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INVERTIBLE ELEMENTS IN THE DIRICHLET SPACE

BY
LEON BROWN

ABSTRACT. It is shown that if a function in the Dirichlet space is invertible then it is cyclic with respect to the operator of multiplication by the identity function.

1. **Introduction.** By the *Dirichlet* space D, we mean the collection of functions analytic in the open unit disc Δ whose derivatives are square summable with respect to area meaure. Equivalently, these are the functions that map Δ onto a region of finite area (counting multiplicity). In order to study D, we introduce the *Bergman* space B. This is the set of functions analytic in Δ that are square integrable with respect to area measure. With the L^2 norm,

$$\|f\|_B^2 = \int_\Delta |f|^2 dA,$$

B is a Hilbert space. D is a Hilbert space with the norm

$$\|f\|_D^2 = |f(0)|^2 + \|f'\|_B^2.$$

In [3] the author and A. L. Shields studied the question of classifying those functions in D which are cyclic with respect to the operator M_z; $M_z f = zf$, that is, those functions f such that polynomial multiples of f are dense in D. In that paper the following question was presented (Question 4, p. 276): If E is a "Banach space of analytic functions" and f is invertible in E must f be cyclic? This question (for the Bergman space) was posed in [8] (see Question 25 on page 114). Harold S. Shapiro [7] used the term "weakly invertible" in place of cyclic. This question can be rephrased as follows: does invertibility imply weak invertibility? In general the answer is no. A counterexample is presented by Shamoyan [6]. For the Dirichlet space the answer was, until now, not known, even under the additional hypothesis that f be bounded (see question 9, page 282 of [3]). Our goal is to solve this problem: for the *Dirichlet* space, every invertible function is weakly invertible (i.e. cyclic).

In the second section we present some miscellaneous results and use Carleson's formula to analyze the "cut-off functions". We prove the main theorem in the third section.

2. **Miscellaneous results and Carleson's formula.** If $f \in D$, let $[f]$ denote the closure in D of polynomial multiples of $f = \overline{\{Pf : P \in \mathcal{P}\}}$, when $\mathcal{P}$ denotes the set of polynomials.

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LEMMA 1. (Richter and Shields [5, Lemma 3]). *If $f \in D$, $\varphi \in D \cap H^\infty$, and $\varphi f \in D$, then $\varphi_r f \longrightarrow \varphi f$, $\left(\varphi_r(z) = \varphi(rz)\right)$, and $\varphi f \in [f]$.*

LEMMA 2. *If $\varphi_n \in H^\infty \cap D$ and $(\varphi_n f)(z) \longrightarrow 1 (z \in \Delta)$ and $\|\varphi_n f\|_D < M$ then f is cyclic.*

PROOF. By Proposition 2 in [3], a sequence g_n in D converges weakly to $g \in D$ if and only if $g_n(z) \longrightarrow g(z) (z \in \Delta)$ and $\|g_n\| \leq M$ for some constant M. Thus $\varphi_n f \longrightarrow 1$ weakly. By Lemma 1, $\varphi_n f \in [f]$. Since $[f]$ is weakly closed we have 1 in $[f]$. Since polynomials are dense in D, 1 is cyclic in D and thus by Proposition 5 in [3], f is cyclic in D.

REMARK: Note that φ_n does not have to be a multiplier of D. However, $H^\infty \not\subset D$ and φ_n must be in D.

We recall a formula of Carleson [4] for the Dirichlet integral of a function f (that is for $\|f'\|_B^2 = \int\int |f'|^2\, dx\, dy$). This formula is the sum of three nonnegative terms, involving respectively the Blashke factor of f, the singular inner factor, and the outer factor. We reproduce only the third of these. We shall write $f(t)$ instead of $f(e^{it})$ for the boundary values of f. The boundary values of f exist because $D \subset H^2$. We introduce the following notation:

$$(*) \qquad I(f) = I(f; x, t) = (\log|f(x+t)| - \log|f(x)|) \cdot (|f(x+t)|^2 - |f(x)|^2).$$

Then from Carleson's formula we have

$$(**) \qquad \frac{1}{8\pi}\int_0^\pi \left(\sin\frac{1}{2}t\right)^{-2} dt \int_{-\pi}^{\pi} I(f; x, t)dx \leq \|f'\|_B^2 (f \in D),$$

with equality when f is an outer function. Note that $I(f; x, t)$ is nonnegative for all x, t since the two terms on the right side of $(*)$ have the same sign. Hence $I(f)$ is unchanged if we replace each of these terms by its absolute value.

DEFINITION: (cutoff functions) If $f \in D$ and f is an outer function then we set

a) $\varphi_n(z) = \varphi[f; n](z) = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|\varphi_n^*(e^{it})|dt\}$ where

$$|\varphi_n^*(e^{it})| = |\varphi_n(t)| = \begin{cases} n & \text{if } |f(t)| \geq n \\ |f(t)| & \text{if } |f(t)| \leq n \end{cases}$$

b) Similarly we define $\phi(f)(z) = \phi(z)$ with

$$|\phi^*(e^{it})| = |\phi(t)| = \begin{cases} |f(t)| & \text{if } |f(t) \geq 1 \\ 1 & \text{if } |f(t)| \leq 1 \end{cases}$$

LEMMA 3.

a) $\varphi_n \in D$ and $\|\varphi_n\|_D \leq \|f\|_D$

b) $\|\phi'\|_B \leq \|f'\|$, so $\phi \in D$.

PROOF. a) $|\varphi_n(0)| = \varphi_n(0) = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\varphi_n(t)|\, dt\} \leq \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(t)|\, dt\}$ $= |f(0)|$. (since f is an outer function).

To complete the proof we show that $\|\varphi'_n\|_B \leq \|f'\|_B$. Since φ_n are outer functions we may compute $\|\phi'_n\|_B$ from (**). Thus it would be sufficient to prove that $I(\varphi_n) \leq I(f)$ for all x, t.

First we show that

$$(1) \qquad \Big|\, |\varphi_n(x+t)|^2 - |\varphi_n(x)|^2 \,\Big| \leq \Big|\, |f(x+t)|^2 - |f(x)|^2 \Big|.$$

We consider the following four cases.

(i) If $|f(x+t)| \leq n$ and $|f(x)| \leq n$ then $\varphi_n(x+t) = |f(x+t)|$ and $\varphi_n(x) = |f(x)|$ and (1) follows.

(ii) If $|f(x+t)| \geq n$ and $|f(x)| \geq n$ then $\varphi_n(x+t) = n$ and $\varphi_n(x) = n$ and (1) follows.

(iii) If $|f(x+t)| \geq n$ and $|f(x)| \leq n$ then we have

$$\begin{aligned} n^2 - |\varphi_n(x)|^2 &\leq |f(x+t)|^2 - |f(x)|^2 \\ &= \Big|\, |f(x+t)|^2 - |f(x)|^2 \Big|. \end{aligned}$$

(iv) If $|f(x+t)| \leq n$ and $|f(x)| \geq n$ then (1) follows in a manner similar to (iii).

The proof that

$$\Big|\log|\varphi_n(x+t)| - \log|\varphi_n(x)|\Big| \leq \Big|\log|(f(x+t)| - \log|f(x)|\Big|$$

is treated in a similar manner. Thus $I(\varphi_n) \leq I(f)$ which completes the proof of a)

b) We again consider four cases, i) and ii) are similar to ii) and i) of a).

(iii) If $|f(x+t)| \geq 1$ and $|f(x)| \leq 1$ then $\Big|\, |\phi(x+t)|^2 - |\phi(x)|^2 \Big| = |f(x+t)|^2 - 1 \leq \Big|\, |f(x+t)|^2 - |f(x)|^2 \Big|$.

(iv) is similar to (iii)

The proof that $|\log|\phi(x+t)| - \log|\phi(x)|| \leq |\log|f(x+t)| - \log|f(x)|\,|$ is treated in a similar manner.

This completes the proof of b).

LEMMA 4. *If f is invertible in D then $\varphi_1 = \varphi[f, 1] \in [f] \cap H^\infty$ and φ_1 is invertible.*

PROOF. We may assume $f(0) > 0$. If f is invertible then f and $1/f$ are outer functions. Let $\psi = \varphi[1/f; 1]$ be cut-off function of $1/f$. Thus $\psi \in D \cap H^\infty$ and $\varphi_1 = f\psi \in D \cap H^\infty$ (Lemma 3). Lemma 1 implies that $\varphi_1 \in [f]$. The fact that $\varphi_1^{-1} = \phi[1/f]$ completes the proof.

3. **The main theorem.**

THEOREM. *If f is invertible in D then f is cyclic in D.*

PROOF. The fact that if $g \in [f]$ and g is cyclic, then f is cyclic and Lemma 4 implies that we may assume that without loss of generality $f \in H^\infty$, $\|f\|_\infty \leq 1$ and $f(0) > 0$. Let $\psi_n = \varphi[1/f; n]$. By Lebesgue's bounded convergence theorem $|(f\psi_n)(t)|$ converge in L^1 to $|f \cdot \frac{1}{f}| = 1$. Thus $(f\psi_n)(z) \longrightarrow 1 (z \in \Delta)$. In particular $(f\psi_n)(0)$ is bounded. We will show that $\|(f\psi_n)'\|_B$ is bounded. Note that $(f\psi_n)' = f\psi_n' + f'\psi_n$. We have

$$\begin{aligned}
\|f\psi_n'\|_B &\leq \|f\|_\infty \|\psi_n'\|_B \leq \|\psi_n'\|_B \\
&\leq \|(1/f)'\|_B \\
|\psi_n(z)| &= \exp\{\frac{1}{2\pi}\int_{2\pi}^{\pi} P_r(\theta - t)\ \log|\psi_n(t)|dt\} \\
&\leq \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\ \log|(1/f)(t)|\ dt\} \\
&= |(1/f)(z)|,\ z \in \Delta.
\end{aligned}$$

Thus $\|f'\psi_n\|_B \leq \|f'/f\|_B = \|ff'/f^2\|_B \leq \|f'/f^2\|_B = \|(1/f)'\|_B$ and we have $\|f\psi_n\|_D$ are uniformly bounded. An application of Lemma 2 completes the proof that f is cyclic.

We remark that this question is still open for the Bergman space. If one assumes that f is in the Nevanlinna class then it is known that if f is invertible in B then f is weakly invertibility in B ([1], [2]).

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Mathematics Department
Wayne State University
Detroit, Michigan 48202