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ASYMPTOTIC EQUIVALENCE OF ORDINARY AND OPERATOR-DIFFERENTIAL EQUATIONS

P.S. SIMEONOV AND D.D. BAINOV

The paper considers problems connected with the asymptotic equivalence of the system of ordinary differential equations

$$\frac{dx}{dt} = f(t,x) \tag{1}$$

and the system of operator differential equations

$$\frac{dy}{dt} = f(t,y) + g(t,y,A_t y) \tag{2}$$

The generality of the operator A_t guarantees a number of its important implementations. By a specific choice of the operator A_t the system (2) can be one of the concentrated delay, a system of distributed delay, a system with maxima, etcetera.

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1. Preliminary Notation

This paper considers the problem of asymptotic equivalence of two non-linear systems of differential equations, one of them being operator-defferential. The proof of the main results will employ some of the ideas presented in [1] and [2] .

Let α be a real number, D be a region of the real Euclidean space R^n with norm $|\cdot|$, and x and y be n-dimensional vector-functions defined on $[\alpha,\infty)$.

Let S denote the space of bounded, continuous n-dimensional vector-functions defined on $[\alpha,\infty)$ with norm $||y|| = \sup\limits_{t\geq\alpha}|y(t)|$.

For each $t \in [\alpha,\infty)$, let an operator $A_t\colon S \to D^m$ be defined.

We shall consider the following two equations:

$$\frac{dx}{dt} = f(t,x) \ , \tag{1}$$

$$\frac{dy}{dt} = f(t,y) + g(t,y,A_t y) \ . \tag{2}$$

DEFINITION 1. The systems (1) and (2) are said to be asymptotically equivalent if whenever one of these systems possesses a bounded solution on the half-line, the other possesses a solution which tends to that of the first as the independent variable t tends to ∞ .

We shall say that condition (A) holds if the following conditions are satisfied:

A1. in the domain $[\alpha,\infty) \times D$ the function f is continuously differentiable, while $f_x(t,x) = \frac{\partial f(t,x)}{\partial x}$ is locally Lipschitzian in X ;

A2. the function $g(t,y,z)$ is continuous in the domain $[\alpha,\infty) \times D \times D^m$.

We shall say that condition (B) holds if the following conditions are satisfied:

B1. for $y \in S$ and fixed, the function $A_t y$ is continuous in $t \in [\alpha,\infty)$;

B2. for any $\epsilon > 0$ and any $\tau > 0$ there exists $\delta = \delta(\epsilon,\tau) > 0$ such that, for $z_1, z_2 \in S$ and $\|z_1 - z_2\| < \delta$, the inequality $|A_t z_1 - A_t z_2| < \epsilon$ holds for each $t \in [\alpha,\tau]$.

Under the assumption that conditions (A) and (B) hold, the equation (2) is an operator-differential one, which, by restrictions on the operator A_t , includes important classes of functional-differential equations, such as

$$\frac{dy}{dt} = f(t,y) + g(t,y,y(\Delta(t))) ,$$

$$\frac{dy(t)}{dt} = f(t,y(t)) + g(t,y(t),y(t-a),y(t-b)) ,$$

$$\frac{dy}{dt} = f(t,y) + g(t,y, \max_{s\in E(t)} y(s)) ,$$

$$\frac{dy}{dt} = f(t,y) + g(t,y, \int_0^t G(t,s)y(s)ds) .$$

Let us denote by $x(t;t_0,y_0)$ the solution of (1), which satisfies the initial condition $x(t_0;t_0,x_0) = x_0$. By $\phi(t,t_0,x_0)$ we denote the fundamental matrix solution of the equation of variations with respect to the solution $x(t;t_0,x_0)$:

$$\frac{dz}{dt} = f_x(t,x(t;t_0,x_0))z .$$

Recall that

$$\phi(t_0,t_0,x_0) = I , \qquad \text{(Unit matrix)}$$

(3)
$$\frac{\partial x(t;t_0,x_0)}{\partial x_0} = \phi(t,t_0,x_0) ,$$

$$\frac{\partial x(t;t_0,x_0)}{\partial t_0} = -\phi(t,t_0,x_0)f(t_0,x_0) .$$

C1. the set $\Omega \subset \mathcal{D}$ is bounded, open, convex and the closure $\bar{\Omega} \subset \mathcal{D}$.

C2. for an arbitrary $t_o \geq \alpha$ and $x_o \in \Omega$ the solution $x(t;t_o,x_o)$ of (1) exists for $t \in [\alpha,t_o]$ and has values in $\mathcal{D}$ (this implies that the corresponding fundamental matrix $\phi(t,t_o,x_o)$ exists in the same region) .

2. Main results

THEOREM 1. *Let conditions (A), (B) and (C) hold. Let $y(t)$ be a solution of (2) with values in Ω for $t \geq \infty \geq \alpha$ and suppose the integral*

$$\int_t^\infty \phi(t,s,y(s))g(s,y(s),A_s y)ds \tag{4}$$

converges uniformy on each bounded subinterval of $[\beta,\infty)$. Then

$$x(t) = y(t) + \int_t^\infty \phi(t,s,y(s))g(s,y(s),A_s y)ds$$

is a solution of (1) on $[\beta,\infty)$, and $|x(t) - y(t)| \to 0$ as $t \to \infty$ if and only if

$$\left|\int_t^\infty \phi(t,s,y(s))g(s,y(s),A_s y)ds\right| \to 0 \quad as \quad t \to \infty .$$

Proof. Since $y(s) \in \Omega$ for $s \geq \beta$, it follows from C2 that the integral $\int_t^T \phi(t,s,y(s))g(s,y(s),A_s y)ds$ is defined for $t \in [\beta,T]$. Also by using (3), we obtain, for $s \geq \beta$

$$\frac{dx(t;s,y(s))}{ds} = \phi(t,s,y(s))\left[- f(s,y(s)) + \frac{dy(s)}{ds}\right]$$
$$= \phi(t,s;y(s))g(s,y(s),A_s y)$$

or

$$x(t;T,y(T)) = y(t) + \int_t^T \phi(t,s,y(s))g(s,y(s),A_s y)ds \quad \text{for} \quad \beta \leq t \leq T .$$

Since the improper integral (4) converges uniformly on each bounded subinterval of $[\beta,\infty)$, then the function

(5) $x(t) \equiv \lim_{T\to\infty} x(t;T,y(T)) = y(t) + \int_t^\infty \phi(t,s,y(s))g(s,y(s),A_s y)ds$

as a uniform limit of solutions of (1), is also a solution of (1) on this subinterval and hence $x(t)$ is a solution of (1) on $[\infty,\alpha)$. Moreover, (5) implies that $|x(t) - y(t)| \to 0$ as $t \to \infty$.

Theorem 1 solves the first part of the problem of asymptotic equivalence.

Now let $x(t)$ be a solution of (1) with values in Ω for $t \geq \alpha$ and without limit points on the boundary of Ω , that is there exists a number $d > 0$ such that for all $t \geq \alpha$ $\{x: |x - x(t)| \leq d\} \subset \Omega$

Theorem 2 investigates the converse problem to that considered in Theorem 1. We shall obtain the result using the following condition (D):

D. If $z(t)$ is a continuous function with values in Ω for $t \geq \alpha$ and if $\alpha \leq t \leq T$, then

$$\int_T^\infty |\phi(t,s,z(s))g(s,z(s),A_s z)|ds < H(T) ,$$

where $H(T) \to 0$, as $T \to \infty$ (we assume, without loss of generality, that $H(T)$ is a continuous and non-increasing function) .

THEOREM 2. *Let conditions (A), (B), (C) and (D) hold and let* $x(t)$ *be a bounded solution of (1) with values in* Ω *for* $t \geq \beta$ *without limit points on the boundary of* Ω .

Then there is a $\beta \geq \alpha$ and a solution $y(t)$ of (2) for $t \geq \beta$, such that $|x(t) - y(t)| \to 0$ as $t \to \infty$.

Proof. We choose $\beta \geq \alpha$ so that $H(\beta) \leq d$. For any integer $n \geq \beta$ we define the set

$$D_n = \{z \in S: |z(t) - x(t)| \leq d , \quad \beta \leq t \leq n$$

and the operator $Q:D_n \to S$ as follows

$$Qz(t) = \begin{cases} x(t) - \int_t^n \phi(t,s,z(s))g(s,z(s),A_s z)ds & \text{for } \beta \le t \le n, \\ x(t) - \int_\beta^n \phi(\beta,s,z(s))g(s,z(s),A_s z)ds & \text{for } d \le t \le \beta, \\ x(t) & \text{for } t \le n . \end{cases}$$

It is easy to see that D_n is a closed, convex, bounded subset of S. We shall show that the operator Q has a fixed point $y_n \in D_n$. For this purpose we shall employ the Schauder Fixed Point Theorem, for whose application it is sufficient to show that: I. $QD_n \subset D_n$; II. Q is continuous; III. $\overline{QD_n}$ is compact in S.

I. The choice of β and condition (D) imply that for $t \ge \alpha$

$$(6) \quad |Qz(t) - x(t)| \le \sup_{\beta\le\tau\le n} \int_\tau^n |\phi(\tau,s,z(s))g(s,z(s),A_s z)|ds \le H(\beta) \le d$$

that is $QD_n \subset D_n$.

II. Let $z_1, z_2 \in D_n$. Then for $t \ge \alpha$ we get the estimate

$$|Qz_1(t) - Qz_2(t)| \le \sup_{\beta\le\tau\le n} \int_\tau^n |\phi(\tau,s,z_1(s))g(s,z_1(s),A_s z_1) - \phi(\tau,s,z_2(s))g(s,z_2(s),A_s z_2)|ds .$$

Taking into account (A), (B) and (C) we conclude that Q is continuous.

III. From (6) it follows that the set QD_n is bounded. Since $Qz(t) = x(t)$ for all $z \in D_n$ and $t \ge n$, it is sufficient to show that the set QD_n is equicontinuous on the interval $[\alpha,n]$.

Let $\varepsilon > 0$ be given. Since $x(t)$ is uniformly continuous on $[\alpha,n]$, there exists $\delta_1 > 0$ such that $\alpha \le t_1 \le t_2 \le n$ and $|t_1 - t_2| < \delta_1$ imply that $|x(t_1) - x(t_2)| < \frac{\varepsilon}{4}$. Then, if $z \in D_n$, $\beta \le t_1 \le t_2 \le n$ and $|t_1 - t_2| < \delta_1$, we have

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$$|Qz(t_1) - Qz(t_2)| \le |x(t_1) - x(t_2)| + \int_{t_1}^{t_2} |\phi(t_1,s,z(s))g(s,z(s),A_s z)|ds$$

$$(7) \qquad + \int_{t_2}^{n} |\phi(t_1,s,z(s)) - \phi(t_2,s,z(s))||g(s,z(s),A_s z)|ds .$$

Then the mean value theorem and the equality $\frac{\partial\phi(t,s,z(s))}{\partial t}$ $= f_x(t,x(t;s,z(s)))\phi(t,s,z(s))$ imply that $\phi(t_2,s,z(s)) - \phi(t_1,s,z(s))$ $= f_x(\tau,x(\tau;s,z(s)))\phi(\tau,s,z(s))(t_2 - t_1)$, where $t_1 < \tau = \tau(s) < t_2$. Therefore, the second integral in (7) does not exceed

$$(8) \quad |t_2 - t_1| \int_{t_2}^{n} |f_x(\tau(s),x(\tau(s);s,z(s)))||\phi(\tau(s),s,z(s))||g(s,z(s),A_s z)|ds .$$

Since the integrands in the first integral of the estimate (7) and in (8) are limited by a common constant, independent of $t_1 \in [\beta,n]$, $t_2 \in [\beta,n]$ and $z \in D_n$, then there exists a $\delta < \delta_1$, such that for $\beta \le t_1 \le t_2 \le n$, $|t_1 = t_2| < \delta$ and $z \in D_n$ the two integrals in (7) are less than $\frac{\varepsilon}{4}$. Hence $|Qz(t_1) - Qz(t_2)| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$.

If $\alpha \le t_1 \le \beta \le t_2 \le n$, $|t_1 - t_2| < \delta$, then $|Q_z(t_1) - Q_z(t_2)|$ $\le |Qz(t_1) - Qz(\beta)| + |Qz(\beta) - Qz(t_2)| = |x(t_1) - x(\beta)| + |Qz(\beta) - Qz(t_2)|$ $< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon$.

Finally, if $\alpha \le t_1 \le t_2 \le \beta$, $|t_1 - t_2| < \delta$, then $|Qz(t_1) - Qz(t_2)|$ $= |x(t_1) - x(t_2)| < \frac{\varepsilon}{4} < \varepsilon$.

Thus, the hypothesis of Schauder Fixed Point Theorem holds and there exists an $y_n \in D_n$ such that $y_n = Qy_n$, that is

$$y_n(t) = x(t) - \int_t^n \phi(t,s,y_n(s))g(s,y_n(s),A_s y_n)ds , \quad \text{for } \beta \le t \le n$$

or

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$$y_n'(t) = f(t,x(t))+g(t,y_n(t),A_t y_n)$$
$$- \int_t^n f_x(t,x(t;s,y_n(s)))\phi(t,s,y_n(s))g(s,y_n(s),A_s y_n)ds$$

Since

$$f(t,y_n(t)) - f(t,x(t)) = f(t,x(t;t,y_n(t))) - f(t,x(t))$$
$$= \int_n^t \frac{d}{ds} f(t,x(t;s,y_n(s)))ds$$
$$= \int_n^t f_x(t,x(t;s,y_n(s)))\phi(t,s,y_n(s))[y_n'(s) - f(s,y_n(s))]ds$$

we have

$$w(t) = - \int_n^t f_x(t,x(t;s,y_n(s)))\phi(t,s,y_n(s))w(s)ds \ , \quad \text{for} \quad \beta \le t \le n$$

where $w(t) = y_n'(t) - f(t,y_n(t)) - g(t,y_n(t),A_t y_n)$. But this implies $w(t) = 0$, thus $y_n(t)$ is a solution of (2) on $[\beta,n]$.

Let $N \ge \beta$ be an integer and consider the sequence y_n , $n = N,N+1,\ldots$ of fixed points. Clearly, $|y_n(t)| \le \sup_{t\ge\beta} |x(t)| + d$ for $t \ge \beta$ and the sequence $\{y_n\}_N^\infty$ is equicontinuous on $[\beta,N]$.

By Ascoli's theorem there is a subsequence $\{y_{n1}\}$ of the y_n's converging uniformly on $[\beta,N]$. Similarly, the functions y_{n1} are solutions of (2) on $[\beta,N + 1]$ for $n1 \ge N + 1$ and the sequence $\{y_{n1}\}$ is equicontinuous on $[\beta,N + 1]$ so there is a subsequence $\{y_{n2}\}$ of the y_{n1}'s converging uniformly on $[\beta,N + 1]$. Clearly on the interval $[\beta,N]$ both subsequences converge to the same limit. Proceeding inductively we define a function $y(t)$ on $[\beta,\infty)$ and a chain of subsequences $\{y_{nk}\}$ such that $\{y_{nk}\}$ converges uniformly to y on $[\beta,N + K]$. The sequence $\{\bar{y}_n\}_1^\infty$, $\bar{y}_n = y_{nn}$ then converges to y uniformly on compact subintervals of $[\beta,\infty)$. Moreover, $y_n(t) \in \Omega$ and $y(t) \in \Omega$ for $t \ge \beta$.

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Then, using condition (D), we obtain that

$$\int_t^\infty \phi(t,s,y(s))g(s,y(s),A_s y)ds$$

exists for $t \geq \beta$. From the estimate

$$|\int_t^\infty \phi(t,s,y(s))g(s,y(s),A_s y)ds - \int_t^n \phi(t,s,\bar{y}_n(s))g(s,\bar{y}_n(s),A_s\bar{y}_n)ds|$$

$$\leq \int_t^m |\phi(t,s,y(s))g(s,y(s),A_s y) - \phi(t,s,\bar{y}_n(s))g(s,\bar{y}_n(s),A_s\bar{y}_n)|ds + 2H(m)$$

$(m > t)$ it follows that

$$\lim_{n\to\infty} \int_t^n \phi(t,s,\bar{y}_n(s))g(s,\bar{y}_n(s),A_s\bar{y}_n)ds = \int_t^\infty \phi(t,s,y(s))g(s,y(s),A_s y)ds$$

uniformly on compact subintervals of $[\beta,\infty)$.

The functions $\bar{y}_n(t)$ are solutions of (2) on $[\beta,N+n]$; consequently $y(t)$ is also solution of (2) on $[\beta,\infty)$ and

$$y(t) = x(t) - \int_t^\infty \phi(t,s,y(s))g(s,y(s),A_s y)ds$$

Hence, $|y(t) - x(t)| \to 0$, as $t \to \infty$.

The following theorem is a corollary of Theorem 1 and Theorem 2.

THEOREM 3. *Let conditions (A), (B), (C) and (D) hold. Then if either of the equations (1) or (2) possesses a bounded solution on a half-line with values in* Ω *and without limit points on the boundary of* Ω *, then the other also possesses such a solution which tends to the former as* $t \to \infty$.

COROLLARY 1. *Let us suppose that the differential equation (1) is linear,* $f(t,x) = A(t)x$ *, where* $A(t) = a_{ij}(t))_1^n$.

Let $g_K, K=1,2,\ldots,n$ be the components of the function g , that is $g_K:[\alpha,\infty)XDXD^m \to R^1$ and the functions $h_K:[\alpha,\infty) \to [0,\infty)$ be such that $|g_K(t,x,y)| \leq h_K(t)$ for $(t,x,y) \in [\alpha,\infty)XDXD^m$ and $K=1,2,\ldots,n$.

Suppose also that for i,j and $K(i\neq j)$ we have

$$\int_t^\infty h_K(\tau)[\exp \int_\tau^t a_{KK}(s)ds]d\tau \to 0 \quad \text{as} \quad t \to \infty$$

and

$$\int_t^\infty |a_{ij}(\tau)|[\exp \int_\tau^t a_{ii}(s)ds]d\tau \to 0 \quad \text{as} \quad t \to \infty .$$

Then, if either of the equations (1) or (2) possesses a bounded solution on a half-line with values in Ω and without limit points on the boundary of Ω , then the other also possesses such a solution which tends to the former as $t \to \infty$.

The proof of Corollary 1 is carried out as in [2].

EXAMPLE 1. Consider the equations

$$\frac{dx}{dt} = - \frac{\sin 2x}{2t} , \tag{9}$$

$$\frac{dy}{dt} = - \frac{\sin 2y}{2t} + g(t,y,A_t y) , \tag{10}$$

where $t \in [1,\infty)$; $x,y \in \Omega = (-\frac{\pi}{2},\frac{\pi}{2})$; $A_t y = \max\limits_{s\in[1,t]} y(s)$;

$$|g(t,y(t),A_t y)| \le h(t) \quad \text{if} \quad y(t) \in \Omega \quad \text{and} \quad \int^\infty h(t)tdt < \infty .$$

We have $x(t;s,z) = \arctan(\frac{s}{t} \tan z)$ and $\phi(t,s,z) = \frac{\partial x}{\partial z}$ $= \frac{st}{s^2 \sin^2 z + t^2 \cos^2 z}$. Then, if $z(s) \in \Omega$ for $s \ge T \ge t \ge 1$ following estimate is valid

$$\int_T^\infty |\phi(t,s,z(s))g(s,z(s),A_s z)|ds \le \int_T^\infty \frac{h(s)s}{t} ds \le \int_T^\infty h(s)sds \equiv H(T) .$$

Since $\lim\limits_{T\to\infty} H(T) = 0$, the conditions of Theorem 3 are fulfilled and equations (9) and (10) are asymptotically equivalent.

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23, Oborishte Str.,
1504 Sofia,
Bulgaria.

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