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ON THE COMMUTATIVITY OF A RING WITH IDENTITY

BY

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ABSTRACT. Let R be a ring with identity. R satisfies one of the following properties for all $x, y \in R$:

(I) $xy^nx^my = x^{m+1}y^{n+1}$ and $mnm!\,n!\,x \neq 0$ except $x = 0$;

(II) $xy^nx^m = x^{m+1}y^{n+1}$ and $mm!\,n!\,x \neq 0$ except $x = 0$;

(III) $x^my^n = y^nx^m$ and $m!\,n!\,x \neq 0$ except $x = 0$;

(IV) $(x^py^q)^n = x^{pn}y^{qn}$ for $n = k, k+1$ and $N(p, q, k)x \neq 0$ except $x = 0$, where $N(p, q, k)$ is a definite positive integer.

Then R is commutative.

1. Let x, y be elements of a ring R. If the following equality

$$(xy)^n = x^ny^n \tag{1}$$

holds for a certain positive integer n, then R need not be a commutative ring. Quite a few papers [1–9] gave additional conditions to make R commutative. [1, 4] discussed $(m, 2)$-rings, i.e. rings in which (1) holds for two consecutive positive integers $n = k$, $k+1$. In this paper, we consider the following equality

$$(x^py^q)^n = x^{pn}y^{qn} \tag{2}$$

where p, q are positive integers. Obviously (2) is a generalization of (1). We obtain a result on the commutativity of a ring with identity, which satisfies (2) for $n = k$, $k+1$. The method of our proof originates from the following generalization of the commutative law:

$$xy^nx^my = x^{m+1}y^{n+1}. \tag{3}$$

2. We need an important lemma.

LEMMA 1. *Let* $I_0^r(x) = x^r$. *If* $k > 1$, *let* $I_k^r(x) = I_{k-1}^r(1+x) - I_{k-1}^r(x)$. *Then* $I_{r-1}^r(x) = \frac{1}{2}(r-1)r! + r!\,x$; $I_r^r(x) = r!$, and $I_j^r(x) = 0$ for $j > r$.

Proof. We first prove that for $k < r+1$

$$I_k^{r+1}(x) = kI_{k-1}^r(1+x) + xI_k^r(x). \tag{4}$$

Obviously $I_0^{r+1}(x) = xI_0^r(x)$.

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If $k=1$, then

$$\begin{aligned} I_1^{r+1}(x) &= I_0^{r+1}(1+x)-I_0^{r+1}(x) \\ &= (1+x)I_0^r(1+x)-xI_0^r(x) \\ &= I_0^r(1+x)+x(I_0^r(1+x)-I_0^r(x)) \\ &= I_0^r(1+x)+xI_1^r(x). \end{aligned}$$

If for $k=m$, we have

$$I_m^{r+1}(x)=mI_{m-1}^r(1+x)+xI_m^r(x).$$

Then

$$\begin{aligned} I_{m+1}^{r+1}(x) &= I_m^{r+1}(1+x)-I_m^{r+1}(x) \\ &= mI_{m-1}^r(1+(1+x))+(1+x)I_m^r(1+x) \\ &\quad -mI_{m-1}^r(1+x)-xI_m^r(x) \\ &= m(I_{m-1}^r(1+(1+x))-I_{m-1}^r(1+x)) \\ &\quad +I_m^r(1+x)+x(I_m^r(1+x)-I_m^r(x)) \\ &= mI_m^r(1+x)+I_m^r(1+x)+xI_{m+1}^r(x) \\ &= (m+1)I_m^r(1+x)+xI_{m+1}^r(x). \end{aligned}$$

Hence (4) holds.

Now we prove that

$$(5) \qquad I_{r-1}^r(x)=\tfrac{1}{2}(r-1)r!+r!\,x; \qquad I_r^r(x)=r!.$$

Let $r=2$. Then

$$\begin{aligned} I_1^2(x) &= I_0^2(1+x)-I_0^2(x) \\ &= (1+x)^2-x^2 \\ &= 1+2x, \end{aligned}$$

and

$$I_2^2(x)=I_1^2(1+x)-I_1^2(x)=2.$$

If for $r=m$, we have

$$I_{m-1}^m(x)=\tfrac{1}{2}(m-1)m!+m!\,x; \qquad I_m^m(x)=m!,$$

Then by (4),

$$\begin{aligned} I_m^{m+1}(x) &= mI_{m-1}^m(1+x)+xI_m^m(x) \\ &= m(\tfrac{1}{2}(m-1)m!+m!\,(1+x))+xm! \\ &= \tfrac{1}{2}m(m+1)!+(m+1)!\,x, \end{aligned}$$

and

$$I_{m+1}^{m+1}(x)=I_m^{m+1}(1+x)-I_m^{m+1}(x)=(m+1)!.$$

Hence (5) holds. It is trivial that $I_j^r(x)=0$ for $j>r$.

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THEOREM 1. *Let R be a ring with identity. If R satisfies* (3) *and* $mnm!\,n!\,x\neq 0$ *except* $x=0$, *then R is commutative.*

Proof. Let $[x, y]=xy-yx$ and $I_j(x)=I_j^m(x)$ for $j=0, 1, 2, \ldots$.
Since $xy^nx^my=x^{m+1}y^{n+1}$, we have

$$x[y^n, x^m]y=0,$$
$$x[y^n, I_0(x)]y=0.$$

Let $x=1+x$ in the above expression. Then we have

$$[y^n, I_1(x)+I_0(x)]y+x[y^n, I_1(x)+I_0(x)]y=0,$$
$$[y^n, I_1(x)]y+[y^n, I_0(x)]y+x[y^n, I_1(x)]y=0.$$

Let $x=1+x$ in the above expression. Then we have

$$2[y^n, I_2(x)]y+2[y^n, I_1(x)]y+x[y^n, I_2(x)]y=0.$$

Let $x=1+x$ in the above expression. Then we have

$$3[y^n, I_3(x)]y+3[y^n, I_2(x)]y+x[y^n, I_3(x)]y=0.$$

Thus letting $x=1+x$ and iterating $m-1$ times we have

$$m[y^n, I_{m-1}(x)]y=m[y^n, \tfrac{1}{2}(m-1)m!+m!\,x]y=0,$$
$$mm!\,[y^n, x]y=0.$$

Now let $y=1+y$, iterate the above equality $n-1$ times, we have

$$mnm!\,n!\,[y, x]=0.$$

By the assumption of the theorem, $[y, x]=0$, R is commutative.

THEOREM 2. *Let R be a ring with identity. If R satisfies the following equality*

$$xy^nx^m=x^{m+1}y^n, \tag{6}$$

and $mm!\,n!\,x\neq 0$ *except* $x=0$, *then R is commutative.*

Proof. Since

$$xy^nx^m=x^{m+1}y^n,$$

we have

$$x[y^n, x^m]=0.$$

Letting $x=1+x$ and iterating $m-1$ times we have

$$mm!\,[y^n, x]=0.$$

Let $y=1+y$ in the above expression. Then we have

$$mm!\,[I_1(y)+I_0(y), x]=0,$$
$$mm!\,[I_1(y), x]=0.$$

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Letting $y=1+y$, iterate $n-1$ times, we have

$$mm!\,n!\,[y,x]=0.$$

By the assumption of the theorem, $[y,x]=0$, R is commutative.

THEOREM 3. *Let R be a ring with identity. If R satisfies the following equality*

$$x^m y^n = y^n x^m, \tag{7}$$

and $m!\ n!\ x\neq 0$ except $x=0$, then R is commutative.

Proof. Trivial.

3. Consider the following equality

$$\sum_{i\in I} x^{s_i}[x^{m_i}, y^{n_i}]y^{t_i}=0, \tag{8}$$

where s_i, m_i, n_i, t_i are positive integers for each i in a finite set I.

THEOREM 4. *Let R be a ring with identity. If R satisfies* (8), *and $N(s_i, m_i, n_i, t_i; I)x\neq 0$ except $x=0$ for a definite positive integer $N(s_i, m_i, n_i, t_i; I)$, then R is commutative.*

Proof. By Lemma 1, $I_s^{s_i}(x)=0$ $(s>s_i)$; $I_m^{m_i}(x)=0$ $(m>m_i)$; $I_n^{n_i}(y)=0$ $(n>n_i)$; $I_t^{t_i}(y)=0$ $(t>t_i)$. It is easily seen that $[x, 1+y]=[1+x, y]=[x, y]$. Therefore, letting $x=1+x$ in (8) and iterating sufficiently large number of times, we have

$$\sum_{i\in I} M_i(s_i, m_i)[x, y^{n_i}]y^{t_i}=0.$$

Let $y=1+y$, iterate sufficiently large number of times, we have

$$\sum_{i\in I} M_i(s_i, m_i)L_i(n_i, t_i)[x, y]=0.$$

Let $N(s_i, m_i, n_i; I)=\sum_{i\in I} M_i(s_i, m_i)L(n_i, t_i)$. Then we finish the proof.

Theorem 4 can be generalized.

THEOREM 5. *Let R be a ring with identity, u_i, v_i $(i\in I)$ be reduced words in x, y with positive exponents. If R satisfies the following equality*

$$\sum_{i\in I} u_i[x^{m_i}, y^{n_i}]v_i=0 \tag{9}$$

for a finite set I, and $N(u_i, m_i, n_i, v_i; I)x\neq 0$ except $x=0$ for a definite positive integer $N(u_i, m_i, n_i, v_i; I)$, then R is commutative.

COROLLARY 1. *Let R be a ring with identity. If R satisfies the following equality*

$$x^s y^n x^m y^t = x^{s+m} y^{n+t}, \tag{10}$$

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and $N(s, m, n, t)x \neq 0$ *except* $x = 0$ *for a definite positive integer* $N(s, m, n, t)$, *then R is commutative.*

Proof. (10) is equivalent to $x^s[y^n, x^m]y^t = 0$.

COROLLARY 2. *Let R be a ring with identity. If R satisfies* (2) *for* $n = k, k+1$, *and* $N(p, q, k)x \neq 0$ *except* $x = 0$ *for a definite positive integer* $N(p, q, k)$, *then R is commutative.*

Proof. Since

$$x^{p(k+1)}y^{q(k+1)} = (x^p y^q)(x^p y^q)^k = x^p y^q x^{pk} y^{qk},$$

we have $x^p[x^{pk}, y^q]y^{qk} = 0$.

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