Canad. Math. Bull. Vol. **19** (2), 1976

ALTERNATING CHEBYSHEV APPROXIMATION WITH A NON-CONTINUOUS WEIGHT FUNCTION

BY
CHARLES B. DUNHAM

Let $[\alpha, \beta]$ be a closed interval and $C[\alpha, \beta]$ be the space of continuous functions on $[\alpha, \beta]$, For g a function on $[\alpha, \beta]$ define

$$\|g\| = \sup\{|g(x)| : \alpha \le x \le \beta\}$$

Let s be a non-negative function on $[\alpha, \beta]$. Let F be an approximating function with parameter space P such that $F(A, .) \in C[\alpha, \beta]$ for all $A \in P$. The Chebyshev problem with weight s is given $f \in C[\alpha, \beta]$, to find a parameter $A^* \in P$ to minimize $e(A) = \|s * (f - F(A, .))\|$ over $A \in P$. Such a parameter A^* is called best and $F(A^*, .)$ is called a best approximation to f.

We are interested in (F, P) such that best approximations are characterized by alternation of their error curve. In the case of ordinary Chebyshev approximation, in which $s = 1$, such (F, P) have been characterized by Rice [**2,** 17–21]. We will show that an alternating theory holds for Chebyshev approximation with respect to weight s for all such (F, P), providing s is upper semi-continuous.

DEFINITION. A function g is *upper semi-continuous* if $\{x : g(x) \ge r\}$ is closed for all real r.

From the definition it is clear that an upper semi-continuous function attains its supremum on any non-empty compact set. In the remainder of the paper s will be assumed to be upper semi-continuous. For $A \in P$, $|f - F(A, .)|$ is continuous, hence $s * |f - F(A, .)|$ is upper semi-continuous and attains its supremum on any non-empty compact subset of $[\alpha, \beta]$. In particular there must exist x such that $s(x) * |f(x) - F(A, x)| = e(A)$.

We show in this paragraph that any non-negative weight function r can be replaced by an equivalent upper semi-continuous weight functon. Let us define

$$s(x) = \max\{r(x), \limsup_{y \to x} r(y)\},$$

then s is non-negative and upper semi-continuous. Further it is easily seen that if g is continuous, $\|s * g\| = \|r * g\|$.

In Rice [2, 3 and 17] and in [1, 225] are defined property Z and property A

Received by the editors November 18, 1975 and, in revised form, September 5, 1975.

https://doi.org/10.4153/CMB-1976-023-9 Published online by Cambridge University Press

(of variable degree). We say that F has *degree* n at A if F has property Z of degree n at A and property A of degree n at A. We assume henceforth that F has a degree at all $A \in P$. To avoid trivial cases, we assume that the number of points at which s is nonzero exceeds the degree of F. By the results of Rice [2, 17–21], F having a degree is both necessary and sufficient for an alternating theory when the weight s is 1. A generalization of a result of de la Vallée-Poussin is

LEMMA 1. *Let F have degree n at A. Let $f - F(A),.)$ alternate in sign on $x_0 < \cdots < x_n$. Then for $F(B,.) \not\equiv F(A,.)$,*

$$\max\{s(x_i) * |f(x_i) - F(B, x_i)| : i = 0, \ldots, n\}$$
$$> \min\{s(x_i) * |f(x_i) - F(A, x_i)| : i = 0, \ldots, n\}.$$

The proof is identical to the proof of the corresponding result in [1].

THEOREM. *Let s be upper semi-continuous. Let F have degree n at A. A necessary and sufficient condition that $F(A,.)$ be best to f is that $s*(f-F(A,.))$ alternate n times.*

Proof. *Sufficiency*; follows directly from Lemma 1.

Necessity: Necessity is proven similarly to the proof of necessity in [1]. Define

$$E(A, x) = w(x, f(x), F(A, x)) = s(x) * (f(x) - F(A, x))$$

In defining η we define

$$\sigma_i = \sup\{(-1)^{i+1} E(A, x) : x \in I_i\}.$$

Let

$$K_i = \{x : x \in I_i, \operatorname{sgn}(f(x) - F(A, x)) = (-1)^{i+1} \quad \text{or} \quad 0\}.$$

By continuity of $f - F(A,.)$, K_i is closed. We have

$$\sigma_i = \sup\{(-1)^{i+1} E(A, x) : x \in K_i\}.$$

By upper semi-continuity of $|s*(f-F(A,.))|$, σ_i is attained on a point y_i of K_i. Define

$$\eta = e(A) - \max\{\sigma_i : i = 0, \ldots, m\},$$

then $\eta > 0$. By upper semi-continuity of $|E(A,.)|$, J_i defined in [1] is closed for all i. We choose $\varepsilon_1 = \eta/(2\|s\|)$, so that $\|F(A,.) - F(B,.)\| < \varepsilon_1$ implies $\|w(.,f, F(A,.)) - w(.,f, F(B,.))\| < \eta/2$. With these modifications the proof of necessity in [1] goes through for the case of this paper.

Best approximations are unique, for if $F(A,.)$ is best, $s*(f-F(A,.))$ alternates and by lemma 2, $F(A,.)$ is uniquely best.

References

1. C. B. Dunham. *Chebyshev approximation with respect to a weight function*, J. Approximation Theory **2** (1969), 223–232.

2. J. R. Rice, *The Approximation of Functions*, Volume **2**, Addison-Wesley, Reading, Mass., 1969.

Computer Science Department,
University of Western Ontario,
London, Ontario, Canada