \in M$ such that $b^{-1}(aR) + a^{-1}(bR) \subseteq \mathfrak{m}$ for some maximal right ideal $\mathfrak{m} \subseteq R$. Choose $c \in M$ such that $aR + bR = cR$. For some $\alpha, \beta, \gamma_1, \gamma_2 \in R$, we have

$$a = c\alpha, \quad b = c\beta, \quad c = a\gamma_1 + b\gamma_2.$$

Then $a(1 - \gamma_1\alpha) = b\gamma_2\alpha$ and $b(1 - \gamma_2\beta) = a\gamma_1\beta$, so $b^{-1}(aR) + a^{-1}(bR) \subseteq \mathfrak{m}$ implies

$$1 - \gamma_1\alpha, \gamma_2\alpha, 1 - \gamma_2\beta, \gamma_1\beta \in \mathfrak{m}.$$

This is impossible, since $R/\mathfrak{m}$ is a division ring. ∎

We now obtain new conditions for the construction of completely prime ideals, expanding the scope of Theorem 3.2. Adapting the terminology of [31], given a subset I of a ring R, we say that I is a *right comparizer* of R if $I \subseteq R$ is a right ideal such that for any right ideals A and B of R we have $A \subseteq B$ or $BI \subseteq A$. The terminology reflects the fact that, in some sense, a right comparizer renders incomparable right ideals comparable. Within the right ideal lattice of any ring, the set of right comparizers is a complete lattice ideal. A ring R is right uniserial if and only if R is a right comparizer of itself. We retain the notation $I^{\omega} = \bigcap_{n=1}^{\infty} I^n$ used in Section 3.

Proposition 4.5 *Let I be a nontrivial, proper ideal of a ring R. Suppose that the following two conditions hold.*

(i) *The ideal I^{ω} is a right waist of R.*
(ii) *The factor ring R/I^{ω} is right annelidan and right Bézout, or right annelidan and right distributive.*

Then I^{ω} is a completely prime ideal if and only if I is not a nilpotent ideal.

Proof Assume, for a contradiction, that the ideal $I \subset R$ is not nilpotent, and that the ideal $I^{\omega} \subset R$ is not completely prime. Then $\overline{R} = R/I^{\omega}$ is right annelidan but not a domain. Since $\overline{R}$ is a right Bézout or right distributive ring, and, by Theorem 2.1(ii), $\mathrm{RZD}(\overline{R})$ is a proper ideal and a right waist of $\overline{R}$, it follows from [31, Proposition 1.5] that $\mathrm{RZD}(\overline{R})$ is a right comparizer of $\overline{R}$. We have $\overline{I}^{\omega} = (\overline{0}) \subsetneqq \mathrm{RZD}(\overline{R})$, and $\mathrm{RZD}(\overline{R})$ is a prime ideal of $\overline{R}$; therefore, we cannot have $\mathrm{RZD}(\overline{R}) \subsetneqq \overline{I}$; hence, $\overline{I} \subseteq \mathrm{RZD}(\overline{R})$.

Consequently, $\overline{I}$ is a right comparizer of $\overline{R}$. By hypothesis, $\overline{I}^{\omega}$ is not a completely prime ideal of $\overline{R}$. By [31, Theorem 2.3], $\overline{I}$ is nilpotent. Therefore, for some $m \in \mathbb{N}$, we have $I^m = I^{\omega}$, whence I^{ω} is an idempotent ideal of R. By Lemma 3.1, $I^{\omega} \subset R$ is a completely prime ideal, a contradiction. ∎

The Bézout and distributive conditions enable us to study right annelidan rings via localization techniques, as the following two theorems show. The first of these, Theorem 4.6, is a broad generalization of the result [21, Corollary (10.24)] on right Ore domains.

For a ring R and a right (resp. left) denominator set S, the right (resp. left) ring of fractions of R with respect to S is denoted by RS^{-1} (resp. $S^{-1}R$).

Theorem 4.6 *Let R be a right Bézout, one-sided annelidan ring. Put*

$$\mathfrak{p} = \mathrm{LZD}(R) \cup \mathrm{RZD}(R).$$

Then $S = R \setminus \mathfrak{p}$ is a right denominator set, the natural map $R \to RS^{-1}$ is injective, and RS^{-1} is a right uniserial ring.

If, in addition, R is left Bézout, then S is also a left denominator set, and the quotient ring RS^{-1} is uniserial.

Proof By Theorem 2.1, $\mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$ or $\mathrm{RZD}(R) \subseteq \mathrm{LZD}(R)$, and so $\mathfrak{p}$ is a completely prime ideal. In particular, S is a multiplicative set, and it is right reversible, because $S \cap \mathrm{LZD}(R) = \varnothing$.

To show that S is right permutable, let $a \in R$ and $s \in S$ be given. Since R is right Bézout, $aR + sR = bR$ for some $b \in R$; say,

$$a = ba_0, \quad s = bs_0, \quad \text{and} \quad ax + sy = b$$

for some $a_0, s_0, x, y \in R$.

If $x \in \mathfrak{p}$, then $1 - xa_0 \in S$, hence $a(1 - xa_0) = sya_0 \in aS \cap sR$. If $x \notin \mathfrak{p}$, then $xs_0 \notin \mathfrak{p}$ (since $\mathfrak{p}$ is completely prime), hence $axs_0 = s(1 - ys_0) \in aS \cap sR$. In either case, $aS \cap sR \neq \varnothing$. Thus, S is a right denominator set.

The kernel of the natural map $R \to RS^{-1}$ is the set

$$\{r \in R : \ rs = 0 \ \text{for some} \ s \in S\}.$$

Since $S \cap \mathrm{RZD}(R) = \varnothing$, this kernel is trivial.

To show that RS^{-1} is right uniserial, pick any two elements of RS^{-1}, which can be written as r_1s^{-1} and r_2s^{-1} for some $r_1, r_2 \in R$ and $s \in S$. Let $r_1R + r_2R = rR$ for some $r \in R$. There exist $x_1, x_2, y_1, y_2 \in R$ such that

$$r_1 = rx_1, \quad r_2 = rx_2, \quad \text{and} \quad r = r_1y_1 + r_2y_2.$$

If $x_1 \in \mathfrak{p}$, then $r_1 = r_2y_2x_1(1 - y_1x_1)^{-1}$, hence $r_1s^{-1} \in (r_2s^{-1})RS^{-1}$. Likewise, if $x_2 \in \mathfrak{p}$ then $r_2s^{-1} \in (r_1s^{-1})RS^{-1}$. In the remaining case, $x_1, x_2 \in S$. Then $(r_1s^{-1})RS^{-1} = (rs^{-1})RS^{-1} = (r_2s^{-1})RS^{-1}$. Therefore, RS^{-1} is right uniserial.

The final statement follows by symmetry. ■

The following theorem extends [43, Theorem 2.2(1)].

Theorem 4.7 *Let R be a right distributive, one-sided annelidan ring. Put*

$$\mathfrak{p} = \mathrm{LZD}(R) \cup \mathrm{RZD}(R).$$

Then $S = R \smallsetminus \mathfrak{p}$ is a right denominator set, the natural map $R \to RS^{-1}$ is injective, and RS^{-1} is a right uniserial ring.

If, in addition, R is left distributive, then S is also a left denominator set, and the quotient ring RS^{-1} is uniserial.

Proof The proof largely follows that of Theorem 4.6; we will briefly outline the needed modifications, which are based on Lemma 4.2.

To show that S is right permutable, given $a \in R$ and $s \in S$, we write $1 = \alpha + \beta$ for some $\alpha \in a^{-1}(sR)$ and $\beta \in s^{-1}(aR)$. Choose $r \in R$ such that $s\beta = ar$. If $r \in S$, then $s\beta \in aS \cap sR$. If $r \notin S$ then $a\alpha \in aS \cap sR$.

To show that RS^{-1} is right uniserial, given $r_1s^{-1}, r_2s^{-1} \in RS^{-1}$ we write $1 = x + y$ for some $x \in r_1^{-1}(r_2R)$ and $y \in r_2^{-1}(r_1R)$. Interchanging r_1 and r_2 if necessary, we can assume $x \notin \mathfrak{p}$. Then $r_1x \in r_2R$ implies $r_1s^{-1} \in (r_2s^{-1})RS^{-1}$. ■

Recall that Corollary 2.2 says that if a right annelidan ring R is a right order in a semisimple ring Q, then R is a right Ore domain, and Q is its division ring of quotients. By contrast, Theorems 4.6 and 4.7 show that if a ring is right Bézout and right annelidan, or right distributive, and right annelidan, then it is a right order in a right uniserial ring. The converse is false.

Example 4.8 Let D be a commutative domain with field of fractions $F \supsetneqq D$, and put

$$Q = F[x]/(x^3), \quad R = D \oplus Dx \oplus Fx^2 \subset Q.$$

Then Q is a commutative uniserial ring, and R is an order in Q. For nonzero $d \in D \setminus \mathrm{U}(D)$, the ideals $dR \subset R$ and $\mathrm{ann}_r^R(x^2) \subset R$ are incomparable, and $x^{-1}(dR) + d^{-1}(xR) \neq R$. Thus, R is not Bézout, distributive, or annelidan.

Nevertheless, a right order in a right uniserial ring will be right annelidan under some natural hypotheses. Theorem 4.11, which follows, significantly extends the scope of Theorems 4.1 and 4.3 while incorporating our localization results. We will make use of the following proposition, which occurred (with a left-right switch) as [31, Lemma 1.6].

Proposition 4.9 *Let R be a left Bézout ring or a left distributive ring. Suppose $\mathfrak{p}$ is a completely prime ideal of R, and $\mathfrak{p} \subseteq \mathrm{rad}(R)$.*

(i) *The ideal $\mathfrak{p}$ is a left waist of R.*
(ii) *Given any $a, b \in R$, we have $Ra \subseteq Rb$ or $\mathfrak{p}b \subseteq \mathfrak{p}a$.*

Remark 4.10 Vis-à-vis Proposition 4.9(i), the following easy fact is frequently useful. Let R be any ring, $\mathfrak{p}$ a completely prime ideal of R, and $x \in R \setminus \mathfrak{p}$. If $\mathfrak{p} \subset R$ is a left waist, then $\mathfrak{p} = \mathfrak{p}x$; if $\mathfrak{p} \subset R$ is a right waist, then $\mathfrak{p} = x\mathfrak{p}$.

We are now ready to characterize the right annelidan condition for Bézout and distributive rings.

Theorem 4.11 *Let R be a Bézout ring or a distributive ring. The following conditions are equivalent.*

(i) *R is right annelidan.*
(ii) *$\mathrm{RZD}(R)$ is a right waist of R.*
(iii) *$\mathrm{RZD}(R) \subseteq \mathrm{rad}(R)$.*
(iv) *$\mathrm{RZD}(R)$ is a left waist of R.*
(v) *The set $S = R \setminus (\mathrm{RZD}(R) \cup \mathrm{LZD}(R))$ is a denominator set, the ring RS^{-1} is uniserial, and $\mathrm{RZD}(RS^{-1}) \subseteq R \subseteq RS^{-1}$.*
(vi) *R is a right order in a right uniserial ring Q such that $\mathrm{RZD}(Q) \subseteq R \subseteq Q$.*
(vii) *R is a subring of a right uniserial ring Q such that $\mathrm{RZD}(Q) \subseteq R \subseteq Q$.*
(viii) *R is a subring of a right annelidan ring Q such that $\mathrm{RZD}(Q) \subseteq R \subseteq Q$.*

Proof (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii): This holds by Theorem 4.1 or 4.3.

(iv) $\Rightarrow$ (iii): This is clear.

(i) $\Rightarrow$ (iv): Use Theorem 2.1(ii) to apply Proposition 4.9(i) to $\mathfrak{p} = \mathrm{RZD}(R)$.

(i) $\Rightarrow$ (v): By Theorem 4.6 or 4.7, S is a denominator set, the natural map $R \to RS^{-1}$ is injective, and the ring RS^{-1} is uniserial. We will regard R as a subring of RS^{-1}.

Suppose $cd^{-1} \in \mathrm{RZD}(RS^{-1})$, where $c \in R$ and $d \in S$. Since $RS^{-1} = S^{-1}R$, we have $a(cd^{-1}) = 0$ for some nonzero $a \in R$. Hence $c \in \mathrm{RZD}(R)$. By (iv), $\mathrm{RZD}(R)$ is a left waist of R; therefore, $c \in Rd$. Thus $cd^{-1} \in R$, which proves that $\mathrm{RZD}(RS^{-1}) \subseteq R$.

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(v) ⇒ (vi) ⇒ (vii) ⇒ (viii): This is trivial.

(viii) ⇒ (i): This holds by Proposition 2.3(i). ∎

In order to apply Theorem 4.11 only assuming, a priori, a one-sided Bézout or distributive hypothesis, we will give a module-theoretic characterization of right Bézout, right annelidan rings and right distributive, right annelidan rings. We first give criteria for the right Bézout or right distributive condition to pass from a ring to a module over a suitable factor ring.

Proposition 4.12 *Let Q be ring, let R be a subring of Q, and let I be a proper ideal of Q such that $I \subseteq R$ and $\operatorname{ann}_r^Q(s) \subseteq I$ for some $s \in I$. If the ring R is right Bézout (resp. right distributive), then the right (R/I)-module Q/I is Bézout (resp. distributive).*

Proof We will use bars to denote images under the canonical maps $Q \to Q/I$ and $R \to R/I$.

Assume R is right Bézout. Let $\overline{q_1}, \overline{q_2} \in \overline{Q}$ be arbitrary. Since $sq_1, sq_2 \in R$, there exist $a, b \in R$ such that $sq_1R + sq_2R = (sq_1a + sq_2b)R$. Because $\operatorname{ann}_r^Q(s) \subseteq I$, it follows that $\overline{q_1}\overline{R} + \overline{q_2}\overline{R} = \overline{q_1a + q_2b}\,\overline{R}$, proving $\overline{Q}_{\overline{R}}$ is Bézout.

Assume R is right distributive. Let $\overline{q_1}, \overline{q_2} \in \overline{Q}$ be arbitrary. By Lemma 4.2, there exist $x, y, z, t \in R$ such that $x + y = 1$, $sq_1x = sq_2z$, and $sq_2y = sq_1t$. Hence, $s(q_1x - q_2z) = s(q_2y - q_1t) = 0$, and so $q_1x - q_2z, q_2y - q_1t \in \operatorname{ann}_r^Q(s) \subseteq I$. Thus, $\overline{q_1}\,\overline{x} \in \overline{q_2}\overline{R}$ and $\overline{q_2}\,\overline{y} \in \overline{q_1}\overline{R}$, so by Lemma 4.2, $\overline{Q}_{\overline{R}}$ is distributive. ∎

And now we give criteria for the right Bézout or right distributive condition to pass from an appropriate module back to the ring.

Proposition 4.13 *Let Q be a right uniserial ring, let R be a subring of Q, and let $\mathfrak{p}$ be a completely prime ideal of Q such that $\mathfrak{p} \subseteq R$. If the right $(R/\mathfrak{p})$-module $Q/\mathfrak{p}$ is Bézout (resp. distributive), then the ring R is right Bézout (resp. right distributive).*

Proof We will use bars to denote images under the canonical maps $Q \to Q/\mathfrak{p}$ and $R \to R/\mathfrak{p}$.

Assume $\overline{Q}$ is a Bézout right $\overline{R}$-module. Let $r_1, r_2 \in R$ be arbitrary; we will show that $r_1R + r_2R \subseteq R$ is a principal right ideal. Since Q is right uniserial, without loss of generality we can assume that $r_1 = r_2q$ for some $q \in Q$. If $q \in \mathfrak{p}$, then $q \in R$, and we are done, so suppose $q \notin \mathfrak{p}$. Since $\overline{Q}_{\overline{R}}$ is Bézout, for some $a \in Q$ we have $\overline{q}\overline{R} + \overline{1}\,\overline{R} = \overline{a}\overline{R}$, *i.e.*, $qR + R = aR + \mathfrak{p}$. Since $q \notin \mathfrak{p}$, we have $a \notin \mathfrak{p}$, so by Remark 4.10, $\mathfrak{p} = a\mathfrak{p} \subseteq aR$. Therefore $qR + R = aR$. It follows that $r_2qR + r_2R = r_2aR$, which is to say, $r_1R + r_2R = r_2aR$. Hence, R is right Bézout.

Assume $\overline{Q}$ is a distributive right $\overline{R}$-module. Let $r_1, r_2 \in R$ be arbitrary; by Lemma 4.2, it suffices to show that $r_1^{-1}(r_2R) + r_2^{-1}(r_1R) = R$. As before, without loss of generality we can assume that $r_1 = r_2q$ for some $q \in Q$. If $q \in \mathfrak{p} \subseteq R$, then $r_1^{-1}(r_2R) = R$, so suppose $q \notin \mathfrak{p}$. By Lemma 4.2, $\overline{q}^{-1}(\overline{1}\,\overline{R}) + \overline{1}^{-1}(\overline{q}\overline{R}) = \overline{R}$, so there exist $x, y \in R$ such that $x + y = 1$, $\overline{x} \in \overline{q}^{-1}(\overline{1}\,\overline{R})$, and $\overline{y} \in \overline{1}^{-1}(\overline{q}\overline{R})$. Then $\overline{q}\,\overline{x} \in \overline{1}\,\overline{R}$ and $\overline{1}\overline{y} \in \overline{q}\overline{R}$, hence $qx \in R$ and $y \in qR + \mathfrak{p}$. Once again, $q \notin \mathfrak{p}$ implies $\mathfrak{p} = q\mathfrak{p} \subseteq qR$, and thus $y \in qR$. Hence, $x \in r_1^{-1}(r_2R)$ and $y \in r_2^{-1}(r_1R)$, so by Lemma 4.2, R is right distributive. ∎

We have established the following generalization of [10, Theorem 2.5(i)].

Corollary 4.14 *Let $\mathfrak{p}$ be a nonzero completely prime ideal of a right uniserial ring Q such that $\mathrm{RZD}(Q) \subseteq \mathfrak{p}$. Let R be a subring of Q that contains $\mathfrak{p}$. Then the following conditions are equivalent.*

(i) *The ring R is right Bézout (resp. right distributive).*
(ii) *The right $(R/\mathfrak{p})$-module $Q/\mathfrak{p}$ is Bézout (resp. distributive).*

Proof (i) ⇒ (ii): This holds by Proposition 4.12.
(ii) ⇒ (i): This holds by Proposition 4.13. ∎

We can now show that under a suitable module-theoretic assumption, the Bézout or distributive condition on R_R can be moved out of the hypotheses of Theorem 4.11 and incorporated into the equivalent conditions (*cf.* (i) and (vii) of Theorem 4.11).

Theorem 4.15 *Let R be a left Bézout (resp. left distributive) ring. The following conditions are equivalent.*

(i) *R is a right Bézout (resp. right distributive) and right annelidan ring.*
(ii) *R is a subring of a right uniserial ring Q such that $\mathrm{RZD}(Q) \subseteq R$, and the right $(R/\mathrm{RZD}(Q))$-module $Q/\mathrm{RZD}(Q)$ is Bézout (resp. distributive).*

Proof (i) ⇒ (ii): Assume (i). By Theorem 4.11(v), R is a subring of the right ring of fractions $Q = RS^{-1}$ of R with respect to the (left and right) denominator set $S = R \smallsetminus (\mathrm{LZD}(R) \cup \mathrm{RZD}(R))$, Q is uniserial, and $\mathrm{RZD}(Q) \subseteq R$. If R is not a domain, then (ii) follows from Corollary 4.14, with $\mathfrak{p} = \mathrm{RZD}(Q)$. If R is a domain, then in lieu of Corollary 4.14 we use [10, Lemma 2.6] (which applies, because S is a left denominator set and thus $Q \cong S^{-1}R$).

(ii) ⇒ (i): Assume (ii). By Proposition 2.3(i), R is right annelidan. If Q is not a domain, we can apply Corollary 4.14 with $\mathfrak{p} = \mathrm{RZD}(Q)$ to obtain (i). If Q is a domain, then the Bézout (resp. distributive) property of the module Q_R passes to the submodule R_R, giving (i). ∎

An illustration of Theorem 4.11 (in which both the Bézout and the distributive hypotheses hold) is given by the elementary example

$$R = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix} \right) : a \in \mathbb{Z},\ b \in \mathbb{Q} \right\}, \qquad Q = RS^{-1} = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix} \right) : a, b \in \mathbb{Q} \right\}.$$

The ring R occurred in [25, Example 3.7], alongside a promise of the localization theory of the present work. Note that the rings R and Q are isomorphic to Dorroh extensions, in suitable categories, of the nonunital ring $\mathbb{Q}$ with zero multiplication. As such, R and Q also illustrate [25, Corollary 4.6] and [25, Corollary 4.2], respectively.

5 A Symmetry Result

We showed that the annelidan condition is not left-right symmetric in [25, Examples 7.3 and 7.4], and we gave conditions under which *right annelidan*is equivalent to *left annelidan* in [25, Corollary 7.6 and Theorem 8.8] (as well as the conditional result [25, Corollary 8.6]). The main theorem of this section extends the latter results, establishing that the annelidan condition is left-right symmetric for Bézout and for distributive rings that satisfy some modest finiteness conditions, of the sort found in [25, Theorem 3.5].

Given a subset I of a ring R, recall that I is a *right comparizer* of R if I is a right ideal of R such that for any right ideals A and B of R we have $A \subseteq B$ or $BI \subseteq A$. If the last condition in the definition can be strengthened to $A \subseteq B$ or $BI \subseteq AI$, then I is said to be a *strong right comparizer* of R. We refer the reader to [31, 32] for properties of right comparizers and strong right comparizers, several examples, and connections to the classification of semiprime segments of a ring.

In order to prove the main results of this section, on right annelidan rings that are Bézout or distributive, we will establish various results on completely prime ideals under somewhat technical assumptions. Typically we will take $\mathfrak{p}$ to be a completely prime ideal, a right waist, and a strong right comparizer in a ring. An exemplar of these conditions is the set of right zero-divisors in a right annelidan ring that is right Bézout or right distributive. We recall that Theorem 2.1(ii) says that the set of right zero-divisors in a right annelidan ring is a completely prime ideal contained in the Jacobson radical. Moreover, we recall from Proposition 4.9 that *every completely prime ideal contained in the Jacobson radical of a right Bézout or right distributive ring is a right waist and a strong right comparizer.*

While the Bézout and distributive conditions are our primary interest, we note that in what M. Ferrero and A. Sant'Ana in [11] define as a ring with right comparability, every completely prime ideal contained in the Jacobson radical is a right waist [11, Lemma 1.3] and a strong right comparizer [11, Remark 2.5]. Thus, many of our results are also applicable to Ferrero and Sant'Ana's rings with right comparability.

The following proposition extends [30, Lemma 6] and [30, Theorem 7].

Proposition 5.1 *Let R be any ring. Suppose $\mathfrak{p}$ is a completely prime ideal, a right waist, and a strong right comparizer of R.*

(i) *For any $a \in R \smallsetminus \operatorname{ann}_\ell^R(\mathfrak{p})$ we have $\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq a\mathfrak{p}$. In particular, $\operatorname{ann}_\ell^R(\mathfrak{p})$ is a right waist of R.*

(ii) *If $\operatorname{ann}_\ell^R(\mathfrak{p}) \neq (0)$, then the ideal*

$$\mathfrak{p}_1 = \operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p})) \subseteq R$$

is completely prime.

Proof (i) Assume, on the contrary, that $\operatorname{ann}_\ell^R(\mathfrak{p}) \nsubseteq a\mathfrak{p}$ for some $a \in R \smallsetminus \operatorname{ann}_\ell^R(\mathfrak{p})$. Choose $b \in \operatorname{ann}_\ell^R(\mathfrak{p}) \smallsetminus a\mathfrak{p}$. Then $b\mathfrak{p} = \{0\}$ whereas $a\mathfrak{p} \neq \{0\}$. So $a\mathfrak{p} \nsubseteq b\mathfrak{p}$, and since $\mathfrak{p}$ is a strong right comparizer, $bR \subseteq aR$. Since $b \in aR \smallsetminus a\mathfrak{p}$, we have $b = ac$ for some $c \in R \smallsetminus \mathfrak{p}$; however, Remark 4.10 implies $\mathfrak{p} = c\mathfrak{p}$, whence

$$\{0\} \neq a\mathfrak{p} = ac\mathfrak{p} = b\mathfrak{p} = \{0\},$$

a contradiction.

We have just shown that for any $a \in R$ we have $aR \subseteq \operatorname{ann}_\ell^R(\mathfrak{p})$ or $\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq a\mathfrak{p} \subseteq aR$. Thus, $\operatorname{ann}_\ell^R(\mathfrak{p})$ is a right waist of R.

(ii) If $\mathfrak{p} = (0)$, then $\mathfrak{p}_1 = (0)$, and we are done. Henceforth assume $\mathfrak{p} \neq (0)$.

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We claim that

$$\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq \mathfrak{p}_1. \tag{5.1}$$

Choose any element $x \notin \mathfrak{p}_1$. By definition of $\mathfrak{p}_1$, there exists $y \in \operatorname{ann}_\ell^R(\mathfrak{p})$ such that $xy \neq 0$. Since $y\mathfrak{p} = \{0\} \neq \mathfrak{p}$, by Remark 4.10 we must have $y \in \mathfrak{p}$. Therefore $x \notin \operatorname{ann}_\ell^R(\mathfrak{p})$, which proves (5.1).

To show that $\mathfrak{p}_1$ is completely prime, assume $a, b \in R$ satisfy $ab \in \mathfrak{p}_1$ and $b \notin \mathfrak{p}_1$. By (5.1), $b \notin \operatorname{ann}_\ell^R(\mathfrak{p})$, so from (i) we obtain $\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq b\mathfrak{p}$. If $ab\mathfrak{p} = \{0\}$, then $a \cdot \operatorname{ann}_\ell^R(\mathfrak{p}) = \{0\}$, whence $a \in \mathfrak{p}_1$ and we are done. So assume $ab\mathfrak{p} \neq \{0\}$. In particular, abR is a nonzero right ideal, and (by virtue of (i)) $\operatorname{ann}_\ell^R(\mathfrak{p})$ is a nonzero right waist, so $abR \cap \operatorname{ann}_\ell^R(\mathfrak{p}) \neq \{0\}$. Thus, there exists $c \in R$ such that

$$0 \neq abc \in \operatorname{ann}_\ell^R(\mathfrak{p}). \tag{5.2}$$

As $ab \in \mathfrak{p}_1$ and $abc \neq 0$, we cannot have $c \in \operatorname{ann}_\ell^R(\mathfrak{p})$, so (i) implies

$$\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq c\mathfrak{p}. \tag{5.3}$$

Now, $b \notin \mathfrak{p}_1$ means that $b \cdot \operatorname{ann}_\ell^R(\mathfrak{p}) \neq \{0\}$, so (5.3) implies $bc\mathfrak{p} \neq \{0\}$. Since $bc \notin \operatorname{ann}_\ell^R(\mathfrak{p})$, another application of (i) gives

$$\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq bc\mathfrak{p}. \tag{5.4}$$

By (5.2), $abc\mathfrak{p} = \{0\}$, and now (5.4) implies $a \cdot \operatorname{ann}_\ell^R(\mathfrak{p}) = \{0\}$. So $a \in \mathfrak{p}_1$. Therefore, $\mathfrak{p}_1$ is completely prime. ∎

We will now show that under appropriate conditions, completely prime ideals and their annihilators will satisfy a double-annihilator condition familiar from the theory of quasi-Frobenius rings. The special case of Proposition 5.2 where R is uniserial was proved by Bessenrodt, Brungs, and Törner in [5, Proposition 2.10].

Proposition 5.2 *Let R be any ring. Suppose $\mathfrak{p}$ is a completely prime ideal and a right waist of R, and suppose $\operatorname{ann}_\ell^R(\mathfrak{p})$ is essential as a left ideal of R.*

(i) *We have $\operatorname{ann}_r^R(\operatorname{ann}_\ell^R(\mathfrak{p})) = \mathfrak{p}$.*
(ii) *If $\mathfrak{p}$ is a strong right comparizer, then $\operatorname{ann}_r^R(\operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}))) = \operatorname{ann}_\ell^R(\mathfrak{p})$.*

Proof (i) Suppose $\operatorname{ann}_r^R(\operatorname{ann}_\ell^R(\mathfrak{p})) \neq \mathfrak{p}$, so there exists $a \in \operatorname{ann}_r^R(\operatorname{ann}_\ell^R(\mathfrak{p})) \setminus \mathfrak{p}$. Hence $\mathfrak{p} = a\mathfrak{p}$. If $x = sa \in Ra \cap \operatorname{ann}_\ell^R(\mathfrak{p})$, then $s\mathfrak{p} = sa\mathfrak{p} = (0)$, and thus $s \in \operatorname{ann}_\ell^R(\mathfrak{p})$. Since $a \in \operatorname{ann}_r^R(\operatorname{ann}_\ell^R(\mathfrak{p}))$, it follows that $x = sa = 0$. Therefore $Ra \cap \operatorname{ann}_\ell^R(\mathfrak{p}) = \{0\}$, which is impossible, since $\operatorname{ann}_\ell^R(\mathfrak{p})$ is an essential left ideal of R.

(ii) Suppose $\operatorname{ann}_r^R(\operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}))) \neq \operatorname{ann}_\ell^R(\mathfrak{p})$, so there exists an element

$$a \in \operatorname{ann}_r^R(\operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}))) \setminus \operatorname{ann}_\ell^R(\mathfrak{p}).$$

By Proposition 5.1(i), $\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq a\mathfrak{p}$; thus,

$$\operatorname{ann}_r^R\left(\operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}))\right) \subseteq \operatorname{ann}_r^R\left(\operatorname{ann}_\ell^R(a\mathfrak{p})\right).$$

If $x = sa \in Ra \cap \operatorname{ann}_\ell^R(\mathfrak{p})$, then $s \in \operatorname{ann}_\ell^R(a\mathfrak{p})$, and $a \in \operatorname{ann}_r^R(\operatorname{ann}_\ell^R(a\mathfrak{p}))$ implies $x = 0$. Hence, $Ra \cap \operatorname{ann}_\ell^R(\mathfrak{p}) = \{0\}$, which, again, is impossible. ∎

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In Proposition 5.2, the assumption that $\operatorname{ann}_\ell^R(\mathfrak{p})$ be essential as a left ideal is indispensable, as the following example shows.

Example 5.3 Let R be the ring defined in [25, Example 7.4]. Namely,

$$R = D[x;\sigma]/(\pi^2 x),$$

where D is the localization $D = F[\pi]_{(\pi)}$ of the polynomial ring $F[\pi]$ with F a field and π an indeterminate such that $\sigma\colon D \to D$ is a ring homomorphism that carries $D \setminus \{0\}$ into $\mathrm{U}(D)$. We write elements $f(x) + (\pi^2 x) \in R$ as $f(x)$.

Let

$$\mathfrak{p} = Rx.$$

As shown in [25], the ring R is left noetherian and left uniserial; any such ring is left duo, so $\mathfrak{p}$ is an ideal. Since $R/\mathfrak{p} \cong D$, the ideal $\mathfrak{p}$ is prime, and in a one-sided duo ring all primes are completely prime. Thus, $\mathfrak{p}$ is a completely prime ideal and a left waist of R; by Proposition 4.9(ii), $\mathfrak{p}$ is a strong left comparizer of R. Thus, all the hypotheses of the opposite version of Proposition 5.2 (including the hypothesis in (ii)) other than $\operatorname{ann}_r^R(\mathfrak{p})$ being an essential right ideal are satisfied.

Nevertheless, a direct calculation gives $\operatorname{ann}_r^R(\mathfrak{p}) = \mathfrak{p}$ and $\operatorname{ann}_\ell^R(\mathfrak{p}) = R\pi^2 + Rx$; therefore,

$$\operatorname{ann}_\ell^R(\operatorname{ann}_r^R(\mathfrak{p})) = R\pi^2 + Rx \neq \mathfrak{p},$$
$$\operatorname{ann}_\ell^R(\operatorname{ann}_r^R(\operatorname{ann}_r^R(\mathfrak{p}))) = R\pi^2 + Rx \neq \operatorname{ann}_r^R(\mathfrak{p}).$$

So the conclusions of both parts of the opposite version of Proposition 5.2 fail. (On the other hand, this example does illustrate the opposite version of Proposition 5.1(ii), with $\mathfrak{p}_1 = \operatorname{ann}_r^R(\operatorname{ann}_r^R(\mathfrak{p})) = \mathfrak{p}$ in this case.)

The conclusion of Proposition 5.2(ii) is equivalent to the statement that $\operatorname{ann}_\ell^R(\mathfrak{p})$ is a right annihilator. We record a quick example showing that this conclusion can fail absent the strong right comparizer assumption.

Example 5.4 Let k be a field, let $F = k\langle x, y\rangle$ be the free algebra over k in two noncommuting indeterminates, and let R be the factor ring

$$R = F/(yx, y^2, x^3, x^2 y).$$

We will continue to write x and y for the images of these elements in R.

Put $\mathfrak{p} = Rx + Ry$. Then $\operatorname{ann}_\ell^R(\mathfrak{p}) = Rx^2 + Ry$. As the unique maximal ideal of a local ring, $\mathfrak{p}$ is completely prime and a right waist. It is easy to see that $\operatorname{ann}_\ell^R(\mathfrak{p})$ is essential as a left ideal of R. Thus, Proposition 5.2(i) applies; however,

$$\operatorname{ann}_r^R\left(\operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}))\right) = \mathfrak{p} \neq \operatorname{ann}_\ell^R(\mathfrak{p}).$$

Evidently, $\mathfrak{p}$ is not a strong right comparizer of R, and the conclusion of Proposition 5.2(ii) fails.

The following "descent property" of strong right comparizers will be instrumental in establishing conditions for every left annihilator of a completely prime ideal to lie within the right annihilator of that ideal, and vice versa, in Theorems 5.6 and 5.9.

Lemma 5.5 *Let R be a ring and let I be a right waist and a strong right comparizer of R. Then every completely prime ideal $\mathfrak{q}$ of R that is contained in I is a right waist and a strong right comparizer of R.*

Proof If $\mathfrak{q} = I$, then there is nothing to prove, so assume $\mathfrak{q} \subsetneq I$. By [31, Proposition 1.9], $\mathfrak{q}$ is a right waist of R, and Remark 4.10 gives $\mathfrak{q} = I\mathfrak{q}$. To show that $\mathfrak{q}$ is a strong right comparizer, assume A and B are right ideals of R such that $A \nsubseteq B$. Then since I is a strong right comparizer, we have $BI \subseteq AI$, and thus $B\mathfrak{q} = BI\mathfrak{q} \subseteq AI\mathfrak{q} = A\mathfrak{q}$, proving $\mathfrak{q}$ is a strong right comparizer of R. ∎

Regarding the somewhat unusual chain condition that appears below in Theorem 5.6, Corollary 5.7, Theorem 5.8, and Theorem 5.13(v), recall that a theorem due to L. W. Small [40] states that any PI-ring with ACC on ideals must satisfy DCC on prime ideals. Whether every noetherian ring satisfies DCC on prime ideals is an open question listed as "Test Problem 3(a)" in the appendix to K. R. Goodearl and R. B. Warfield's book [16]. This chain condition comes up naturally in noncommutative algebraic geometry, as in the work of J. P. Bell, D. Rogalski, and S. J. Sierra in [3], where they establish the Dixmier–Moeglin characterization of primitive ideals in certain noetherian algebras under hypotheses that include DCC on prime ideals.

Theorem 5.6 *Let R be any ring. Suppose $\mathfrak{p}$ is a completely prime right ideal, a right waist, and a strong right comparizer of R. If R satisfies the descending chain condition on completely prime ideals and $\operatorname{ann}_\ell^R(\mathfrak{p})$ is essential as a left ideal of R, then $\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq \operatorname{ann}_r^R(\mathfrak{p})$.*

Proof First, note that $\mathfrak{p}$ must actually be an ideal, since any right waist is contained in the Jacobson radical, and any completely prime one-sided ideal contained in the Jacobson radical is a two-sided ideal by [12, Lemma 2.5].

Suppose, for a contradiction, that $\operatorname{ann}_\ell^R(\mathfrak{p}) \nsubseteq \operatorname{ann}_r^R(\mathfrak{p})$. By Proposition 5.1(i), the ideal $\operatorname{ann}_\ell^R(\mathfrak{p})$ is a right waist of R; thus, $\operatorname{ann}_r^R(\mathfrak{p}) \subsetneq \operatorname{ann}_\ell^R(\mathfrak{p})$. Let us define a sequence of ideals $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2, \dots$ by

$$\mathfrak{p}_0 = \mathfrak{p} \quad \text{and} \quad \mathfrak{p}_n = \operatorname{ann}_\ell^R\left(\operatorname{ann}_\ell^R(\mathfrak{p}_{n-1})\right) \text{ for each } n \in \mathbb{N}.$$

Claim. For every $n \geqslant 0$, the ideal $\mathfrak{p}_n$ is completely prime, $\mathfrak{p}_{n+1} \subsetneq \mathfrak{p}_n \subseteq \mathfrak{p}$, and $\operatorname{ann}_r^R(\mathfrak{p}_n) \subsetneq \operatorname{ann}_\ell^R(\mathfrak{p}_n)$.

From the claim it will follow that

$$\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \mathfrak{p}_3 \supsetneq \cdots$$

is an infinite strictly descending chain of completely prime ideals, contradicting the DCC hypothesis and completing the proof.

We prove the claim by induction on n. Having already shown that $\operatorname{ann}_r^R(\mathfrak{p}) \subsetneq \operatorname{ann}_\ell^R(\mathfrak{p})$, the $n = 0$ case of the claim requires only that we prove

$$\operatorname{ann}_\ell^R\left(\operatorname{ann}_\ell^R(\mathfrak{p})\right) \subsetneq \mathfrak{p}. \tag{5.5}$$

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But if (5.5) is false, then since $\mathfrak{p}$ is a right waist, we have $\mathfrak{p} \subseteq \operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}))$, and by taking the right annihilator of both ideals in this containment, we obtain

$$\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq \operatorname{ann}_r^R\left(\operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}))\right) \subseteq \operatorname{ann}_r^R(\mathfrak{p}),$$

a contradiction. So the claim holds when $n = 0$.

Now assume, as inductive hypothesis, that $\mathfrak{p}_n$ is a completely prime ideal that satisfies $\mathfrak{p}_{n+1} \subsetneqq \mathfrak{p}_n \subseteq \mathfrak{p}$ and $\operatorname{ann}_r^R(\mathfrak{p}_n) \subsetneqq \operatorname{ann}_\ell^R(\mathfrak{p}_n)$. Then $\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq \operatorname{ann}_\ell^R(\mathfrak{p}_n) \subseteq \operatorname{ann}_\ell^R(\mathfrak{p}_{n+1})$ and thus $\operatorname{ann}_\ell^R(\mathfrak{p}_n)$ and $\operatorname{ann}_\ell^R(\mathfrak{p}_{n+1})$ are essential as left ideals of R. Since $\mathfrak{p}_n \subseteq \mathfrak{p}$, Lemma 5.5 implies that $\mathfrak{p}_n$ is a right waist and a strong right comparizer of R, and thus by Proposition 5.1(ii), the ideal $\mathfrak{p}_{n+1}$ is completely prime. Furthermore, since $\mathfrak{p}_{n+1} \subseteq \mathfrak{p}$, it follows from Lemma 5.5 that $\mathfrak{p}_{n+1}$ is a right waist and a strong right comparizer of R; and as we have noted earlier, $\operatorname{ann}_\ell^R(\mathfrak{p}_n)$ and $\operatorname{ann}_\ell^R(\mathfrak{p}_{n+1})$ are essential as left ideals of R. So all the hypotheses of Proposition 5.2 apply to both $\mathfrak{p}_n$ and $\mathfrak{p}_{n+1}$.

From $\mathfrak{p}_{n+1} \subsetneqq \mathfrak{p}_n$ we obtain $\operatorname{ann}_\ell^R(\mathfrak{p}_n) \subseteq \operatorname{ann}_\ell^R(\mathfrak{p}_{n+1})$. This containment must be strict; otherwise, taking the right annihilator of both sides and applying Proposition 5.2(i) would give $\mathfrak{p}_n = \mathfrak{p}_{n+1}$. By Proposition 5.2(ii),

$$\operatorname{ann}_r^R(\mathfrak{p}_{n+1}) = \operatorname{ann}_r^R\left(\operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}_n))\right) = \operatorname{ann}_\ell^R(\mathfrak{p}_n) \subsetneqq \operatorname{ann}_\ell^R(\mathfrak{p}_{n+1}).$$

Now repeating the argument used to prove the $n = 0$ case, with $\mathfrak{p}$ replaced by $\mathfrak{p}_{n+1}$, shows that $\mathfrak{p}_{n+2} = \operatorname{ann}_\ell^R(\operatorname{ann}_\ell^R(\mathfrak{p}_{n+1})) \subsetneqq \mathfrak{p}_{n+1}$.

Thus, $\mathfrak{p}_{n+1}$ is a completely prime ideal that satisfies $\mathfrak{p}_{n+2} \subsetneqq \mathfrak{p}_{n+1} \subseteq \mathfrak{p}$ and $\operatorname{ann}_r^R(\mathfrak{p}_{n+1}) \subsetneqq \operatorname{ann}_\ell^R(\mathfrak{p}_{n+1})$. This completes the induction, proving the claim and with it the theorem. ∎

Evidently, Theorem 5.6 holds under somewhat weaker hypotheses: it is enough to assume DCC on the subset of completely prime ideals that satisfy various additional conditions invoked in the proof. A similar observation will apply to Theorem 5.9 below.

An immediate consequence of Theorem 5.6 is the following corollary.

Corollary 5.7 *Let R be any ring with left uniform dimension* $\operatorname{u.dim}({}_R R) = 1$. *Suppose $\mathfrak{p}$ is a completely prime right ideal, a right waist, and a strong right comparizer of R. If R satisfies the descending chain condition on completely prime ideals, then* $\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq \operatorname{ann}_r^R(\mathfrak{p})$.

It is known that in a uniserial ring with DCC on prime ideals, the left and right annihilators of any completely prime ideal coincide [5, Corollary 8.11]. We can now generalize this result to Bézout annelidan rings and to distributive annelidan rings under a relaxed chain condition.

Theorem 5.8 *Let R be a Bézout ring or a distributive ring. Suppose that R satisfies the descending chain condition on completely prime ideals.*

(i) *If R is right annelidan, then for any completely prime right ideal $\mathfrak{p}$ of R we have* $\operatorname{ann}_\ell^R(\mathfrak{p}) \subseteq \operatorname{ann}_r^R(\mathfrak{p})$.

(ii) *If R is annelidan, then for any completely prime ideal $\mathfrak{p}$ of R we have* $\operatorname{ann}_\ell^R(\mathfrak{p}) = \operatorname{ann}_r^R(\mathfrak{p})$.

Proof (i) According to the left-hand versions of [26, Proposition 3.2] and [26, Proposition 3.3], a lineal ring that is left Bézout or left distributive has left uniform dimension 1. If $\mathfrak{p} \nsubseteq \mathrm{RZD}(R)$, then $\mathrm{ann}_\ell^R(\mathfrak{p}) = (0)$ and there is nothing to prove. If $\mathfrak{p} \subseteq \mathrm{RZD}(R)$, then by Theorem 2.1(ii) we have $\mathfrak{p} \subseteq \mathrm{rad}(R)$; since $\mathfrak{p}$ is a completely prime ideal contained in the Jacobson radical of a right Bézout or right distributive ring, the right-hand version of Proposition 4.9 shows that $\mathfrak{p}$ is a right waist and a strong right comparizer. So Corollary 5.7 applies.

(ii) This follows using symmetry. ∎

It would be of interest to know whether Corollary 5.7 remains true if the hypothesis "DCC on completely prime ideals" is changed to "ACC on completely prime ideals." If so, the latter could be added to the list of chain conditions in our main symmetry result, Theorem 5.13. We do not know the answer and leave this as an open problem.

Question If R is a ring with left uniform dimension 1 that satisfies the ascending chain condition on completely prime ideals, and $\mathfrak{p}$ is a completely prime right ideal, a right waist, and a strong right comparizer of R, must we have $\mathrm{ann}_\ell^R(\mathfrak{p}) \subseteq \mathrm{ann}_r^R(\mathfrak{p})$?

We can, however, prove an "ACC" analogue of Corollary 5.7 with modified hypotheses and a reversed containment of right and left annihilators.

Theorem 5.9 *Let R be a ring with left uniform dimension* $\mathrm{u.dim}(_R R) = 1$. *Suppose* $\mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$, *and suppose that* $\mathrm{RZD}(R)$ *is a right waist and a strong right comparizer of R. If R satisfies the ascending chain condition on completely prime ideals, then for every completely prime ideal $\mathfrak{p}$ of R, we have* $\mathrm{ann}_r^R(\mathfrak{p}) \subseteq \mathrm{ann}_\ell^R(\mathfrak{p})$.

Proof Assume the conclusion fails. By the ACC hypothesis, we can choose $\mathfrak{p}$ that is maximal among completely prime ideals of R satisfying $\mathrm{ann}_r^R(\mathfrak{p}) \nsubseteq \mathrm{ann}_\ell^R(\mathfrak{p})$.

Since $\mathrm{ann}_r^R(\mathfrak{p}) \neq (0)$, we must have $\mathfrak{p} \subseteq \mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$. Lemma 5.5 implies that $\mathfrak{p}$ is a right waist and a strong right comparizer of R. Thus, Proposition 5.1(i) applies, showing that $\mathrm{ann}_\ell^R(\mathfrak{p})$ is a right waist of R. Hence,

$$\mathrm{ann}_\ell^R(\mathfrak{p}) \subsetneqq \mathrm{ann}_r^R(\mathfrak{p}). \tag{5.6}$$

We claim that $\mathrm{ann}_\ell^R(\mathfrak{p}) \neq (0)$. Indeed, assume $\mathrm{ann}_\ell^R(\mathfrak{p}) = (0)$. Taking left annihilators of $\mathfrak{p} \subseteq \mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$, we obtain $\mathrm{ann}_\ell^R(\mathrm{RZD}(R)) = (0)$. By [28, Proposition 1.2] and its proof (with a left-right switch), for any ring R satisfying $\mathrm{u.dim}(_R R) = 1$, the set $\mathrm{RZD}(R)$ is a completely prime ideal of R and $\mathrm{ann}_r^R(\mathrm{LZD}(R)) \subseteq \mathrm{ann}_\ell^R(\mathrm{RZD}(R))$. Hence, $\mathrm{ann}_r^R(\mathrm{LZD}(R)) = (0)$. From $\mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$, we obtain $\mathrm{ann}_r^R(\mathrm{RZD}(R)) = (0)$. Since $\mathrm{ann}_r^R(\mathfrak{p}) \neq (0)$, we cannot have $\mathfrak{p} = \mathrm{RZD}(R)$. So $\mathfrak{p} \subsetneqq \mathrm{RZD}(R)$. Pick $\alpha \in \mathrm{RZD}(R) \smallsetminus \mathfrak{p}$. Since $\mathfrak{p}$ is a right waist of R, we have $\mathfrak{p} \subset \alpha R$. Then $\mathrm{ann}_\ell^R(\alpha R) \subseteq \mathrm{ann}_\ell^R(\mathfrak{p}) = (0)$, contradicting $\alpha \in \mathrm{RZD}(R)$. This proves the claim.

By Proposition 5.1(ii), the ideal

$$\mathfrak{p}_1 = \mathrm{ann}_\ell^R(\mathrm{ann}_\ell^R(\mathfrak{p}))$$

is completely prime. Since $(0) \neq \mathrm{ann}_\ell^R(\mathfrak{p})$, we know that $\mathfrak{p}_1 \subseteq \mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$. Lemma 5.5 implies that $\mathfrak{p}_1$ is a right waist and a strong right comparizer of R.

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From $\mathrm{u.dim}({}_R R) = 1$, it follows that $\mathrm{ann}_\ell^R(\mathfrak{p})$ is essential as a left ideal of R. So Proposition 5.2(ii) tells us that

$$\mathrm{ann}_r^R(\mathfrak{p}_1) = \mathrm{ann}_\ell^R(\mathfrak{p}). \tag{5.7}$$

Combining (5.6) and (5.7) gives $\mathrm{ann}_r^R(\mathfrak{p}_1) \subsetneqq \mathrm{ann}_r^R(\mathfrak{p})$, which precludes $\mathfrak{p}_1 \subseteq \mathfrak{p}$. Therefore, since $\mathfrak{p}$ is a right waist, $\mathfrak{p} \subsetneqq \mathfrak{p}_1$.

By the maximal choice of $\mathfrak{p}$, we have $\mathrm{ann}_r^R(\mathfrak{p}_1) \subseteq \mathrm{ann}_\ell^R(\mathfrak{p}_1)$. By equation (5.7), $(0) \neq \mathrm{ann}_\ell^R(\mathfrak{p}) \subseteq \mathrm{ann}_\ell^R(\mathfrak{p}_1)$. By Proposition 5.2(i),

$$\mathfrak{p}_1 = \mathrm{ann}_r^R\left(\mathrm{ann}_\ell^R(\mathfrak{p}_1)\right) \subseteq \mathrm{ann}_r^R\left(\mathrm{ann}_\ell^R(\mathfrak{p})\right) = \mathfrak{p},$$

which contradicts $\mathfrak{p} \subsetneqq \mathfrak{p}_1$. ■

For a certain class of right annelidan rings R, we can now prove a necessary and sufficient condition for $\mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$.

Theorem 5.10 *Let R be a Bézout ring or a distributive ring. Suppose that R is right annelidan and satisfies the ascending chain condition on completely prime ideals. Then* $\mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$ *if and only if* $\mathrm{ann}_r^R(\mathfrak{p}) \subseteq \mathrm{ann}_\ell^R(\mathfrak{p})$ *for every completely prime ideal* $\mathfrak{p}$ *of R.*

Proof As in the proof of Theorem 5.8, we deduce that $\mathrm{u.dim}(R_R) = \mathrm{u.dim}({}_R R) = 1$.

"Only if": This follows from Theorem 5.9 in the same way that Theorem 5.8 follows from Corollary 5.7.

"If": Assume that $\mathrm{ann}_r^R(\mathfrak{p}) \subseteq \mathrm{ann}_\ell^R(\mathfrak{p})$ for every completely prime ideal $\mathfrak{p}$ of R. In particular,

$$\mathrm{ann}_r^R(\mathrm{RZD}(R)) \subseteq \mathrm{ann}_\ell^R(\mathrm{RZD}(R)). \tag{5.8}$$

Suppose, for a contradiction, that $\mathrm{LZD}(R) \nsubseteq \mathrm{RZD}(R)$.

By Theorem 2.1(ii), $\mathrm{RZD}(R)$ is a right waist of R, so $\mathrm{RZD}(R) \subsetneqq \mathrm{LZD}(R)$. Pick $a \in \mathrm{LZD}(R) \setminus \mathrm{RZD}(R)$. By Theorem 4.11(iv), $\mathrm{RZD}(R)$ is a left waist of R; therefore, $\mathrm{RZD}(R) \subseteq Ra$. Hence, $\mathrm{ann}_r^R(\mathrm{RZD}(R)) \neq (0)$. Note that $\mathrm{RZD}(R)$ is a strong left comparizer of R, by Proposition 4.9(ii). Thus,

$$\begin{aligned}
\mathrm{LZD}(R) &= \mathcal{Z}(R_R) \quad \text{(by Theorem 2.1)}\\
&\subseteq \mathrm{ann}_\ell^R(\mathrm{ann}_\ell^R(\mathrm{RZD}(R))) \quad \text{(by [28, Lemma 1.1])}\\
&\subseteq \mathrm{ann}_\ell^R(\mathrm{ann}_r^R(\mathrm{RZD}(R))) \quad \text{(by equation (5.8))}\\
&= \mathrm{RZD}(R) \quad \text{(by the opposite version of Proposition 5.2(i))},
\end{aligned}$$

a contradiction. ■

Corollary 5.11 *Let R be a Bézout ring or a distributive ring. Suppose that R is right annelidan and either right noetherian or left noetherian. Then* $\mathrm{ann}_r^R(\mathfrak{p}) \subseteq \mathrm{ann}_\ell^R(\mathfrak{p})$ *for every completely prime ideal* $\mathfrak{p}$ *of R.*

Proof By [25, Theorem 3.5], $\mathrm{LZD}(R)$ is a nil ideal. Now apply Theorem 5.10. ■

Proposition 13 of [27] states that if R is a uniserial ring with ACC on prime ideals, then $\mathrm{LZD}(R) = \mathrm{RZD}(R)$ if and only if $\mathrm{ann}_r^R(\mathfrak{p}) = \mathrm{ann}_\ell^R(\mathfrak{p})$ for every completely prime ideal $\mathfrak{p}$ of R. Theorem 5.10 and its dual yield the following generalization.

Corollary 5.12 *Let R be an annelidan Bézout ring or an annelidan distributive ring. Suppose that R satisfies the ascending chain condition on completely prime ideals. Then* $\mathrm{LZD}(R) = \mathrm{RZD}(R)$ *if and only if* $\mathrm{ann}_r^R(\mathfrak{p}) = \mathrm{ann}_\ell^R(\mathfrak{p})$ *for every completely prime ideal* $\mathfrak{p}$ *of* R.

In the context of Corollary 5.12 (as well as Theorem 5.13, which follows), it is worth mentioning that rings R for which $\mathrm{LZD}(R) = \mathrm{RZD}(R)$ are studied in [13] under the name "eversible rings." S. P. Redmond has proved [35, Theorem 2.3] that the condition $\mathrm{LZD}(R) = \mathrm{RZD}(R)$ is necessary and sufficient for the directed zero-divisor graph of the noncommutative ring R to be connected.

We come now to the main result of this section. In [25] it was shown that the annelidan condition, although not left-right symmetric in general, is left-right symmetric for principally injective rings. We will now prove that the annelidan condition is left-right symmetric in an additional ten cases, where *Bézout* or *distributive* is paired up with any of five chain conditions.

Theorem 5.13 *Let R be a Bézout ring or a distributive ring. Suppose R satisfies any one of the following chain conditions:*

(i) *the ascending chain condition on right annihilators of elements,*
(ii) *the ascending chain condition on principal right ideals,*
(iii) *the ascending chain condition on left annihilators of elements,*
(iv) *the ascending chain condition on principal left ideals,*
(v) *the descending chain condition on completely prime ideals.*

Then R is left annelidan if and only if R is right annelidan, and in this case, $\mathrm{LZD}(R) = \mathrm{RZD}(R)$.

Proof By symmetry, it suffices to show that if R is right annelidan, then $\mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$, since R must then be left annelidan by Theorem 4.1 or 4.3. By [25, Theorem 3.5], for a right annelidan ring R that satisfies condition (i), (ii), (iii), or (iv), the upper and lower nilradical of R both coincide with $\mathrm{LZD}(R)$. In this case, $\mathrm{LZD}(R) \subseteq \mathrm{RZD}(R)$. Now assume R is right annelidan and satisfies condition (v).

As in the proof of Theorem 5.8, we deduce that $\mathrm{u.dim}(R_R) = \mathrm{u.dim}({}_RR) = 1$. By Theorem 5.8(i),

$$\mathrm{ann}_\ell^R\left(\mathrm{RZD}(R)\right) \subseteq \mathrm{ann}_r^R\left(\mathrm{RZD}(R)\right). \tag{5.9}$$

In the left annelidan ring R^{op}, the completely prime ideal $\mathfrak{p} = \mathrm{LZD}(R^{\mathrm{op}})$ is contained in $\mathrm{rad}(R^{\mathrm{op}})$; since R^{op} is right Bézout or right distributive, $\mathfrak{p}$ is a right waist and a strong right comparizer. Thus, we can apply Corollary 5.7 to obtain $\mathrm{ann}_\ell^{R^{\mathrm{op}}}(\mathrm{LZD}(R^{\mathrm{op}})) \subseteq \mathrm{ann}_r^{R^{\mathrm{op}}}(\mathrm{LZD}(R^{\mathrm{op}}))$, equivalently,

$$\mathrm{ann}_r^R\left(\mathrm{RZD}(R)\right) \subseteq \mathrm{ann}_\ell^R\left(\mathrm{RZD}(R)\right). \tag{5.10}$$

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By [28, Proposition 1.2(ii)],

$$\operatorname{ann}_\ell^R\left(\operatorname{RZD}(R)\right) = \operatorname{ann}_r^R\left(\operatorname{LZD}(R)\right). \tag{5.11}$$

Combining (5.9), (5.10), and (5.11) yields

$$\operatorname{ann}_r^R\left(\operatorname{LZD}(R)\right) = \operatorname{ann}_r^R\left(\operatorname{RZD}(R)\right). \tag{5.12}$$

Assume, for a contradiction, that $\operatorname{LZD}(R) \nsubseteq \operatorname{RZD}(R)$. Pick $a \in \operatorname{LZD}(R) \smallsetminus \operatorname{RZD}(R)$. From Theorem 2.1(ii), Theorem 4.11(iv), and Remark 4.10, we obtain $\operatorname{RZD}(R) = \operatorname{RZD}(R) \cdot a$. Therefore, $\{0\} \subsetneqq \operatorname{ann}_r^R(a) \subseteq \operatorname{ann}_r^R(\operatorname{RZD}(R))$. By the left-right dual of Proposition 5.2(i), we have $\operatorname{ann}_\ell^R(\operatorname{ann}_r^R(\operatorname{RZD}(R))) = \operatorname{RZD}(R)$; however, by equation (5.12),

$$\operatorname{LZD}(R) \subseteq \operatorname{ann}_\ell^R\left(\operatorname{ann}_r^R(\operatorname{LZD}(R))\right) = \operatorname{ann}_\ell^R\left(\operatorname{ann}_r^R(\operatorname{RZD}(R))\right) = \operatorname{RZD}(R),$$

a contradiction. ■

Theorem 5.13 extends [42, Theorem 2.2], which states that $\operatorname{LZD}(R) = \operatorname{RZD}(R)$ for uniserial rings R that satisfy the descending chain condition on prime ideals.

6 A Coda on Pseudo-valuation Rings

A ring R is called a *right pseudo-valuation ring* (also known as a *right pseudo-chain ring*) if for any $a \in R \smallsetminus \operatorname{U}(R)$ and for any right ideals $\mathfrak{A}$ and $\mathfrak{B}$ of R, we have

$$\mathfrak{A} \subseteq \mathfrak{B} \quad \text{or} \quad \mathfrak{B}a \subseteq \mathfrak{A}.$$

This class of rings was introduced in [29] as a noncommutative generalization of commutative pseudo-valuation rings, which have been frequently studied, beginning with [18,19]. See [2] for a survey of the literature. Right pseudo-valuation rings—like right annelidan rings, right Bézout rings, and right distributive rings—are a natural generalization of right uniserial rings. We note that a ring is right uniserial if and only if it is a right Bézout right pseudo-valuation ring, if and only if it is a right distributive right pseudo-valuation ring.

A right pseudo-valuation ring need not be right annelidan, nor conversely. Nevertheless, there are a number of similarities between these two classes of rings. A common sufficient condition is: if R is a local ring whose maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^2 = 0$, then R is annelidan and a pseudo-valuation ring. Common necessary conditions: the Köthe Conjecture has an affirmative answer for both the class of right annelidan rings and the class of right pseudo-valuation rings, the upper and lower nilradicals coincide for any right annelidan or right pseudo-valuation ring, and $\operatorname{RZD}(R) \subset R$ is a completely prime ideal whenever R is right annelidan or a right pseudo-valuation ring.

Theorem 6.1 *Let R be a right pseudo-valuation ring. Then R is local. Let $\mathfrak{m}$ be the maximal ideal of R.*

(i) *If $\operatorname{ann}_\ell^R(\mathfrak{m}) = (0)$, then R is right annelidan.*
(ii) *If $\operatorname{RZD}(R) \subsetneqq \mathfrak{m}$, then R is right annelidan.*

Proof By [29, Theorem 1.1], R is local. To prove (i), suppose R is not right annelidan. By [25, Proposition 2.1(v)], there exist $a, b \in R$ such that $\operatorname{ann}_\ell^R(a) \notin \operatorname{ann}_\ell^R(b)$ and $aR \notin bR$. Choose $y \in \operatorname{ann}_\ell^R(a) \setminus \operatorname{ann}_\ell^R(b)$. By [29, Theorem 1.1(e)], $b\mathfrak{m} = a\mathfrak{m}$. Therefore, $yb\mathfrak{m} = \{0\}$, and since $yb \neq 0$, it follows that $\operatorname{ann}_\ell^R(\mathfrak{m}) \neq (0)$. The proof of (i) is complete.

Obviously, $\mathrm{RZD}(R) \subsetneqq \mathfrak{m}$ implies $\operatorname{ann}_\ell^R(\mathfrak{m}) = (0)$, and thus (ii) is an immediate consequence of (i). ∎

In particular, if R is a pseudo-valuation ring whose Jacobson radical contains a regular element, then R is annelidan. This corollary can also be deduced from [29, Theorem 3.7] and Proposition 2.3(ii).

For any ideal $\mathfrak{m}$ of a ring R, as we observed in the proof of Theorem 6.1, if $\mathrm{RZD}(R) \subsetneqq \mathfrak{m}$ then $\operatorname{ann}_\ell^R(\mathfrak{m}) = (0)$. The converse is false, even when R is a uniserial ring and $\mathfrak{m}$ is its maximal ideal. Indeed, as shown in [9] (see also [7, Theorem 4.3]), there exists a prime uniserial ring R that is not a domain whose maximal ideal $\mathfrak{m}$ is its only completely prime ideal. Since R is prime, $\operatorname{ann}_\ell^R(\mathfrak{m}) = (0)$; nonetheless, $\mathrm{RZD}(R) = \mathfrak{m}$ (*e.g.,* by Theorem 2.1(ii)).

Recall that a ring R is called *right Kasch* if every simple right R-module is isomorphic to a right ideal of R.

Corollary 6.2 *A right pseudo-valuation ring is right Kasch or right annelidan.*

Proof Suppose R is a right pseudo-valuation ring that is not right annelidan. Then R is a local ring; let $\mathfrak{m}$ denote its maximal ideal. By Theorem 6.1, there exists an element $a \in \operatorname{ann}_\ell^R(\mathfrak{m}) \setminus \{0\}$. Therefore, $a\mathfrak{m} = \{0\}$, and the unique simple right R-module is $R/\mathfrak{m} \cong aR$, whence R is right Kasch. ∎

The literature on pseudo-valuation rings is strangely bereft of examples that are not annelidan. The following two examples show that in Theorem 6.1(i) the hypothesis $\operatorname{ann}_\ell^R(\mathfrak{m}) = (0)$ is indispensable, even for finite commutative local rings.

Example 6.3 Let $K = \{1, a, b, c\}$ be the Klein 4-group, and let R be the group algebra $R = \mathbb{F}_2 K$. We will show that R is a (commutative) pseudo-valuation ring that is not annelidan.

Put $\mathfrak{m} = \operatorname{rad}(R)$, the augmentation ideal of this local group algebra. Note that $\mathfrak{m}^2$ is the unique minimal ideal of R. According to [29, Theorem 1.1(d)], in order to prove R is a pseudo-valuation ring, it is enough to show that for all $x, y \in \mathfrak{m}$ either $xR \subseteq yR$ or $y\mathfrak{m} \subseteq x\mathfrak{m}$. If $z \in \mathfrak{m}^2$, then $z\mathfrak{m} = (0)$, and if $z \in \mathfrak{m} \setminus \mathfrak{m}^2$ then $z\mathfrak{m} = \mathfrak{m}^2$. Thus, given any $x, y \in \mathfrak{m}$, we have $x\mathfrak{m} = y\mathfrak{m}$ except in two cases:

(i) $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $y \in \mathfrak{m}^2$;
(ii) $x \in \mathfrak{m}^2$ and $y \in \mathfrak{m} \setminus \mathfrak{m}^2$.

In case (i), $y\mathfrak{m} \subset x\mathfrak{m}$. In case (ii), $xR \subset yR$. Thus, R is a pseudo-valuation ring.

On the other hand, the ideals $\operatorname{ann}_r^R(1 + a)$ and $\operatorname{ann}_r^R(1 + b)$ are incomparable; therefore, R is not lineal, and in particular it is not annelidan.

Example 6.4 Let

$$R = \mathbb{F}_2[x, y]/(xy, x^3, y^3, x^2 - y^2).$$

(We will write x and y for their images in the factor ring R.) Then R is a commutative local ring with maximal ideal $\mathfrak{m} = (x, y)$. Again $\mathfrak{m}^2$ is the unique minimal ideal of R, and it follows as in Example 6.3 that R is a pseudo-valuation ring. Since $\mathrm{ann}_r^R(x)$ and $\mathrm{ann}_r^R(y)$ are incomparable, R is not lineal (and thus not annelidan).

Examples 6.3 and 6.4 show that *pseudo-valuation* does not imply *lineal* (much less *annelidan*). *Lineal* does not imply *pseudo-valuation*, even among local rings, as our final three examples show.

Example 6.5 Define the free algebra $R_0 = \mathbb{F}_2\langle a, b, c, d\rangle$, and let I be the ideal of R_0 generated by the following set of elements:

$$\begin{array}{llll} a^2, & ab+ba, & ab+ac, & ac+b^2+ca, \\ ad, & bc+cb+da, & bd+c^2+db, & bd+cd, \\ cd+dc, & d^2, & xyz \ \text{ for all } \ x,y,z \in \{a,b,c,d\}. & \end{array}$$

Let $R = R_0/I$. We will write a, b, c, d for the images of these elements in R.

It was proved in [26, Example 5.10] that R is lineal, and obviously R is local with maximal ideal $\mathfrak{m} = aR + bR + cR + dR$. Since $bc \in bcR \setminus aR$ and $ab \in a\mathfrak{m} \setminus bc\mathfrak{m}$, by [29, Theorem 1.1(d)] the ring R is not a right pseudo-valuation ring. Since $bc \in Rbc \setminus Rd$ and $bd \in \mathfrak{m}d \setminus \mathfrak{m}bc$, the ring R is not a left pseudo-valuation ring.

Example 6.6 Let A be a local domain that is not a division ring; let M be the maximal ideal of A. Fix an integer $n \geqslant 2$, and put $R = A[x]/(x^n)$. Then R is local, with maximal ideal $\mathfrak{m}$ generated by the images in R of M and x. By [26, Proposition 2.4], the ring R is lineal. By [25, Proposition 2.3], the ring R is not right or left annelidan. Since $\mathrm{RZD}(R) \subsetneqq \mathfrak{m}$, Theorem 6.1 implies that R is not a right or left pseudo-valuation ring.

Example 6.7 Let R be the ring defined in Example 5.4. We claim that R is lineal but neither left nor right annelidan, and that R is neither a left pseudo-valuation ring nor a right pseudo-valuation ring. Of course, we deduced in Example 5.4 that R is a local ring whose Jacobson radical $\mathfrak{p}$ is not a strong right comparizer; therefore, by [29, Theorem 1.1(d)], R is not a right pseudo-valuation ring.

Given any nonzero element $\alpha \in yR + \mathfrak{p}^2$, we have $\mathrm{ann}_r^R(\alpha) = \mathfrak{p}$. Given any $\alpha \in \mathfrak{p}$ that is not contained in $yR + \mathfrak{p}^2$, we have $\mathrm{ann}_r^R(\alpha) = \mathfrak{p}^2$. Thus, every right annihilator in R is equal to $\{0\}$, $\mathfrak{p}^2$, $\mathfrak{p}$, or R. Consequently, R is lineal.

Now, $\mathrm{ann}_r^R(x) = \mathfrak{p}^2$ is not a right waist, since it is incomparable with yR. Moreover, $\mathrm{ann}_\ell^R(x) = yR + \mathfrak{p}^2$ is not a left waist, since it is incomparable with Rx. Thus, R is neither right nor left annelidan.

Finally, $\mathfrak{p}$ is the sole prime ideal of R, and $\mathfrak{p}x$ is not a left waist, since it is incomparable with Ry. By [29, Theorem 1.1(h)], R is not a left pseudo-valuation ring.

Of course, the ring in Example 6.7 is Kasch, so even within the class of lineal rings, the converse of Corollary 6.2 is false.

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