ON A DISCRETE ANALOGUE OF INEQUALITIES OF OPIAL AND YANG

Cheng-Ming Lee

(received August 29, 1967)

In a recent paper [2], Wong proved the following

THEOREM 1. *Let* $\{U_i\}_1^\infty$ *be a non-decreasing sequence of non-negative numbers, and let* $U_o = 0$. *Then we have*

$$(1) \quad \sum_{i=1}^{n} (U_i - U_{i-1})U_i^p \leq (n+1)^p(p+1)^{-1} \sum_{i=1}^{n} (U_i - U_{i-1})^{p+1} \quad \text{for } p \geq 1.$$

Yang [3] proved the following integral inequality:

THEOREM 2. *If* $y(x)$ *is absolutely continuous on* $a \leq x \leq X$, *with* $y(a) = 0$, then

$$(2) \qquad \int_a^X |y^p y'^q| dx \leq q(p+q)^{-1}(X-a)^p \int_a^X |y'(x)|^{p+q} dx$$

for $p \geq 1$ and $q \geq 1$.

The purpose of this note is to obtain a discrete analogue of (2) which includes the inequality (1) as a special case. In fact, we are going to prove

THEOREM 3. *Let* $\{U_i\}_1^\infty$ *be a non-decreasing sequence of non-negative numbers, and let* $U_o = 0$. *If*

$$p > 0\,,\ q > 0\,,\ p+q \geq 1 \quad \text{or} \quad p < 0, \quad q < 0\,,$$

https://doi.org/10.4153/CMB-1968-010-7 Published online by Cambridge University Press

then

(3a) $$\sum_{1}^{n} (U_i - U_{i-1})^q U_i^p \leq K_n \sum_{i=1}^{n} (U_i - U_{i-1})^{p+q} ,$$

where $K_o = q(p+q)^{-1}$ and for $n = 1, 2, 3, \ldots,$

$$K_n = \max \{K_{n-1} + pn^{p-1}(p+q)^{-1} , \quad q(n+1)^p(p+q)^{-1}\} .$$

If

$p > 0$, $q < 0$, $p+q \leq 1$, $p + q \neq 0$ or $p < 0$, $q > 0$, $p+q \geq 1$,

then

(3b) $$\sum_{1}^{n} (U_i - U_{i-1})^q U_i^p \geq C_n \sum_{1}^{n} (U_i - U_{i-1})^{p+q} ,$$

where

$$C_o = q(p+q)^{-1} \text{ and for } n = 1, 2, 3, \ldots,$$

$$C_n = \min \{ C_{n-1} + pn^{p-1}(p+q)^{-1} , \quad q(n+1)^p(p+q)^{-1}\} .$$

In particular, we have

(4) $$\sum_{1}^{n} (U_i - U_{i-1})^q U_i^p \leq q(n+1)^p(p+q)^{-1} \sum_{1}^{n} (U_i - U_{i-1})^{p+q}$$

for $p \geq 1$, $q \geq 1$;

(5a) $$\sum_{1}^{n} (U_i - U_{i-1})^q U_i^p \leq K_n'' \sum_{1}^{n} (U_i - U_{i-1})^{p+q}$$

for $p \leq 0$, $q < 0$;

(5b) $$\sum_{1}^{n} (U_i - U_{i-1})^q U_i^p \geq K_n'' \sum_{1}^{n} (U_i - U_{i-1})^{p+q}$$

https://doi.org/10.4153/CMB-1968-010-7 Published online by Cambridge University Press

for $p \geq 0$, $p+q < 0$, where $K_1'' = 1$ and for $n = 2, 3, 4, \ldots,$

$$K_n'' = 1 + p(p+q)^{-1} \sum_{i=2}^{n} i^{p-1} .$$

Proof. Let $X_i = (U_i - U_{i-1})^{p+q}$ for $i = 1, 2, 3, \ldots,$ $p+q \neq 0$, so that $(U_i - U_{i-1})^q = X_i^{qk}$, where $k = (p+q)^{-1}$. Since $U_i = \sum_{j=1}^{i} (U_j - U_{j-1})$, by Hölder's inequality we have

$$U_i \leq i^{1-k} (\sum_{j=1}^{i} X_j)^k \equiv D_i \qquad \text{if } p+q \geq 1 ,$$

and

$$U_i \geq D_i \quad \text{if } p+q < 0 \text{ or } 0 < p+q \leq 1 .$$

Therefore, $U_i^p \leq D_i^p$ and hence

$$\sum_{1}^{n} (U_i - U_{i-1})^q U_i^p \leq \sum_{1}^{n} X_i^{qk} D_i^p$$

if $p \geq 0$, $p+q \geq 1$ or $p \leq 0$ and either $p+q < 0$ or $0 < p+q \leq 1$; while $U_i^p \geq D_i^p$ and hence

$$\sum_{i=1}^{n} (U_i - U_{i-1})^q U_i^p \geq \sum_{i=1}^{n} X_i^{qk} D_i^p$$

if $p \leq 0$, $p+q \geq 1$ or $p \geq 0$ and either $p+q < 0$ or $0 < p+q \leq 1$. Thus, (3a), (3b) will follow if we can prove

(6a) $$\sum_{i=1}^{n} X^{qk} D_i^p \leq K_n \sum_{i=1}^{n} X_i \qquad \text{for } pq > 0 ,$$

and

(6b) $$\sum_{1}^{n} X^{qk} D^{p} \geq C_n \sum_{i=1}^{n} X_i \quad \text{for} \quad pq < 0 \quad .$$

We prove (6a) by induction on n. Clearly it holds for $n = 1$ since $K_1 \geq 1$. Assume that it holds for n, and observe that

(*) $$\sum_{i=1}^{n+1} X_i^{qk} D_i^{p} \leq K_n \sum_{i=1}^{n} X_i + X_{n+1}^{qk} D_{n+1}^{p} \quad .$$

Now, note that $X_i \geq 0$ for all $i \geq 1$, so that by a classical theorem [1] of arithmetic and geometric means, we have for $pq > 0$,

$$X_{n+1}^{qk} D_{n+1}^{p} = (n+1)^{p} \{X_{n+1}^{qk} [(n+1)^{-1} \sum_{i=1}^{n+1} X_i]^{pk}\}$$

$$\leq (n+1)^{p} \{qkX_{n+1} + pk(n+1)^{-1} \sum_{i=1}^{n+1} X_i\} \equiv E_{n+1}$$

since $pk+qk = 1$. Hence from (*) we get

$$\sum_{i=1}^{n+1} X_i^{qk} D_i^{p} \leq K_n \sum_{i=1}^{n} X_i + qk(n+1)^{p} X_{n+1} + pk(n+1)^{p-1} \sum_{i=1}^{n+1} X_i$$

$$\leq K_{n+1} \sum_{i=1}^{n+1} X_i$$

since $K_n \geq qk(n+1)^{p}$ and $K_{n+1} \geq K_n + pk(n+1)^{p-1}$, which proves (6a). Note that for $pq < 0$, one can easily see that $X_{n+1}^{qk} D_{n+1}^{p} \geq E_{n+1}$, so that (4b) will follow by proceeding as above, and the proofs of (3a) and (3b) are completed.

To see (4), consider $K_n' = q(n+1)^{p}(p+q)^{-1}$ for $p \geq 1$, $q \geq 1$. We have $K_1' = q\,2^{p}(p+q)^{-1} \geq 1$, and

https://doi.org/10.4153/CMB-1968-010-7 Published online by Cambridge University Press

$$K'_{n+1} - K'_n = q(p+q)^{-1}[(n+2)^p - (n+1)^p]$$

$$\geq q(p+q)^{-1}[(n+1)^p + p(n+1)^{p-1} - (n+1)^p] \geq p(p+q)^{-1}(n+1)^{p-1},$$

where we used the Bernoulli inequality. Thus (4) follows from the proof of (3a). Also, (5a), (5b) follows from the facts:

$$K''_{n+1} - K''_n = p(n+1)^{p-1}(p+q)^{-1}, \quad \text{and}$$

$$K''_n \geq 1 \geq q(n+1)^p(p+q)^{-1} \quad \text{for} \quad p < 0 \quad \text{and} \quad q < 0,$$

but $K''_n \leq 1 \leq q(n+1)^p(p+q)^{-1}$ for $p \geq 0$ and $p+q < 0$:

Thus we complete the proof of Theorem 3.

We remark that (3a) [or (4)] becomes (1) when $q = 1$ and $p \geq 1$. Also, note that (3a) is true even for $0 < p < 1$ when $q = 1$, but (1) fails to hold for $p < 1$.

ACKNOWLEDGEMENT. Thanks are due to Dr. Paul R. Beesack for many helpful suggestions and for his guidance and encouragement throughout the course of this research.

REFERENCES

1. Hardy, Littlewood and Pólya, Inequalities p.17.

2. James S. W. Wong, A discrete analogue of Opial's inequality. Can. Math. Bull. 10 (1967) 115-118.

3. Gou-Sheng Yang, On a certain result of Z. Opial, Jap. J. of Math. 42(1966) 78-83.

Carleton University