BULL. AUSTRAL. MATH. SOC. MOS 76C20, 76C99

VOL. 6 (1972), 145-156.

Diffraction of planetary waves by a semi-infinite plate

P.F. Siew and D.G. Hurley

In this paper the diffraction of a planetary wave by a semi-infinite plate of arbitrary inclination is investigated using a β-plane approximation. The Wiener-Hopf technique is used to obtain an integral representation of the solution and an asymptotic description of the diffracted wave is obtained by the method of steepest descent.

1. Introduction

This paper is concerned with the planetary waves that can occur in a thin layer of fluid on the surface of a rotating sphere. These waves are of interest both because of their applications in meteorology and oceanography and in their own right as an example of anisotropic waves.

Anisotropic waves can exhibit properties that are quite different from those of the more well-known isotropic waves such as light waves and sound waves, and it is highly desirable that for each type of wave there should be available an appreciable number of solutions that display their properties. An example of the need for such solutions is provided by the work of Robinson [7] and [8] and Hurley [1] who investigated the diffraction of internal gravity waves and obtained some quite unexpected results.

Many of the properties of planetary waves have been established by

Received 5 October 1971. The second author acknowledges helpful discussions he had with Professor V.T. Buchwald concerning energy fluxes after the work described in the paper had been completed apart from the determination of the constant C in equation (22).

Longuet-Higgins in a notable series of papers [3], [4], [5], [6], but problems in which diffraction is important do not appear to have been considered. In the present paper, the simplest such problem is considered, namely the diffraction of planetary waves by a half-plane.

2. Basic equations

Consider a thin layer of liquid of uniform depth h on the surface of a sphere that is rotating with angular velocity Ω about a diameter. Longuet-Higgins [5] has shown that the well-known β-plane approximation holds for planetary waves in the liquid provided that their absolute wave number n is large. In this case, [6], the velocity components are nearly geostrophic and a stream function

(1) $$\psi = -\frac{g\zeta}{f}$$

exists such that

(2) $$u \simeq \frac{\partial\psi}{\partial y}, \quad v \simeq -\frac{\partial\psi}{\partial x}.$$

Here (x, y) are rectangular co-ordinates with x increasing to the east and y to the north, (u, v) are the corresponding velocity components, ζ the surface elevation assumed small, g the acceleration due to gravity and $f = 2\Omega\sin\chi$ the Coriolis parameter where χ is the latitude. The function ψ satisfies the equation

(3) $$\left\{\frac{\partial}{\partial t}(\nabla^2-a^2) + \beta\frac{\partial}{\partial x}\right\}\psi = 0,$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $a^2 = \frac{f^2}{gh}$, and it has been assumed that

$$f = f_0 + \beta y,$$

where f_0 and β are constants.

If $\frac{L^2f^2}{gh}$ is small, where L is the wave length, (3) reduces to

(4) $$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0,$$

in which case the planetary waves are referred to as being divergenceless.

The condition that the plane wave

(5) $$\psi = e^{i(lx+my-\sigma t)}$$

should satisfy (3) yields the dispersion relation

(6) $$(l+\gamma)^2 + m^2 = \gamma^2 - a^2 ,$$

where $\gamma = \frac{\beta}{2\sigma}$. The wave-number locus is a circle, centre $(-\gamma, 0)$, as shown in Figure 1.

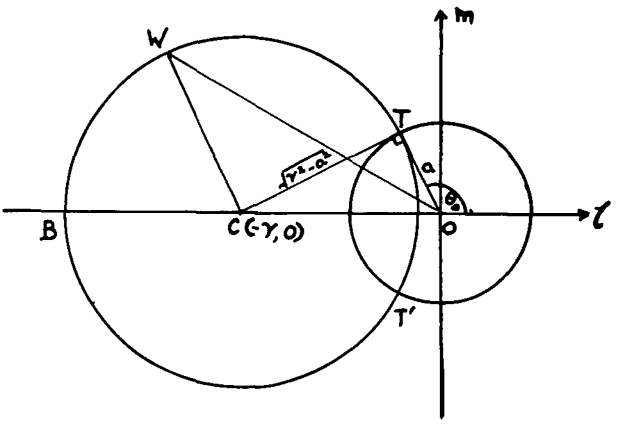

Figure 1. Wave-number locus as defined by equation (6).

Waves whose wave number vectors *OW* lie on the arc *TBT'* are called planetary waves [5] and their group velocity is in the direction *WC* .

3. Formulation of the problem

Suppose that there is a semi-infinite impermeable plate inclined at an angle α $(-\pi < \alpha < \pi)$ to the Oy axis as shown in Figure 2 on page 148.

In terms of rectangular axes *OXY* with *OY* along the barrier, the equations corresponding to (2) and (3) are

(7) $$U \simeq \frac{\partial\psi}{\partial Y} , \quad V \simeq - \frac{\partial\psi}{\partial X}$$

and

(8) $$\frac{\partial}{\partial t}\left[\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial Y^2} - a^2\psi\right] + \beta\cos\alpha \frac{\partial\psi}{\partial X} - \beta\sin\alpha \frac{\partial\psi}{\partial Y} = 0 ,$$

where (U, V) are the velocity components in the X and Y directions respectively.

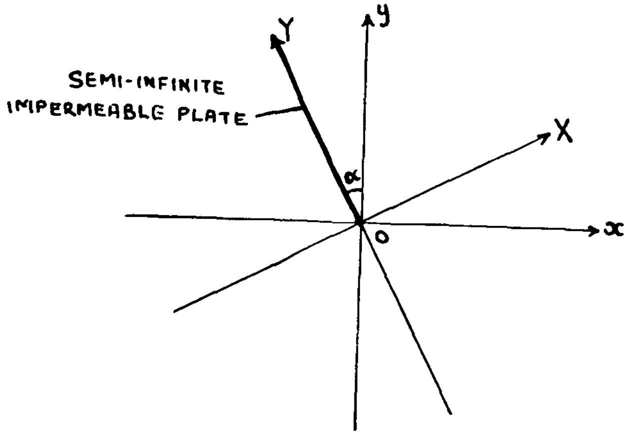

Figure 2. Notations.

Let the total stream function be

(9) $$\Psi = \{\psi_j+\psi(X, Y)\}e^{-i\sigma t} ,$$

where

(10) $$\psi_j = e^{iK\cos(\theta-\alpha)X+iK\sin(\theta-\alpha)Y}$$

represents the incident wave whose wave number vector has magnitude K and is inclined at an angle θ to the Ox axis. Figure 1 shows that θ must satisfy the relation $\theta_0 \leq \theta \leq 2\pi-\theta_0$ where $\theta_0 = \pi - \arcsin\left(\frac{\sqrt{\gamma^2-a^2}}{\gamma}\right)$.

It follows from (8) that

(11) $$\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial Y^2} - a^2\psi + 2i\gamma\cos\alpha \frac{\partial\psi}{\partial X} - 2i\gamma\sin\alpha \frac{\partial\psi}{\partial Y} = 0 .$$

On the plate the total normal velocity must vanish so that

(12) $$\frac{\partial\psi}{\partial Y}(0, Y) = -\frac{\partial\psi_j}{\partial Y} = -iK\sin(\theta-\alpha)e^{iK\sin(\theta-\alpha)Y} , \quad Y > 0 .$$

Also, ψ must satisfy the Sommerfeld radiation condition, and the singularities of $\frac{\partial\psi}{\partial X}$ and $\frac{\partial\psi}{\partial Y}$ at the origin must be no worse than R^{δ} where $R = (X^2+Y^2)^{\frac{1}{2}}$ and $\delta > -1$.

4. Method of solution

Let

(13) $$\overline{\psi}(X, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(X, Y) e^{i\lambda Y} dY$$

be the Fourier transform of ψ with respect to Y . Applying (13) to (11) gives

$$\frac{\partial^2\overline{\psi}}{\partial X^2} + i2\gamma\cos\alpha \frac{\partial\overline{\psi}}{\partial X} - (\lambda^2+2\gamma\lambda\sin\alpha+a^2)\overline{\psi} = 0 ,$$

so that

(14) $$\overline{\psi}(X, \lambda) = A(\lambda)\exp\left\{-i\gamma\cos\alpha X-\sqrt{(\lambda+\gamma\sin\alpha)^2-b^2}|X|\right\} ,$$

where $b^2 = \gamma^2 - a^2$ and use has been made of the continuity of $\frac{\partial\psi}{\partial Y}$ and hence of $\overline{\psi}$ at $X = 0$. Here the branch of $\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)}$, where $\lambda_1 = -\gamma\sin\alpha + b$ and $\lambda_2 = -\gamma\sin\alpha - b$, is that which is real positive as $\lambda \to +\infty$ and Lighthill's method [2] of putting

(15) $$\sigma = \sigma_0 + i\varepsilon \quad \text{where} \quad 0 < \varepsilon \ll 1$$

and then taking the limit $\varepsilon \to 0$ is used to determine the branch cuts. The result is shown in Figure 3 on page 150.

To determine $A(\lambda)$ in (14) we introduce two functions, $f(Y)$ and $g(Y)$, by the equations

(16) $$\frac{\partial\psi}{\partial Y}(0, Y) = \begin{cases} f(Y) & (Y < 0) \\ -iK\sin(\theta-\alpha)e^{iK\sin(\theta-\alpha)Y} & (Y > 0) \end{cases}$$

and

(17) $$\frac{\partial\psi}{\partial X}(0+, Y) - \frac{\partial\psi}{\partial X}(0-, Y) = \begin{cases} 0 & (Y < 0) \\ g(Y) & (Y > 0) \end{cases} ,$$

where $\frac{\partial\psi}{\partial X}(0\pm, Y)$ denotes the limit of $\frac{\partial\psi}{\partial X}(X, Y)$ as $X \to \pm 0$.

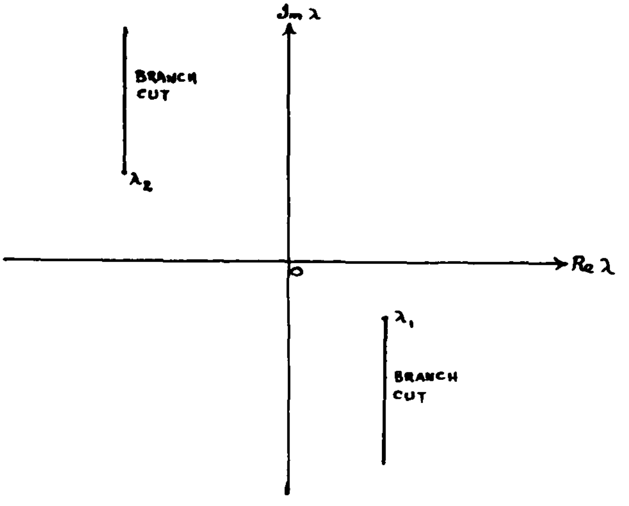

Figure 3. Location of branch cuts.

Transforming (16) and using (14) gives

(18) $$\lambda A(\lambda) = i\bar{f}_{-}(\lambda) + \frac{iK\sin(\theta-\alpha)}{\sqrt{2\pi}}\left(\frac{1}{\lambda+K\sin(\theta-\alpha)}\right)_{+} ,$$

where subscripts "+" denote a function regular for

$$\text{Im}\lambda > \varepsilon'\left\{\frac{\gamma_0\cos^2\theta}{\sqrt{\gamma_0^2\cos^2\theta-a^2}} - \cos\theta\right\} \text{ where } \gamma = \frac{\beta}{2\sigma_0^2}(\sigma_0-i\varepsilon) = \gamma_0 - i\varepsilon' ,$$

and "-" a function regular for $\text{Im}\lambda < \text{Im}\lambda_2$.

(14) and the transform of (17) give

(19) $$-2\sqrt{(\lambda+\gamma\sin\alpha)^2-b^2}\, A(\lambda) = \bar{g}_{+}(\lambda) .$$

(18) and (19) imply

(20) $$i\bar{f}_{-}(\lambda) + \frac{iK\sin(\theta-\alpha)}{\sqrt{2\pi}}\left(\frac{1}{\lambda+K\sin(\theta-\alpha)}\right)_{+} = \frac{-\lambda\bar{g}_{+}(\lambda)}{2\sqrt{(\lambda+\gamma\sin\alpha)^2-b^2}} .$$

Writing

$$\frac{\sqrt{\lambda-\lambda_2}}{\lambda+K\sin(\theta-\alpha)} = \left[\frac{\sqrt{\lambda-\lambda_2}-\sqrt{-\lambda_2-K\sin(\theta-\alpha)}}{\lambda+K\sin(\theta-\alpha)}\right]_- + \left(\frac{\sqrt{-\lambda_2-K\sin(\theta-\alpha)}}{\lambda+K\sin(\theta-\alpha)}\right)_+ ,$$

(20) gives, after rearrangement,

$$(21)\quad \left[i\bar{f}(\lambda)\sqrt{\lambda-\lambda_2} + \frac{iK\sin(\theta-\alpha)}{\sqrt{2\pi}}\left\{\frac{\sqrt{\lambda-\lambda_2}-\sqrt{-\lambda_2-K\sin(\theta-\alpha)}}{\lambda+K\sin(\theta-\alpha)}\right\}\right]_-$$

$$= \left[\frac{-\lambda\bar{g}(\lambda)}{2\sqrt{\lambda-\lambda_1}} - \frac{iK\sin(\theta-\alpha)\sqrt{-\lambda_2-K\sin(\theta-\alpha)}}{\lambda+K\sin(\theta-\alpha)}\right]_+$$

$$= E(\lambda)\ , \text{ say,}$$

where $E(\lambda)$ is an entire function. Now the condition on the behaviour of $\frac{\partial\psi}{\partial X}$ and $\frac{\partial\psi}{\partial Y}$ near the origin (see end of §3) implies $E(\lambda) = O(\lambda^q)$, $0 < q < 1$, as $\lambda \to \infty$, so by the extended form of Liouville's Theorem, $E(\lambda)$ is a polynomial of degree less than one, hence, a constant C, say.

(21), (18), (14) and the Fourier inversion formula give

$$(22)\quad \psi(X,\ Y) = \frac{C}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\exp\{-i\gamma\cos\alpha X-\sqrt{(\lambda+\gamma\sin\alpha)^2-b^2}\,|X|-i\lambda Y\}}{\lambda\sqrt{\lambda+\gamma\sin\alpha+b}}\,d\lambda$$

$$+ \frac{iK\sin(\theta-\alpha)\sqrt{\gamma\sin\alpha+b-K\sin(\theta-\alpha)}}{2\pi}\times$$

$$\int_{-\infty}^{\infty}\frac{\exp\{-i\gamma\cos\alpha X-\sqrt{(\lambda+\gamma\sin\alpha)^2-b^2}\,|X|-i\lambda Y\}}{\lambda\sqrt{\lambda+\gamma\sin\alpha+b}[\lambda+K\sin(\theta-\alpha)]}\,d\lambda\ ,$$

where the indentations along the $\mathrm{Re}\lambda$ axis are below $\lambda = \lambda_2$ and above $\lambda = \lambda_1$, $\lambda = 0$ and $\lambda = -K\sin(\theta-\alpha)$, and the unknown constant C will be determined in the next section.

5. Asymptotic behaviour of solution

The method of steepest descent is used to determine the behaviour of (22) at large distances from the origin. Putting $X = R\cos\tau$ and $Y = R\sin\tau$, it is found that each integrand in (22) has a single saddle point at

$$(23)\qquad \lambda_s = -\gamma\sin\alpha + b\sin\tau\ ,$$

https://doi.org/10.1017/S0004972700044348 Published online by Cambridge University Press

and that the path of steepest descent through it is given by

$$(24)\qquad \nu = -\frac{[(\mu+\gamma\sin\alpha)\sin\tau-b][(\mu+\gamma\sin\alpha)-b\sin\tau]}{|\cos\tau|[(\mu+\gamma\sin\alpha-b\sin\tau)^2+b^2\cos^2\tau]^{\frac{1}{2}}},$$

where $\lambda = \mu + i\nu$.

This path crosses the $\mathrm{Re}\lambda$ axis at $\lambda = \lambda_s$ and $\lambda = \lambda_0 = \frac{b}{\sin\tau} - \gamma\sin\alpha$. Figure 4 shows the integration path Γ and the paths of steepest descent Γ_s, Γ_s^*, and Γ_0^* corresponding to $\sin\tau > 0$, $\sin\tau < 0$ and $\sin\tau = 0$ respectively for various values of $\sin\alpha$.

Thus

$$(25)\qquad \psi \sim \eta\left\{C + \frac{iK\sin(\theta-\alpha)\sqrt{\gamma\sin\alpha+b-K\sin(\theta-\alpha)}}{\sqrt{2\pi}[b\sin\tau-\gamma\sin\alpha+K\sin(\theta-\alpha)]}\right\}\frac{\exp[-iR\{\gamma\cos(\alpha+\tau)+b\}+i\frac{\pi}{4}]}{R^{\frac{1}{2}}} + \psi_R + \psi_P + O\left(\frac{e^{-i\gamma R}}{R^{\frac{3}{2}}}\right),$$

where $\eta = \frac{|\cos\tau|}{(b\sin\tau-\gamma\sin\alpha)\sqrt{1+\sin\tau}}$ and ψ_R (ψ_P) is the contribution from the pole at $-K\sin(\theta-\alpha)$ (0) and must be included if the point $-K\sin(\theta-\alpha)$ (0) is crossed as Γ is deformed into Γ_s, Γ_s^* or Γ_0^* .

It may be shown that $-K\sin(\theta-\alpha)$ always lies between λ_1 and λ_2 , and hence ψ_R is non-zero if

$$(26)\qquad -K\sin(\theta-\alpha) < \lambda_s ,$$

an inequality which is satisfied for the regions of the Oxy plane that are denoted by I and II in Figure 5 on page 154. Region II is the shadow region bounded by the plate and the extension of the line VO which is in the direction of the group velocity of the incident wave. In it $\psi_R = -\psi_j$ as expected.

In region I the residue calculation gives

$$(27)\qquad \psi_R = -\exp\{-i2\gamma\cos\alpha X-iK\cos(\theta-\alpha)X+iK\sin(\theta-\alpha)Y\} .$$

https://doi.org/10.1017/S0004972700044348 Published online by Cambridge University Press

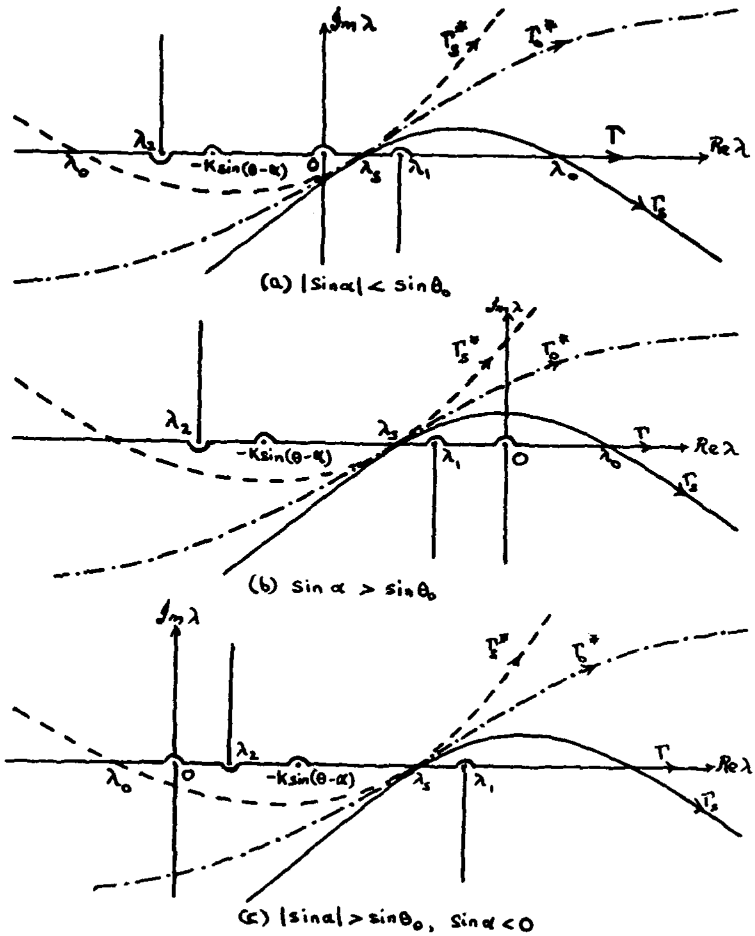

Figure 4. Paths of steepest descent for various values of $\sin\alpha$. —→— denote the path for $\sin\tau > 0$, — · → · — for $\sin\tau = 0$ and — → — for $\sin\tau < 0$.

This represents a reflected wave with wave number vector as shown in Figure 6 and the boundary of region *I* is a ray from the origin in the direction of the group velocity of this wave, again as expected.

Each of the integrands in (22) has a pole at $\lambda = 0$, so that ψ_p is non-zero when either

(28a) $$\sin\tau > \frac{\gamma\sin\alpha}{b} = \frac{\sin\alpha}{\sin\theta_0}\,, \quad |\sin\alpha| < \sin\theta_0$$

https://doi.org/10.1017/S0004972700044348 Published online by Cambridge University Press

or

(28b) $$\sin\tau > \frac{\sin\theta_0}{\sin\alpha}\ , \quad |\sin\alpha| > \sin\theta_0\ .$$

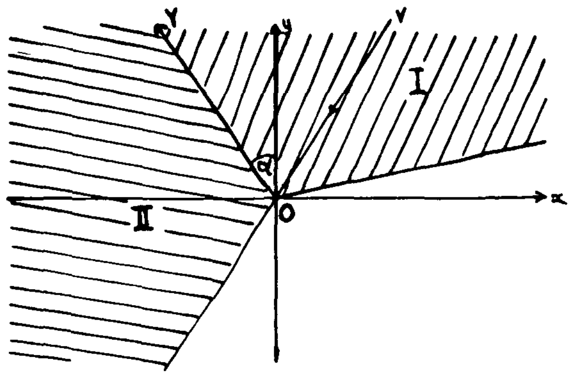

Figure 5. The shadow region (II) and the region (I) in which the reflected wave is found, the plate being along *OY* . The group velocity of the incident wave is in the direction *VO* .

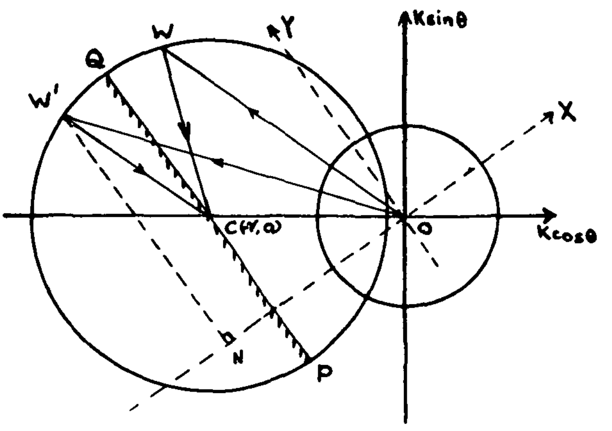

Figure 6. Wave number locus with *OXY* axes superimposed. *OW* and *OW'* correspond to the incident and reflected wave number vectors. *PQ* is parallel to *OY* . $|ON| = 2\gamma\cos\alpha + K\cos(\theta-\alpha)$ is the *X*-component of the wave number in the reflected wave.

When (28a) is satisfied,

(29a) $$\psi_P = D\exp\left\{-i\gamma\cos\alpha X - i\gamma\sqrt{\cos^2\alpha-\cos^2\theta_0}\,|X|\right\}\ ,$$

where $D = \frac{\sqrt{\gamma\sin\alpha+b}-K\sin(\theta-\alpha)-i\sqrt{2\pi}C}{\sqrt{\gamma\sin\alpha+b}}$. (29a) represents plane waves found on both sides of the plate that have wave number vectors normal to the plate irrespective of the direction of the incident wave.

When (28b) is satisfied,

$$\psi_P = D\exp\left\{-\gamma\sqrt{\cos^2\theta_0-\cos^2\alpha}\,|X| - i\gamma\cos\alpha X\right\}, \tag{29b}$$

which represents edge waves, independent of the Y-direction but decaying exponentially away from the barrier.

Now if, following Longuet-Higgins [5], it is assumed that the energy flux in a wave is the product of its group velocity and the energy density associated with it, then it can be shown that the incident and reflected waves have energy fluxes equal in magnitude and equally inclined to, but on either side of the normal to the plate. If it is also assumed that the energy fluxes of two waves is the sum of that due to each, then the nett energy flux of the incident and reflected waves in the direction of the normal to the plate is zero. Since the group velocity of the wave represented by (29a) is not parallel to the plate it will give rise to a normal energy flux unless

$$D = 0\ , \quad C = -\frac{i\sqrt{\gamma\sin\alpha+b}-K\sin(\theta-\alpha)}{\sqrt{2\pi}}\ , \quad \psi_P = 0\ . \tag{30}$$

Since the plate should not act as a source of energy it then follows that C is determined by (30) when ψ_P is given by (29a). It also seems likely that C is determined by (30) when ψ_P is given by (29b).

With the help of (30) we now obtain from (22):

$$\psi(X,\ Y) = \frac{-i\sqrt{\gamma\sin\alpha+b}-K\sin(\theta-\alpha)}{2\pi}\times \int_{-\infty}^{\infty}\frac{\exp\{-i\gamma\cos\alpha X-\sqrt{(\lambda+\gamma\sin\alpha)^2-b^2}\,|X|-i\lambda Y\}}{[\lambda+K\sin(\theta-\alpha)]\sqrt{\lambda+\gamma\sin\alpha+b}}\,d\lambda \tag{31}$$

with the property

$$\Psi(0,\ Y) = 0\ , \quad (Y > 0) \tag{32}$$

which is consistent with the boundary conditions imposed in Section 3.

(25) now gives that the diffracted wave is

$$(33) \quad \psi_s \simeq \frac{-i|\cos\tau|\sqrt{\gamma\sin\alpha+b-K\sin(\theta-\alpha)}\exp\left\{-iRb\left[1+\frac{\gamma}{b}\cos(\alpha+\tau)\right]+i\frac{\pi}{4}\right\}}{\sqrt{2\pi}[b\sin\tau-\gamma\sin\alpha+K\sin(\theta-\alpha)]\sqrt{1+\sin\tau}R^{\frac{1}{2}}} + O\left(\frac{e^{-i\gamma R}}{R^{\frac{3}{2}}}\right),$$

which together with the time dependence factor $e^{-i\sigma t}$ represents waves with hyperbolic crests moving towards the West, the Ox-axis being the transverse axis of the hyperbolae. The result may be applicable to the effect of a promontory on planetary waves in the ocean.

The corresponding result for the divergenceless case may be obtained from the foregoing if we put $a = 0$ wherever it occurs. The asymptotic form of the diffracted wave has crests which are parabolae rather than hyperbolae.

References

[1] D.G. Hurley, "Internal waves in a wedge-shaped region", *J. Fluid Mech.* 43 (1970), 97-120.

[2] M.J. Lighthill, "Studies in magneto-hydrodynamic waves and other anisotropic wave motions", *Phil. Trans. Roy. Soc. London Ser. A* 252 (1960), 397-430.

[3] M.S. Longuet-Higgins, "Planetary waves on a rotating sphere", *Proc. Roy. Soc. Ser. A.* 279 (1964), 446-473.

[4] M.S. Longuet-Higgins, "On group velocity and energy flux in planetary wave motions", *Deep Sea Research* 11 (1964), 35-42.

[5] M.S. Longuet-Higgins, "Planetary waves on a rotating sphere. II", *Proc. Roy. Soc. Ser. A* 284 (1965), 40-68.

[6] M.S. Longuet-Higgins and A.E. Gill, "Resonant interactions between planetary waves", *Proc. Roy. Soc. Ser. A* 299 (1967), 120-140.

[7] R.M. Robinson, "The effects of a vertical barrier on internal waves", *Deep Sea Research* 16 (1969), 421-429.

[8] R.M. Robinson, "The effects of a corner on a propagating internal gravity wave", *J. Fluid Mech.* 42 (1970), 257-267.

Department of Mathematics,
University of Western Australia,
Nedlands, Western Australia.