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ON THE KUIPER-KUO THEOREM

CHUAN I. CHU

ABSTRACT. In this note we shall give a simple and more direct proof of the Kuiper-Kuo Theorem. Also, we shall simplify Kuiper's proof of the Morse Lemma.

1. **Introduction.** In the studying of C^0- or C^1-equivalence of jets, Kuiper [5] and Kuo [6] constructed vector fields and local flows to obtain the required homeomorphism or diffeomorphism.

In this note we shall use the technique by Bochner [1] to give an explicit formula of the vector field which is simpler than those used by Kuiper and Kuo. This vector field also provides us a method to show that two jets are C^0-equivalent.

As an application of this vector field, we shall give a simple proof of Kuiper's version of the Morse lemma [4].

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2. **The Kuiper-Kuo Theorem.** For a C^k function f: $\mathbb{R}^n \longrightarrow \mathbf{R}$, let ∇f: $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ be the unique C^{k-1} mapping from $\mathbf{R}^n$ into $\mathbf{R}^n$ defined by $df(x)(y) = \nabla f(x) \cdot y$ for all $y \in \mathbf{R}^n$. Here $\nabla f(x) \cdot y$ is the usual inner product in $\mathbf{R}^n$.

We denote by $J^k(n,1)$ the space of all k-jets at 0 of all C^k functions f: $\mathbf{R}^n \longrightarrow \mathbf{R}$ such that $f(0) = 0$.

THEOREM 2.1. *Let $f \in J^k(n,1)$ satisfy the Kuiper-Kuo condition*

$$\|\nabla f(x)\| \geqq c\|x\|^{k-\delta} \tag{1}$$

for all x in a neighborhood U of 0, where $0 < c$ and $0 < \delta \leqq 1$ are constants. Let $g: U \longrightarrow \mathbf{R}$ be a C^2 function such that $g(x) = O(\|x\|^{k+1}), \nabla g(x) = O(\|x\|^k)$. Then $f + g$ is C^0-equivalent to f. That is, there exists a local homeomorphism ψ at 0 such that $(f+g)(\psi(x)) = f(x)$. Here ψ is defined on some neighborhood U_1 of 0, $U_1 \subseteq U$.

PROOF. For $0 \leqq t \leqq 1$, $\|x\|$ small, define $B(0,t) = 0$, and

$$B(x,\ t) = \frac{g(x)}{\|\nabla f(x)\|^2 + t\nabla g(x) \cdot \nabla f(x)} \nabla f(x),\ x \neq 0 \tag{2}$$

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Note that for $\|x\|$ small,

$$\begin{aligned}\|\nabla f(x)\|^2 + t\nabla g(x)\cdot\nabla f(x) &\geqq \|\nabla f(x)\|^2 - t\|\nabla f(x)\|\,\|\nabla g(x)\| \\ &\geqq \|\nabla f(x)\|(\|\nabla f(x)\| - \|\nabla g(x)\|) \\ &\geqq c'\|x\|^{2k-2\delta}.\end{aligned}$$

Here c' is some positive constant. Thus $B(x,t)$ is well defined for $\|x\|$ small and $0 \leqq t \leqq 1$. Also B is C^1 at (x,t) for $x \neq 0$, $\|x\|$ small.

Next, we consider for $x \neq 0$, $\|x\|$ small,

$$\begin{aligned}\frac{\|B(x,t)\|}{\|x\|} &\leqq \frac{|g(x)|}{\|x\|(\|\nabla f(x)\| - \|\nabla g(x)\|)} \\ &\leqq c_1\frac{|g(x)|}{\|x\|^{k+1-\delta}} \qquad (3)\\ &\leqq c_2\|x\|^{\delta}.\end{aligned}$$

Here c_1 and c_2 are positive constants. Thus B is uniformly continuous for $0 \leqq t \leqq 1$.

Note that this also shows that B is differentiable at $(0,t)$ and $dB(0,t) = 0$ for all t.

Consider the differential equation

$$\frac{\partial\phi}{\partial t}(x,\ t) = -B(\phi(x,\ t),t),\ \phi(x,\ 0) = x. \qquad (4)$$

Since B is continuous, (4) has a solution. We have to show that (4) has a unique solution.

Suppose $x \neq 0$. Then (4) has a unique solution $\phi(x,t)$ with initial condition x since $B(x,t)$ is C^1 for $x \neq 0$.

Then from (3) we have

$$\left\|\frac{\partial\phi}{\partial t}(x,\ t)\right\| = \|B(\phi(x,\ t),\ t)\| \leqq a\|\phi(x,\ t)\|$$

for some positive constant a and $\|x\|$ small.

Now

$$\begin{aligned}-\frac{\partial}{\partial t}(\|\phi(x,\ t)\|^2) &= -2\phi(x,\ t)\cdot\frac{\partial\phi}{\partial t}(x,\ t) \\ &\leqq \|\phi(x,\ t)\|^2 + \left\|\frac{\partial\phi}{\partial t}(x,\ t)\right\|^2 \\ &\leqq (1+a^2)\|\phi(x,\ t)\|^2.\end{aligned}$$

Thus

$$\frac{\partial}{\partial t}(\|\phi(x,\ t)\|^2) \geqq -b\|\phi(x,\ t)\|^2,$$

here $b = 1+a^2$. Hence $\|\phi(x,t)\| \geqq e^{-bt}\|x\|$. Thus a solution curve with initial condition $x \neq 0$ will not meet a solution curve with initial condition $x = 0$. Thus the solution curve of (4) is unique and hence ϕ is continuous. (c.f. Hartman [2])

Define $F(x,t) = f(x) + tg(x)$ for $x \in U, 0 \leqq t \leqq 1$.

Then it is straightforward to see that $d/dt\,\big(F(\phi(x,t),t)\big) \equiv 0$. Thus $F(\phi(x,t),t) \equiv$ const. Hence $\psi(x) = \phi(x,1)$ yields the required local homeomorphism. ∎

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THEOREM 2.2. *If $\delta = 1$ then the assumption of g can be taken to be: g is C^2 and $g(x) = o(\|x\|^k)$ and $\nabla g(x) = o(\|x\|^{k-1})$.*

REMARK 2.3. Theorem 2.1 and Theorem 2.2 lead to the C^0-sufficiency of k-jets for C^{k+1} functions and C^k functions respectively. For C^0-sufficiency of jets, we refer to the works of Koike [3], Kuiper [5] and Kuo [6].

REMARK 2.4. The vector field $B(x, t)$ could also be applied to some g such that $g(x) = O(\|x\|^k)$ as well.

Let $f(x_1, x_2) = x_1^4 + x_2^4 \in J^4(2, 1)$ and $g(x_1, x_2) = bx_1^2x_2^2$ with $0 < b < 2$. Then it is easy to see that

$$\| \nabla f(x_1, x_2)\| \geqq 2(x_1^2 + x_2^2)^{3/2} > b(x_1^2 + x_2^2)^{3/2} \geqq \| \nabla g(x_1, x_2)\|$$

if $(x_1, x_2) \neq (0, 0)$. In this case, $B(x, t)$ is defined and uniformly continuous for $0 \leqq t \leqq 1$ and $\|x\| < 1$ say.

Hence $x_1^4 + x_2^4$ is C^0-equivalent to $x_1^4 + x_2^4 - bx_1^2x_2^2$.

3. **C^k-smoothness of B.** In this section, we shall discuss some sufficient conditions that yield the C^k-differentiability of B.

PROPOSITION 3.1. *Let $f \in J^k(n, 1)$ be such that $\| \nabla f(x)\| \geqq c\|x\|^{k-1}$, $f(x) = O(\|x\|^{k_0})$ for $\|x\|$ small and $k_0 \leqq k$. Let p be a real number such that $p \geqq k - k_0$. Suppose g is C^2 defined on a small neighborhood of 0 with the property that $g(x) = o(\|x\|^{k+p})$, $\nabla g(x) = o(\|x\|^{k+p-1})$ and $d(\nabla g)(x) = o(\|x\|^{k+p-2})$. Then B is C^1.*

PROOF. B is clearly C^1 for $x \neq 0$, $\|x\|$ small. Also we have seen in the proof of Theorem 2.1 that $\partial B/\partial x\ (0, t) = 0$. To show $\partial B/\partial x\ (x, t)$ is continuous at $(0, t)$ we consider for $x \neq 0$, $y \in \mathbf{R}^n$ that

$$\frac{\partial B}{\partial x}(x,\ t)(y) = \frac{dg(x)(y)}{\| \nabla f(x)\|^2 + t \nabla g(x) \cdot \nabla f(x)} \nabla f(x) + \frac{g(x)}{\| \nabla f(x)\|^2 + t \nabla g(x) \cdot \nabla f(x)} \times d(\nabla f)(x)(y)$$
$$- g(x) \frac{2 \nabla f(x) \cdot d(\nabla f)(x)(y) + td(\nabla g)(x)(y) \cdot \nabla f(x) + t \nabla g(x) \cdot d(\nabla f)(x)(y)}{\left(\| \nabla f(x)\|^2 + t \nabla g(x) \cdot \nabla f(x)\right)^2} \nabla f(x).$$

Thus,

$$\left\|\frac{\partial B}{\partial x}(x,\ t)\right\| \leqq \frac{\| \nabla g(x)\|}{\| \nabla f(x)\| - \| \nabla g(x)\|} + \frac{|g(x)|\ \|d \nabla f(x)\|}{\| \nabla f(x)\|^2 + t \nabla g(x) \cdot \nabla f(x)}$$
$$+ \frac{2|g(x)|\ \|d(\nabla f)(x)\|}{\left(\| \nabla f(x)\| - \| \nabla g(x)\|\right)^2} + \frac{|g(x)|\ \|d \nabla g(x)\|}{\left(\| \nabla f(x)\| - \| \nabla g(x)\|\right)^2}$$
$$+ \frac{\| \nabla g(x)\|\ \|d(\nabla f)(x)\|\ |g(x)|}{\| \nabla f(x)\| \left(\| \nabla f(x)\| - \| \nabla g(x)\|\right)^2}.$$

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Then it is easy to see that the right hand side of the above inequality is $o(\|x\|^{p+k_o-k})$. Thus, $\partial B/\partial x\,(x,t) \longrightarrow 0$ uniformly for $0 \leqq t \leqq 1$. Therefore, $\partial B/\partial x\,(x,t)$ is continuous at $(0,t)$. Similarly, we can show that $\partial B/\partial t$ is continuous. Thus B is C^1. ∎

COROLLARY 3.2. *Suppose $f \in J^k(n,1)$ is homogeneous of degree k. Assume that f and g satisfy the conditions in Proposition* 3.1. *Then f is C^1-equivalent to $f+g$. Hence f is C^1-sufficient for C^{k+p} functions* (cf. Theorem 2 of Kuiper [5]).

LEMMA 3.3. *Let U be an open set in $\mathbf{R}^n$ containing 0. Suppose that Q: $U \longrightarrow \mathbf{R}$ is C^s $(s \geqq 1)$ and such that $|Q(x)| \geqq c\|x\|^r$ for $x \in U, c > 0, r > 0$. Assume that $Q(x) = O(\|x\|^{r_o})$ where $0 < r_o \leqq r$. Let k be a positive integer such that $k \geqq 2^s r - (2^s-1)r_o + s$. For C^k map P: $U \longrightarrow \mathbf{R}^m$ such that $j^k(P) = 0$, define H: $U \longrightarrow \mathbf{R}^m$ by*

$$H(x) = \frac{P(x)}{Q(x)},\ x \neq 0,\ H(0) = 0.$$

Then H is C^s.

PROOF. We prove by induction on s. First, we assume that $s = 1$.

Clearly, H is C^1 at $x \neq 0, x \in U$. Now for $x \neq 0, x \in U$, consider

$$\frac{\|H(x)\|}{\|x\|} = \frac{\|P(x)\|}{\|x\|\ |Q(x)|} \leqq \frac{\|P(x)\|}{c\|x\|^{r+1}}.$$

Since $j^k(P) = 0$, $o(P(x)) = k \geqq 2r - r_o + 1 \geqq r + 1$. Hence

$$\frac{\|H(x)\|}{\|x\|} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow 0.$$

This shows that H is differentiable at 0 and $dH(0) = 0$.

Now for $x \neq 0, x \in U$, we have

$$dH(x) = \frac{dP(x)}{Q(x)} - \frac{dQ(x)}{Q^2(x)}\,P(x)$$

and hence

$$\|dH(x)\| \leqq \frac{\|dP(x)\|}{|Q(x)|} + \frac{\|dQ(x)\|}{|Q(x)|^2}\,\|P(x)\|.$$

Again, since $k \geqq 2r - r_o + 1$, $\|dH(x)\| \longrightarrow 0$ as $x \longrightarrow 0$. This shows that H is C^1.

Assume the lemma holds for $s-1, s \geqq 2$. Suppose that $k \geqq 2^s r - (2^s - 1)r_o + s$. Then since $k \geqq 2(r - r_o) + r_o + 1, H$ is C^1.

Define H_1, H_2: $U \longrightarrow L(\mathbf{R}^n, \mathbf{R}^m)$ (= the linear space of all linear maps from $\mathbf{R}^m$ into $\mathbf{R}^m$) by

$$H_1(x) = \frac{dP(x)}{Q(x)} \qquad x \neq 0,\ H_1(0) = 0$$

$$H_2(x) = \frac{dQ(x)}{Q^2(x)}\,P(x) \qquad x \neq 0,\ H_2(0) = 0.$$

Since $o(P(x)) = k$, $o(dP(x)) = k-1$. Also, by assumption we have $k - 1 \geqq 2^{s-1}(r - r_o) + r_o + s - 1$. Hence, by the induction hypothesis, H_1 is C^{s-1}. Also from the fact that $o(P(x)dQ(x)) = k + r_o - 1, O(Q^2(x)) = 2r_o$ and $k + r_o - 1 \geqq 2^{s-1}(2r) - (2^{s-1} - 1)(2r_o) + s - 1$ we have, by the induction hypothesis, H_2 is C^{s-1}. Thus from $dH(x) = H_1(x) - H_2(x)$ we have that dH is C^{s-1}. That is, H is C^s. ∎

COROLLARY 3.4. *Let $f \in J^r(n,1)$ satisfy $\| \nabla f(x)\| \geqq c\|x\|^{r-1}$ and $f(x) = O(\|x\|^{r_o})$ for $\|x\|$ small, $0 < r_o \leqq r, c > 0, r \geqq 2$. Let $k \geqq 2^{s+1}(r - r_o) + s + r_o - 1$ and g be C^k with $j^k g = 0$. Then $B(x,t)$ defined by* (2) *is C^s for $0 \leqq t \leqq 1$ and $\|x\|$ small. Here we assume $s \geqq 1$.*

PROOF. Take $Q(x,t) = \| \nabla f(x)\|^2 + t \nabla g(x) \cdot \nabla f(x)$ and $P(x) = g(x) \nabla f(x)$. Then apply the above lemma to $B(x,t)$ and $\partial B/\partial t\,(x,t)$. ∎

REMARK 3.5. Lemma 3.3 is a sufficient condition for general mappings. However, in our case, $B(x,t)$ is rather special. We can apply the technique used by Taken [7] to improve the condition to $k \geqq s(r - r_o) + s + r - 1$. For $s = 1$, this is already shown in the proof of Proposition 3.1.

COROLLARY 3.6. *Suppose $f \in J^r(n,1)$ is homogeneous of degree r satisfying $\| \nabla f(x)\| \geqq c\|x\|^{r-1}$ for $\|x\|$ small. Let $k \geqq r$ and g be C^k with $j^k g = 0$. Then B defined by* (2) *is C^{k-r+1}.*

COROLLARY 3.7. *Same as in Corollary* 3.6 *with $r = r_o = 2$. Then B is C^{k-1}.*

EXAMPLE 3.8. $-x_1^2 + x_2^2 + x_2^3$ is C^1-equivalent to $-x_1^2 + x_2^2$. However, $-x_1^2 + x_2^2 + x_2^{7/2}$ is C^2-equivalent to $-x_1^2 + x_2^2$.

THEOREM 3.9 (KUIPER-MORSE). *Let $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}$ be a C^k function, $k \geqq 2$. Assume that 0 is an isolated non-degenerate critical point of f, $f(0) = 0$. Put $f_2 = j^2(f)$. Then f is C^{k-1}-equivalent to f_2. That is, there exists a neighborhood U of 0 in $\mathbf{R}^n$ and a C^{k-1} diffeomorphism $\psi\colon U \longrightarrow \psi(U)$ such that $f(x) = f_2(\psi(x))$ for $x \in U$.*

PROOF. Let $f_k = j^k(f)$ and $g = f - f_k$. If $g \equiv 0$, then $f = f_k$; so we can apply Part A of Kuiper [4].

Assume that $g \not\equiv 0$. Then g is C^k and $j^k g = 0$. Since f_2 is non-degenerate, $\| \nabla f_2(x)\| \geqq c\|x\|$ for $\|x\|$ small, $c > 0$. Hence $\| \nabla f_k(x)\| \geqq c_1\|x\|$ for some constant $c_1 > 0$ and $\|x\|$ small.

Define

$$B(x,\ t) = \frac{g(x)}{\| \nabla f_k(x)\|^2 + t \nabla g(x) \cdot \nabla f_k(x)} \nabla f_k(x) \text{ for } x \neq 0$$

and $B(0,t) = 0, 0 \leqq t \leqq 1$.

Then by Corollary 3.4, B is C^{k-1}. Hence its local flow ϕ is also C^{k-1}. (c.f. Hartman [2]). That is, $f = f_k + g$ is C^{k-1}-equivalent to f_k. By part A of Kuiper [4], f_k is C^{k-1}-equivalent to f_2. Hence by transitivity, f is C^{k-1}-equivalent to f_2. ∎

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Department of Mathematics
Hong Kong Baptist College
224 Waterloo Road, Kowloon
Hong Kong