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Equivalence of L_p Stability and Exponential Stability of Nonlinear Lipschitzian Semigroups

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Abstract. L_p stability and exponential stability are two important concepts for nonlinear dynamic systems. In this paper, we prove that a nonlinear exponentially bounded Lipschitzian semigroup is exponentially stable if and only if the semigroup is L_p stable for some $p > 0$. Based on the equivalence, we derive two sufficient conditions for exponential stability of the nonlinear semigroup. The results obtained extend and improve some existing ones.

1 Introduction

The abstract Cauchy problem in a Banach space X

$$(1.1)\qquad \begin{cases} u'(t) = Au(t), & t > 0, \\ u(0) = x \in C \subseteq X, \end{cases}$$

has been widely discussed by means of a semigroup of operators (see, for instance, [1–3, 5–7, 13, 14, 16, 17]), where C is a closed subset of X. If A is linear, the abstract Cauchy problem (1.1) is well posed in the sense of the classical solution for $x \in D(A)$ if and only if $(A, D(A))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X. Moreover, Pazy [15] showed that the following asymptotic result holds for linear strongly continuous semigroups.

Theorem 1.1 *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on Banach space X. Then the following two statements are equivalent:*

$$(D_1)\qquad \int_0^\infty \|T(t)x\|^p dt < \infty, \quad \text{for } x \in X \text{ and some } p \geq 1$$

$$(D_2)\qquad \|T(t)\| \leq Me^{-at}, \quad \text{for some } M \geq 1 \text{ and } a > 0.$$

For more information on the result, we can refer to [4, 8, 9, 14, 20].

If A is nonlinear, Peng [17] proved that the abstract Cauchy problem (1.1) is well posed in the sense of a strong solution introduced in [2] if and only if A generates an

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exponentially bounded Lipschitzian semigroup on $\overline{D(A)}$. As we all know, for non linear systems it is very important (and difficult) to investigate the asymptotic behaviors of their solution semigroups. Consequently, it is natural to ask whether the equivalence in Theorem 1.1 still holds for the nonlinear Lipschitzian semigroup. However, a counterexample was provided in [18] to illustrate that the equivalence may not hold. It means that the condition (D_1) is too weak to be equivalent to exponential stability for nonlinear semigroups. Conseqently, only some "strengthened-version" of condition (D_1) may be equivalent to exponential stability of nonlinear semigroups. In fact, Ichikawa [10] proved the equivalence of a nonlinear semigroup corresponding to the semilinear version of (1.1), *i.e.,*

$$\begin{cases} u'(t) = (A+B)u(t), & t > 0, \\ u(0) = x \in X, \end{cases} \tag{1.2}$$

where A generates a linear strongly continuous semigroup and $B\colon X \to X$ is a Lipschitz continuous operator with $B(0) = 0$. Particularly, he gave the following equivalence result of L_p stability and exponential stability for a nonlinearly perturbed semigroup corresponding to (1.2).

Theorem 1.2 ([10, Corollary 2.1]) *Let $g(\,\cdot\,)$ be a positive continuous function on $[0,\infty)$. Suppose that $X_t = X_0 = X, t \geq 0$ and $T(t,s) = T(t-s)$ for some one-parameter semigroup of nonlinear operators $(T(t))_{t\geq 0}$ generated by $A+B$ on X_0 satisfies*

$$\|T(t,s)x\| \leq g(t-s)\|x\|, \quad x \in X_0, t \geq s.$$

Then the two statements below are equivalent:

(N_1') $\displaystyle\int_0^\infty \|T(t)x\|^p dt \leq K^p\|x\|^p,\ x \in X_0,$ *for some $p > 0$ and $K > 0$;*

(N_2') $\|T(t)x\| \leq Me^{-at}\|x\|,\ x \in X_0,$ *for some $M \geq 1$ and $a > 0$.*

To the best of our knowledge, there are few papers investigating the asymptotic behaviors of nonlinear semigroups except [10]. Motivated by this, we attempt to give an equivalence characterization of L_p stability and exponential stability for nonlinear Lipschitzian semigroup associated with (1.1) investigated in recent years by [11, 12, 18, 19]. Actually, it is also significant to characterize the equivalence from the point of view of Lyapunov's direct method ,because there exist some situations where one may easily find Lyapunov functions that are not strictly positive but assure L_p stability, *i.e.*, (N_1') holds.

The paper is arranged as follows: in Section 2 we give some useful notions and basic results. In Section 3, we show the equivalence of L_p stability and exponential stability of a nonlinear exponentially bounded Lipschtizian semigroup. Based on the equivalence, we derive two further sufficient conditions for exponential stability of the nonlinear semigroup.

2 Preliminaries

Throughout this paper, X and Y are assumed to be Banach spaces over the same coefficient field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). Let C and D be respective subsets of X and Y. An operator T from C into D is called Lipschitz continuous if there exists a real constant $M > 0$ such that $\|Tx - Ty\| \leq M\|x - y\|$ for all $x, y \in C$, where the constant M is usually called the Lipschitz constant of T. The minimum Lipschitz constant of T, denoted by $L(T)$, can be computed by

$$L(T) = \sup_{\substack{x,y\in C\\ x\neq y}} \frac{\|Tx - Ty\|}{\|x - y\|}.$$

It is easy to check that the nonnegative functional $L(\cdot)$ is a seminorm of the space $\mathrm{Lip}(C, D)$ of Lipschitz operators from C into D. Obviously, $L(T)$ is just the operator norm $\|T\|$ when the operator T reduces to the linear case.

Definition 2.1 ([19]) A one-parameter family $(T(t))_{t\geq 0}$ of Lipschitz operators from C into itself is called a *Lipschitzian semigroup* on C if it possesses the following two properties: (i) $T(0) = I$ (the identity operator on X), $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$; and (ii) the mapping $t \mapsto T(t)x$ is continuous at $t = 0$ for every $x \in C$.

Furthermore, if there exist two real constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $L(T(t)) \leq Me^{\omega t}$ for all $t \geq 0$, then the Lipschitzian semigroup $(T(t))_{t\geq 0}$ is said to be *exponentially bounded*.

Definition 2.2 ([19]) Let $(T(t))_{t\geq 0}$ be a Lipschitzian semigroup on C, and let

$$D(A) = \Big\{ x \in C : \text{ the limit } \lim_{t\to 0^+} \frac{T(t)x - x}{t} \text{ exists} \Big\}.$$

If $D(A)$ is not empty, then we say that $(T(t))_{t\geq 0}$ possesses an *infinitesimal generator* A, which is defined by

$$A\colon D(A) \subset C \longrightarrow X, \quad Ax = \lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

In this case, we also say that A *generates* the Lipschitzian semigroup $(T(t))_{t\geq 0}$.

Definition 2.3 An exponentially bounded Lipschitzian semigroup $(T(t))_{t\geq 0}$ on C is called

(i) L_p *stable* if there exist some constants $p > 0$ and $K > 0$ such that

$$\int_0^\infty \|T(t)x - T(t)y\|^p dt \leq K^p \|x - y\|^p \text{ for all } x, y \in C \text{ and } t \geq 0;$$

(ii) *exponentially stable* if there exist constants $a > 0$ and $M \geq 1$ such that

$$L(T(t)) \leq Me^{-at} \text{ for all } t \geq 0.$$

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Remark 2.4 It should be pointed out that exponential stability of Lipschitzian semigroup $(T(t))_{t\geq 0}$ is discussed in the sense of seminorm $L(\cdot)$. Since $L(\,\cdot\,)$ is a nonlinear extension of linear operator norm $\|\cdot\|$, our definition can reduce to one of exponential stability of linear strongly continuous semigroups [5, 7, 14, 16]. In addition, the nonlinear exponential stability in [10] is the special case of Definition 2.3(ii).

Definition 2.5 ([11]) A continuous function $u(\,\cdot\,)$ from $[0,\infty)$ into X is said to be a *global solution* to (1.1) if $u(0)=x$, $u(t)\in C$ for $t\geq 0$ and $u(t)$ satisfies $u'(t)=Au(t)$ for $t\geq 0$.

Theorem 2.6 ([11]) *Suppose that A is a continuous operator from C into X. Then A is the infinitesimal generator of exponentially bounded Lipschitzian semigroup $(T(t))_{t\geq 0}$ on C satisfying the estimate*

$$L(T(t))\leq Me^{\omega t} \text{ for } M\geq 1, t\geq 0 \text{ and } \omega\in\mathbb{R}$$

if and only if it satisfies the following two conditions:

(A_1) $\liminf_{h\downarrow 0} d(x+hAx, C)/h=0$ *for all* $x\in C$.

(A_2) *There exists a nonnegative functional V on $X\times X$ satisfying*

(V_1) $|V(x,y)-V(\hat{x}-\hat{y})|\leq M(\|x-\hat{x}\|+\|y-\hat{y}\|)$, *for* $(x,y),(\hat{x},\hat{y})\in X\times X$

(V_2) $\|x-y\|\leq V(x,y)\leq M\|x-y\|$, *for* $x,y\in C$

such that

$$D_+V(x,y)(Ax,Ay)\leq \omega V(x,y) \quad \text{for } x,y\in C.$$

Here the symbol $D_+V(x,y)(\xi,\eta)$ is defined by

$$D_+V(x,y)(\xi,\eta)=\liminf_{h\downarrow 0}(V(x+h\xi,y+h\eta)-V(x,y))/h \text{ for } (x,y),(\xi,\eta)\in X\times X.$$

In this case, for each $x\in C$, the abstract Cauchy problem (1.1) *has a unique global solution $u(t)=T(t)x$ for $t\geq 0$.*

3 The Main Results

Let Z be a Banach space and let $T(t,s)$, $t\geq s\geq 0$, be a family of nonlinear operators with domain $Y_s\subset Z$ with the following properties:

$$\begin{aligned}
&T(t,s)Y_s\subset Y_t \text{ for } t\geq s,\\
&T(t,t)y=y, t\geq 0 \text{ for } y\in Y_t,\\
&T(t,u)T(u,s)=T(t,s) \text{ on } Y_s \text{ for } s\leq u\leq t,\\
&T(\,\cdot\,,s)y \text{ is continuous on } [s,\infty) \text{ for each } y\in Y_s.
\end{aligned} \tag{3.1}$$

Lemma 3.1 ([10, Lemma 2.1]) *Let $0<r<1$, $L>0$ and let n be a nonnegative integer. Then $nL\leq t\leq(n+1)L$ implies $e^{-at}\leq r^n\leq(1/r)e^{-at}$, $a=-(\ln r)/L>0$.*

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Theorem 3.2 *Let $g(\cdot)$ be a positive continuous function on $[0,\infty)$. Suppose that $T(t,s)$ satisfy the condition*

$$(B)\quad \|T(t,s)x - T(t,s)y\| \le g(t-s)\|x-y\| \text{ for all } x,y \in Y_s, t \ge s.$$

Then the following two statements are equivalent:

$$(N_1)\quad \int_s^{\infty} \|T(t,s)x - T(t,s)y\|^p dt \le K^p \|x-y\|^p \text{ for some } p>0 \text{ and } K>0,$$

$$(N_2)\quad \|T(t,s)x - T(t,s)y\| \le Me^{-a(t-s)}\|x-y\| \text{ for some } M \ge 1 \text{ and } a>0,$$

where $x, y \in Y_s, t \ge s \ge 0$.

Proof Obviously, it is enough to prove only that (N_1) implies (N_2). The proof idea comes mainly from [10]. For any $0 \le s < t$ and $x, y \in Y_s$, we have

$$\begin{aligned}
(3.2)\qquad \|T(t,s)x - T(t,s)y\|^p \int_s^t g^{-p}(t-u)\,du
&= \int_s^t g^{-p}(t-u)\|T(t,s)x - T(t,s)y\|^p\,du\\
&= \int_s^t g^{-p}(t-u)\|T(t,u)T(u,s)x - T(t,u)T(u,s)y\|^p\,du\\
&\le \int_s^t g^{-p}(t-u)g^p(t-u)\|T(u,s)x - T(u,s)y\|^p\,du\\
&= \int_s^t \|T(u,s)x - T(u,s)y\|^p du\\
&\le K^p\|x-y\|^p.
\end{aligned}$$

We take $\tilde{L} > 0$ arbitrarily and define J by $J^p = \int_0^{\tilde{L}} g^{-p}(u)\,du$. Then $J > 0$ and for any $t - s \ge \tilde{L}$, from (3.2) we derive

$$(3.3)\qquad \|T(t,s)x - T(t,s)y\| \le (K/J)\|x-y\|.$$

Combined with (B), (3.3) implies that there exists a constant $R > 0$ such that

$$(3.4)\qquad \|T(t,s)x - T(t,s)y\| \le R\|x-y\| \text{ for all } t \ge s \ge 0 \text{ and } x,y \in Y_s.$$

Now let $t > s, x, y \in Y_s$ and we consider

$$\begin{aligned}
(t-s)\|T(t,s)x - T(t,s)y\|^p &= \int_s^t \|T(t,s)x - T(t,s)y\|^p\,du\\
&\le R^p \int_s^t \|T(u,s)x - T(u,s)y\|^p\,du\\
&\le (RK)^p\|x-y\|^p.
\end{aligned}$$

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Consequently, we derive

$$\|T(t,s)x - T(t,s)y\| \le KR\|x-y\|/(t-s)^{1/p}, x, y \in Y_s.$$

Then for each $0 < r < 1$ we can choose a number $L = L(r) > 0$ such that

$$\|T(t,s)x - T(t,s)y\| \le r\|x-y\|, x, y \in Y_s \text{ whenever } t-s \ge L. \tag{3.5}$$

Now let $t - s \ge L$. Then there exists an integer $n \ge 1$ such that $nL \le t-s < (n+1)L$. Using the semigroup property (3.1) and (3.5) n times and then (3.4), we derive

$$\|T(t,s)x - T(t,s)y\| \le r^n R\|x-y\| \text{ for all } x, y \in Y_s.$$

According to Lemma 3.1, we have

$$\|T(t,s)x - T(t,s)y\| \le \tilde{M}e^{-at}\|x-y\|, \text{ for all } x, y \in Y_s, t-s \ge L, \tag{3.6}$$

where $\tilde{M} = R/r$ and $a = -(\ln r)/L > 0$. Combining (3.6) and (3.4), we conclude that

$$\|T(t,s)x - T(t,s)y\| \le Me^{-at}\|x-y\|, x, y \in Y_s, t \ge s,$$

where $M = \max(\tilde{M}, Re^{aL})$. ■

Corollary 3.3 *For a one-parameter exponentially bounded Lipschitzian semigroup $(T(t))_{t\ge 0}$ on X, the following two statements are equivalent:*

(i) *$(T(t))_{t\ge 0}$ is L_p stable;*
(ii) *$(T(t))_{t\ge 0}$ is exponentially stable.*

Proof Let $X_t = X_0 = X, t \ge s \ge 0$ and $T(t,s) = T(t-s)$. For exponentially bounded Lipschitzian semigroup $(T(t))_{t\ge 0}$, without loss of generality, we assume $(T(t))_{t\ge 0}$ to satisfy $L(T(t)) \le Me^{\omega t}$, where $M \ge 1$ and $\omega \in \mathbb{R}$. Define $g(t) = Me^{\omega t}$ for $t \in [0,\infty)$. Then $g(\cdot)$ is a positive continuous function on $[0,\infty)$ and $(T(t))_{t\ge 0}$ satisfies $L(T(t-s)) \le g(t-s)$ for $t \ge s$, *i.e.*, the assumption (B) holds. Consequently, from Theorem 3.2 we immediately derive that (i) is equivalent to (ii). ■

Remark 3.4 On the one hand, Corollary 3.3 partially extends the Datko–Pazy theorem on linear strongly continuous semigroups in [15] to the nonlinear case. On the other hand, Corollary 3.3 improves [10, Corollary 2.1] which requires the zero element to be a common fixed point of the nonlinear semigroup $T(t)$ for all $t \ge 0$.

Generally speaking, it is difficult or even impossible to derive an analytical expression of the solution operator $T(t)$ of the abstract Cauchy problem (1.1) associated with a nonlinear operator A. Consequently, the significance of Corollary 3.3 is mainly theoretical. As we all know, it is straightforward to discuss asymptotic behaviors of solutions of (1.1) associated with the nonlinear operator A by information of A. Based on this, we present the following two sufficient conditions for the exponential stability of the nonlinear solution semigroup corresponding to (1.1) by Corollary 3.3.

Theorem 3.5 *Let X be a Hilbert space and $(T(t))_{t\geq 0}$ be a Lipschitzian semigroup on $C \subseteq X$ which is generated by A according to Theorem 2.6. Assume that there exists a nonnegative function $v(x)$ on C with the following properties:*

(i) *$c\|x\|^p \leq v(x) \leq \|x\|^p$ for some $c < 1$ and all $x \in C$ and $p > 0$;*
(ii) *$v(x)$ is Fréchet differentiable and $\langle v'(x-y), Ax-Ay\rangle \leq -\beta\|x-y\|^p$ for $x, y \in C$ and some $\beta > 0$.*

Then $(T(t))_{t\geq 0}$ is exponentially stable. In particular, there exists a constant $M \geq 1$ such that $L(T(t)) \leq Me^{-\beta t}$.

Proof Let $w(t) = T(t)x - T(t)y$, where $T(t)x$ and $T(t)y$ are global solutions(in the sense of Definition 2.5) of the abstract Cauchy problem (1.1) for all $x, y \in C$ and $t \geq 0$. From (ii) and (i) we have

$$\begin{aligned}\left(e^{\beta t}v(w(t))\right)' &= \beta e^{\beta t}v(w(t)) + e^{\beta t}\langle v'(T(t)x - T(t)y), AT(t)x - AT(t)y\rangle \\ &\leq \beta e^{\beta t}\|w(t)\|^p - \beta e^{\beta t}\|w(t)\|^p = 0.\end{aligned}$$

Integrating $(e^{\beta t}v(w(t)))'$ from 0 to t, we derive that $v(w(t)) \leq e^{-\beta t}v(x-y)$ for $t \geq 0$ and $x, y \in C$. From (i) we immediately derive that

$$\int_0^\infty \|T(t)x - T(t)y\|^p dt \leq \frac{1}{c\beta}\|x-y\|^p.$$

Consequently, from Corollary 3.3 we can conclude that $(T(t))_{t\geq 0}$ is exponentially stable, *i.e.*, there exists a constant $M \geq 1$ such that $L(T(t)) \leq Me^{-\beta t}$. ■

Corollary 3.6 *Let X be a Hilbert space and $(T(t))_{t\geq 0}$ be a Lipschitzian semigroup on $C \subseteq X$ which is generated by A according to Theorem 2.6. Assume that there exists a nonnegative linear bounded operator P from X into itself such that*

$$2Re\langle P(x-y), Ax-Ay\rangle \leq -a\|x-y\|^2 \text{ for all } x, y \in C \text{ and some } a > 0.$$

Then $(T(t))_{t\geq 0}$ is exponentially stable, that is, there exists a constant $M \geq 1$ such that $L(T(t)) \leq Me^{-at}$.

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Probems.* Monographs in Mathematics 96. Birkhäuser Verlag, Basel, 2001.
[2] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces.* Noordhoff, Leiden, 1976.
[3] A. Belleni-Morante and A. C. Mcbride, *Applied Nonlinear Semigroups: An Introduction.* John Wiley & Sons, Chichester, 1998.
[4] R. Datko, *Extending a theorem of A. M. Liapunov to Hilbert space.* J. Math. Anal. Appl. **32**(1970), 610–616. http://dx.doi.org/10.1016/0022-247X(70)90283-0
[5] K. J. Engel and R. Nagel, *One-parameter Semigroups of Linear Evolution Equations.* Graduate Texts in Mathematics 194. Springer-Verlag, New York, 2000.
[6] H. O. Fattorini, *The Cauchy Prolem.* Encyclopedia of Mathematics and its Applications18. Addison-Wesley, Reading, MA, 1983.

https://doi.org/10.4153/CMB-2011-070-0 Published online by Cambridge University Press

[7] J. A. Goldsein, *Semigroups of Linear Operators and Applications.* Oxford Mathematical Monographs, Oxford University Press, New York, 1985.
[8] W. Hahn, *Stability of Motion.* Die Grundlehren der mathematischen Wissenschaften 138. Springer-Verlag, New York, 1967.
[9] A. Ichikawa, *Stability and optimal control of stochastic evolution equations.* In: Distributed Parameter Control Systems: Theory and Application, Chapter 5, Pergamon Press, New York, 1982.
[10] ———, *Equivalence of L_p stability and exponential stability for a class of nonlinear semigroups.* Nonlinear Anal. **8**(1984), no. 7, 805–815. http://dx.doi.org/10.1016/0362-546X(84)90078-6
[11] Y. Kobayshi and N. Tanaka, *Semigroups of Lipschitz operators.* Adv. Differential Equations **6**(2001), no. 5, 613–640.
[12] ———, *Convergence and approximation of semigroups of Lipschitz operators.* Nonlinear Anal. **61**(2005), no. 5, 781–821. http://dx.doi.org/10.1016/j.na.2005.01.040
[13] I. Miyadera, *Nonlinear Semigroups.* Translations of Mathematical Monographs 109. American Mathematical Society, Providence, RI, 1992.
[14] J. M. A. A. van Neerven, *The Asymptotic Behviour of Semigroups of Linear Operators.* Operator Theory: Advances and Applications 88. Birkhäuser Verlag, Basel, 1996.
[15] A. Pazy, *On the applicability of Lyapunov's theorem in Hilbert space.* SIAM J. Math. Anal. **3**(1972), 291–294. http://dx.doi.org/10.1137/0503028
[16] ———, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences 44. Springer-Verlag, New York, 1983.
[17] J. G. Peng, *Theoretical Researchs on Nonlinear Lipschitz Opertors and their Applications.* (in Chinese), Ph.D. dissertation, Xi'an Jiaotong University, 1998.
[18] J. G. Peng and Z. B. Xu, *On asymptotic behaviours of nonlinear semigroup of Lipschitz operators with applications.*(in Chinese), Acta Math. Sinica **45**(2002), no. 6, 1099–1106.
[19] ———, *A novel dual approach to nonlinear semigroups of Lipschitz operators.* Trans. Amer. Math. Soc. **357**(2005), no. 1, 409–424. http://dx.doi.org/10.1090/S0002-9947-04-03635-9
[20] J. Zabczyk, *Remarks on the control of discrete-time distributed parameter systems.* SIAM J. Control **12**(1974), 721–735. http://dx.doi.org/10.1137/0312056

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