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ANALYSIS ON SPARSE PARTS IN THE MAXIMAL IDEAL SPACE OF H^∞

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ABSTRACT. Analysis on sparse parts of the Banach algebra of bounded analytic functions is given. It is proved that Sarason's theorem for QC-level sets cannot be generalized to general Douglas algebras.

0. **Introduction.** Let D be the open unit disc and let H^∞ be the space of bounded analytic functions on D. With the supremum norm $\|\cdot\|_\infty$, H^∞ becomes a Banach algebra. We denote by L^∞ the space of bounded measurable functions on the unit circle ∂D with respect to the Lebesgue measure. By identifying a function in H^∞ with its boundary function, we may consider that H^∞ is an essentially supremum norm closed subalgebra of L^∞. A norm closed subalgebra B with $H^\infty \subset B \subset L^\infty$ is called *a Douglas algebra*. By Sarason [14], $H^\infty + C$ is the smallest Douglas algebra, where C is the space of continuous functions on ∂D. We denote by $M(B)$ the maximal ideal space of B with the weak*-topology. Then we can consider that $M(L^\infty) \subset M(B) \subset M(H^\infty) = M(H^\infty + C) \cup D$, and $M(L^\infty)$ becomes the Shilov boundary for every Douglas algebra B. We identify a function with its Gelfand transform. For a point ζ in $M(H^\infty)$, there is a representing measure μ_ζ on $M(L^\infty)$; $\int_{M(L^\infty)} f\, d\mu_\zeta = f(\zeta)$ for every $f \in H^\infty$. We denote by $\operatorname{supp}\mu_\zeta$ the closed support set of μ_ζ. The pseudo-hyperbolic metric ρ on $M(H^\infty)$ is defined as follows;

$$\rho(\zeta,\xi) = \sup\{|f(\xi)| \,; f \in H^\infty, \|f\|_\infty \leq 1, f(\zeta) = 0\}.$$

The set $P(\zeta) = \{\xi \in M(H^\infty) \,;\, \rho(\zeta,\xi) < 1\}$ is called *a Gleason part*. If $P(\zeta) \neq \{\zeta\}$, in [9] Hoffman proved that there is a continuous one to one map L_ζ from (another) open unit disc Δ onto $P(x)$ such that $f \circ L_\zeta \in H^\infty(\Delta)$ for every $f \in H^\infty$. To avoid the confusion, we use Δ as the domain of Hoffman's map L_ζ, and we define $L^\infty(\partial\Delta)$, $(H^\infty + C)(\Delta)$ and $M\big(H^\infty(\Delta)\big)$ as on D.

A function ϕ in H^∞ is called *inner* if $|\phi| = 1$ on $M(L^\infty)$. For a sequence $\{z_n\}_n$ in D with $\sum_{n=1}^\infty 1 - |z_n| < \infty$, a function

$$\psi(z) = \prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|}\frac{z - z_n}{1 - \bar{z}_n z} \qquad z \in D$$

is called a *Blaschke product* and $\{z_n\}_n$ is called the *zero sequence* of ψ. Moreover if

$$\inf_k \prod_{n:n\neq k}\left|\frac{z_k - z_n}{1 - \bar{z}_n z_k}\right| > 0 \text{ and } \lim_{k\to\infty}\prod_{n:n\neq k}\left|\frac{z_k - z_n}{1 - \bar{z}_n z_k}\right| = 1,$$

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then ψ is called *interpolating* and *sparse* respectively. Put

$$\delta(\psi) = \inf_k \prod_{n:n\neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right|.$$

For $f \in H^\infty$, we denote by $Z(f)$ the zero set of f on $M(H^\infty)$; $Z(f) = \{\zeta \in M(H^\infty) ; f(\zeta) = 0\}$. For a subset E of $M(H^\infty)$, we denote by $\mathrm{cl}\, E$ or $\bar{E}$ the weak*-closure of E in $M(H^\infty)$. If ψ is an interpolating Blaschke product with zeros $\{z_n\}_n$, then $\mathrm{cl}\{z_n\}_n = Z(\psi)$ and this set is homeomorphic to the Čech compactification of the discrete set (see [8, p. 205]), and if $\zeta \in Z(\psi)$ then $P(\zeta) \neq \{\zeta\}$ [9, Theorem 5.5].

In this paper, *we fix a sparse Blaschke product b and a point x in $Z(b)\backslash D$*. By [9, p. 107], there is a constant α with $|\alpha| = 1$, depending on b and x, such that $(b \circ L_x)(w) = \alpha w$ for $w \in \Delta$. For the sake of simplicity, in this paper we assume $\alpha = 1$, that is,

$$(b \circ L_x)(w) = w \text{ for every } w \in \Delta.$$

By Budde [2], there is a continuous extension

$$\hat{L}_x \colon M\big(H^\infty(\Delta)\big) \longrightarrow \overline{P(x)}$$

such that $(h \circ L_x)\hat{} = h \circ \hat{L}_x$ on $M\big(H^\infty(\Delta)\big)$ for every $h \in H^\infty$, and $\hat{L}_x$ becomes a homeomorphic map. For each $f \in H^\infty(\Delta)$, identifying D and Δ, $f \circ b \in H^\infty$ and $(f \circ b) \circ L_x(w) = f \circ (b \circ L_x)(w) = f(w)$ for $w \in \Delta$, so that we have $(f \circ b) \circ L_x = f$ on Δ. Hence

$$(\#) \qquad (f \circ b) \circ \hat{L}_x = f \text{ on } M\big(H^\infty(\Delta)\big).$$

This means that $H^\infty|_{\overline{P(x)}}$ is the same space with $H^\infty(\Delta)$ via the map $\hat{L}_x$. Put

$$\partial = \hat{L}_x\big(M(L^\infty(\partial\Delta))\big) \subset \overline{P(x)}.$$

Then ∂ becomes the Shilov boundary for the restriction algebra $H^\infty|_{\overline{P(x)}}$. For $\zeta \in \overline{P(x)}$, we denote by λ_ζ the representing measure on ∂ for $H^\infty|_{\overline{P(x)}}$. Put

$$H^\infty_{\mathrm{supp}\,\mu_x} = \{f \in L^\infty ; f|_{\mathrm{supp}\,\mu_x} \in H^\infty|_{\mathrm{supp}\,\mu_x}\}.$$

Since $\mathrm{supp}\,\mu_x$ is a weak peak set for H^∞ [8, p. 207], $H^\infty_{\mathrm{supp}\,\mu_x}$ is a Douglas algebra and

$$M(H^\infty_{\mathrm{supp}\,\mu_x}) = \{\zeta \in M(H^\infty) ; \mathrm{supp}\,\mu_\zeta \subset \mathrm{supp}\,\mu_x\} \cup M(L^\infty),$$

and also $P(x) \subset M(H^\infty_{\mathrm{supp}\,\mu_x})$.

We denote by I the set of inner functions ϕ on D such that $\phi \circ L_x$ is inner on Δ, that is, $|\phi| = 1$ on ∂. By (#), $(J \circ b) \circ \hat{L}_x = J$ on $M\big(H^\infty(\Delta)\big)$ for inner functions J on Δ. Since $J \circ b$ is an inner function (see [4, p. 442]), $I \circ \hat{L}_x$ coincides with the set of all inner functions on Δ. For a subset Γ of L^∞, we denote by $[\Gamma]$ the closed subalgebra of L^∞ generated by functions in Γ. Put

$$B_1 = [H^\infty_{\mathrm{supp}\,\mu_x}, \bar{b}] \text{ and } B_2 = [H^\infty_{\mathrm{supp}\,\mu_x}, \bar{\phi} ; \phi \in I].$$

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Then B_1 and B_2 are Douglas algebras, and

$$H^\infty_{\operatorname{supp}\mu_x} \subsetneq B_1 \subsetneq B_2 \subsetneq L^\infty.$$

For a Douglas algebra B, put $\mathrm{QC}_B = B \cap \bar{B}$, where $\bar{B}$ is the set of complex conjugate functions which are contained in B. For $\zeta \in M(L^\infty)$, the set

$$\{\xi \in M(L^\infty)\,;\, f(\xi) = f(\zeta) \text{ for every } f \in \mathrm{QC}_B\}$$

is called a QC_B-*level set*. For a function $g \in L^\infty$, we put

$$N_B(g) = \mathrm{cl}\big[\bigcup\{\operatorname{supp}\mu_\zeta\,;\,\zeta \in M(B), g|_{\operatorname{supp}\mu_\zeta} \notin H^\infty|_{\operatorname{supp}\mu_\zeta}\}\big].$$

When $B = H^\infty + C$, we abbreviate as QC and $N(g)$.

In [15], Sarason proved that if $f, g \in L^\infty$ and either $f|_{\operatorname{supp}\mu_\zeta} \in H^\infty|_{\operatorname{supp}\mu_\zeta}$ or $g|_{\operatorname{supp}\mu_\zeta} \in H^\infty|_{\operatorname{supp}\mu_\zeta}$ for each $\zeta \in M(H^\infty + C)$, then $f|_Q \in H^\infty|_Q$ or $g|_Q \in H^\infty|_Q$ for each QC-level set Q. In [12], the author proved that under the same condition, $N(f) \cap N(g) = \emptyset$, and gave several applications.

Our purpose of this paper is to show that the above results cannot be generalized to the Douglas algebra B_1, that is, there are two inner functions I and J, and a QC_{B_1}-level set Q such that

(a) $|I(\zeta)| = 1$ or $|J(\zeta)| = 1$ for every $\zeta \in M(B_1)$;
(b) both $I|_Q$ and $J|_Q$ are not constant;
(c) $N_{B_1}(\bar{I}) \cap N_{B_1}(\bar{J}) \neq \emptyset$.

We prove this theorem in Section 4. Sections 1, 2 and 3 are preparations for proving our main theorem. In Section 1, we shall prove that if $\zeta \in M(H^\infty_{\operatorname{supp}\mu_x}) \backslash \overline{P(x)}$ then there is a Blaschke product ψ such that $|\psi(\zeta)| = 1$ and $\psi = 0$ on $\overline{P(x)}$, and if $\phi \in I$ then $|\phi| = 1$ on $M(H^\infty_{\operatorname{supp}\mu_x}) \backslash \overline{P(x)}$. As a consequence, ∂ is the topological boundary of the set $\overline{P(x)}$ in $M(H^\infty_{\operatorname{supp}\mu_x})$. In Section 2, we study $\operatorname{supp}\mu_\zeta$ and $\operatorname{supp}\lambda_\zeta$ for $\zeta \in \overline{P(x)}$. We prove that $\operatorname{supp}\mu_\zeta = \mathrm{cl}[\bigcup\{\operatorname{supp}\mu_\xi\,;\,\xi \in \operatorname{supp}\lambda_\zeta\}]$. In Section 3, we study the Douglas algebra B_2, and prove that Sarason and author's theorems are true for B_2.

1. **Basic results.** Budde [2] (see also [7, p. 5]) proved the following lemma.

LEMMA 1. $P(x) = \{\zeta \in M(H^\infty_{\operatorname{supp}\mu_x})\,;\,|b(\zeta)| < 1\}$.

Hence $P(x)$ is an open subset of $M(H^\infty_{\operatorname{supp}\mu_x})$. Using the idea of Gorkin [5, Theorem 2.2], we can prove the following theorem. For the sake of completeness we give its proof.

THEOREM 1. *Let y be a point in $M(H^\infty_{\operatorname{supp}\mu_x}) \backslash \overline{P(x)}$. Then there is a Blaschke product ψ such that $|\psi(y)| = 1$ and $\psi = 0$ on $P(x)$.*

To prove Theorem 1, we use the following lemmas due to Hoffman [9]. For two subsets E_1 and E_2 of $M(H^\infty)$, put $\rho(E_1, E_2) = \inf\{\rho(\zeta, \xi)\,;\,\zeta \in E_1, \xi \in E_2\}$.

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LEMMA 2. *Let ϕ be an interpolating Blaschke product and $\delta(\phi) \geq \delta > 0$. Then there exist $r = r(\delta)$, $0 < r < 1$, and $\lambda = \lambda(\delta)$, $0 < \lambda < 1$, such that*

$$\{\zeta \in M(H^\infty)\ ;\ |\phi(\zeta)| < r\} \subset \Big\{\zeta \in M(H^\infty)\ ;\ \rho\big(\zeta, Z(\phi)\big) \leq \lambda\Big\}.$$

We may take as $r(\delta) \longrightarrow 1$ and $\lambda(\delta) \longrightarrow 1 \quad (\delta \longrightarrow 1)$.

LEMMA 3. *The pseudo-hyperbolic metric ρ is lower semi-continuous on $M(H^\infty) \times M(H^\infty)$.*

For a Blaschke product ψ with zeros $\{z_n\}_{n=1}^\infty$, a subproduct with zeros $\{z_n\}_{n=k}^\infty$ is called a *tail* of ψ.

PROOF OF THEOREM 1. Since $y \notin \overline{P(x)}$, there is an open subset U of $M(H^\infty)$ such that $y \in U$ and $\bar{U} \cap \overline{P(x)} = \emptyset$. Then $\rho(x, \bar{U}) = 1$. Take δ_n such that $0 < \delta_n < 1$, $\delta_n \longrightarrow 1$ and $\prod_{n=1}^\infty r(\delta_n) > 0$, where $r(\delta_n)$ is a constant given in Lemma 2. By Lemma 3, there is an open subset W_n of $M(H^\infty)$ such that $x \in W_n$ and $\lambda(\delta_n) < \rho(\bar{W}_n, \bar{U})$. Let b_n be a sparse Blaschke subproduct of b with the zero sequence $W_n \cap D \cap Z(b)$. Then $x \in Z(b_n) \subset \bar{W}_n$ by [8, p. 205]. Since b is sparse, by considering tails of $b_n, n = 1, 2, \ldots$, we may assume that $\delta(b_n) > \delta_n$ and $\psi = \prod_{n=1}^\infty b_n$ is a Blaschke product. Since $b_n(x) = 0$, $\psi = 0$ on $\overline{P(x)}$. Since $\lambda(\delta_n) < \rho\big(Z(b_n), \bar{U}\big)$, by Lemma 2, $|b_n| \geq r(\delta_n)$ on $\bar{U}$. Hence

$$\inf_{z \in D \cap U} \Big|\Big(\prod_{n=k}^\infty b_n\Big)(z)\Big| = \inf_{z \in D \cap U} \prod_{n=k}^\infty |b_n(z)| \geq \prod_{n=k}^\infty r(\delta_n).$$

By Lemma 1, $|b(y)| = 1$, so that $|b_n(y)| = 1$. Since $y \in \bar{U} = \overline{D \cap U}$,

$$\begin{aligned} |\psi(y)| = \Big|\Big(\prod_{n=k}^\infty b_n\Big)(y)\Big| &\geq \inf_{z \in D \cap U} \Big|\Big(\prod_{n=k}^\infty b_n\Big)(z)\Big| \\ &\geq \prod_{n=k}^\infty r(\delta_n) \longrightarrow 1 \quad (k \longrightarrow \infty). \end{aligned}$$

To prove Theorem 2, we need a following lemma.

LEMMA 4 [16]. *For every inner function I, there is an interpolating Blaschke product J such that $\{\zeta \in M(H^\infty)\ ;\ |J(\zeta)| = 1\} = \{\zeta \in M(H^\infty)\ ;\ |I(\zeta)| = 1\}$.*

THEOREM 2. *If $\phi \in I$, then $|\phi| = 1$ on $M(H^\infty_{\operatorname{supp}\mu_x}) \backslash \overline{P(x)}$.*

PROOF. First we shall prove when ϕ is interpolating. To prove our assertion, suppose not. Then there is a point y in $M(H^\infty_{\operatorname{supp}\mu_x}) \backslash \overline{P(x)}$ such that $|\phi(y)| < 1$. Then ϕ is not invertible in $H^\infty_{\operatorname{supp}\mu_y}$ and there is a point ζ in $M(H^\infty_{\operatorname{supp}\mu_y})$ such that $\phi(\zeta) = 0$. Here we have $\operatorname{supp}\mu_\zeta \subset \operatorname{supp}\mu_y$. By Theorem 1, there is a Blaschke product ψ such that $|\psi(y)| = 1$ and $\psi = 0$ on $\overline{P(x)}$. Since $\psi = \psi(y)$ on $\operatorname{supp}\mu_y$, $|\psi(\zeta)| = 1$, so that $\zeta \in M(H^\infty_{\operatorname{supp}\mu_x}) \backslash \overline{P(x)}$. Hence there is a subproduct ϕ_1 of ϕ such that $\phi_1(\zeta) = 0$ and $Z(\phi_1) \cap \overline{P(x)} = \emptyset$ (see [10]). Since $\phi \circ \hat{L}_x$ is inner, $\phi_1 \circ \hat{L}_x$ is also inner. Since $\overline{P(x)} = \hat{L}_x\big(M(H^\infty(\Delta))\big)$, $\phi_1 \circ \hat{L}_x$ does not vanish on $M\big(H^\infty(\Delta)\big)$. Therefore $\phi_1 \circ \hat{L}_x = c$ for some constant c with $|c| = 1$,

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that is, $\phi_1 = c$ on $\overline{P(x)}$. Since $c = \phi_1(z) = \int_{M(L^\infty)} \phi_1 \, d\mu_x$, $\phi_1 = c$ on $\operatorname{supp} \mu_x$. Since $\operatorname{supp} \mu_\zeta \subset \operatorname{supp} \mu_y \subset \operatorname{supp} \mu_x$, $\phi_1(\zeta) = \int_{M(L^\infty)} \phi_1 \, d\mu_\zeta = c$. This is a contradiction.

Next suppose that ϕ is a general inner function in I. By Lemma 4, there is an interpolating Blaschke product I such that

$$\{\zeta \in m(H^\infty) \, ; \, |I(\zeta)| = 1\} = \{\zeta \in M(H^\infty) \, ; \, |\phi(\zeta)| = 1\}.$$

Since $\phi \in I$, $|\phi| = 1$ on ∂. Hence $|I| = 1$ on ∂ and $I \in I$. By the first paragraph, $|I| = 1$ on $M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \overline{P(x)}$, so that $|\phi| = 1$ on $M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \overline{P(x)}$.

The following theorem shows that ∂, not $\overline{P(x)} \setminus P(x)$, is the topological boundary of $\overline{P(x)}$ in $M(H^\infty_{\operatorname{supp} \mu_x})$.

THEOREM 3. $\partial = \overline{P(x)} \cap \operatorname{cl}[M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \overline{P(x)}]$.

PROOF. Let $\zeta \in \overline{P(x)} \setminus \partial$. Then $\zeta = \hat{L}_x(\eta)$ for some $\eta \in M\big(H^\infty(\Delta)\big) \setminus M\big(L^\infty(\partial\Delta)\big)$. By [8, p. 179], there is an inner function J on Δ such that $|J(\eta)| < 1$. Since $(J \circ b) \circ \hat{L}_x = J$, $|(J \circ b)(\zeta)| < 1$. Since $J \circ b \in I$, by Theorem 2 $|J \circ b| = 1$ on $M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \overline{P(x)}$, so that $\zeta \notin \operatorname{cl}[M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \overline{P(x)}]$. Hence

$$\partial \supset \overline{P(x)} \cap \operatorname{cl}[M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \overline{P(x)}].$$

To prove the converse inclusion, suppose that $\xi \in \partial$ and $\xi \notin \operatorname{cl}[M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \overline{P(x)}]$. We shall show a contradiction. Here we have

$$M(H^\infty_{\operatorname{supp} \mu_\xi}) = \{y \in M(H^\infty_{\operatorname{supp} \mu_x}) \, ; \, \operatorname{supp} \mu_y \subset \operatorname{supp} \mu_\xi\} \cup M(L^\infty).$$

Let $y \in M(H^\infty_{\operatorname{supp} \mu_x})$ with $\operatorname{supp} \mu_y \subset \operatorname{supp} \mu_\xi$ and $y \neq \xi$. Since $I \circ \hat{L}_x$ is the set of all inner functions on Δ, I separates the points in $\overline{P(x)}$ [4, p. 428]. If $y \in \overline{P(x)}$ then $\phi(y) \neq \phi(\xi)$ for some $\phi \in I$. Since $|\phi(\xi)| = 1$, $\phi = \phi(\xi)$ on $\operatorname{supp} \mu_\xi$. Hence $\phi(y) = \phi(\xi)$. This contradiction implies that $y \notin \overline{P(x)}$. Since $\xi \notin \operatorname{cl}[M(H^\infty_{\operatorname{supp} \mu_x}) \setminus \overline{P(x)}]$, ξ is an isolated point in $M(H^\infty_{\operatorname{supp} \mu_\xi})$. By Shilov's idempotent theorem, there is a function h in $H^\infty_{\operatorname{supp} \mu_\xi}$ such that $h(\xi) = 1$ and $h = 0$ on $M(H^\infty_{\operatorname{supp} \mu_\xi}) \setminus \{\xi\}$. Since $M(L^\infty) \subset M(H^\infty_{\operatorname{supp} \mu_\xi}) \setminus \{\xi\}$, $1 = h(\xi) = \int_{M(L^\infty)} h \, d\mu_\xi = 0$. This is the desired contradiction.

2. **Support sets.** Let u be a complex valued bounded harmonic function on D. By [1, Proposition 6], u can be extended continuously on $M(H^\infty)$; we use the same symbol u, and

$$u(\zeta) = \int_{M(L^\infty)} u \, d\mu_\zeta \text{ for } \zeta \in M(H^\infty). \tag{1}$$

For $v \in L^\infty$, the function $v(z) = \int_{M(L^\infty)} v \, d\mu_z$ for $z \in D$ is harmonic, so that $v(z)$ can be extended on $M(H^\infty)$, and its extended function coincides with the original v on $M(L^\infty)$. Therefore we identify a function in L^∞ with its harmonic extension on D.

For each point $\eta \in M\big(H^\infty(\Delta)\big)$, we denote by σ_η its representing measure on $M\big(L^\infty(\partial\Delta)\big)$. Put $\zeta = \hat{L}_x(\eta)$. Since $\hat{L}_x$ is a homeomorphism from $M\big(L^\infty(\partial\Delta)\big)$ onto ∂, there

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is a probability measure λ on ∂ such that $\int_\partial f\,d\lambda = \int_{M(L^\infty(\partial\Delta))} f\circ\hat{L}_x\,d\sigma_\eta$ for every $f \in C(\partial)$, the space of continuous functions on ∂. For $f \in H^\infty$, we have $\int_\partial f\,d\lambda = f\circ\hat{L}_x(\eta) = f(\zeta)$. Hence $\lambda = \lambda_\zeta$, the representing measure on ∂ for the point ζ, and $\operatorname{supp}\lambda_\zeta = \hat{L}_x(\operatorname{supp}\sigma_\eta)$. Since a real bounded harmonic function v has the form $v = \log|g|$ for some invertible function g in H^∞ [8, p. 182], $v\circ\hat{L}_x$ is harmonic on Δ, and by (1) and (#),

$$v(\zeta) = (v\circ\hat{L}_x)(\eta) = \int_{M(L^\infty(\partial\Delta))} v\circ\hat{L}_x\,d\sigma_\eta = \int_\partial v\,d\lambda_\zeta\ ; \text{ and}$$
$$(v\circ b)\circ\hat{L}_x = \log|(g\circ b)\circ\hat{L}_x| = \log|g| = v.$$

Hence

$$u(\zeta) = \int_\partial u\,d\lambda_\zeta \text{ for } \zeta \in \overline{P(x)} \text{ and } u \in L^\infty\ ; \tag{2}$$

$$(u\circ b)\circ\hat{L}_x = u \text{ on } M\big(H^\infty(\Delta)\big) \text{ for } u \in L^\infty(\partial\Delta). \tag{3}$$

For $\zeta \in M(H^\infty)$, $H^\infty(\Delta) \ni f \longrightarrow (f\circ b)(\zeta)$ is a nonzero homomorphism, hence there is a point η in $M\big(H^\infty(\Delta)\big)$ such that $f(\eta) = (f\circ b)(\zeta)$. We put $\eta = \hat{b}(\zeta)$. By [4, p. 441], $\hat{b}\colon M(H^\infty) \longrightarrow M\big(H^\infty(\Delta)\big)$ is a continuous map, and

$$(u\circ b)(\zeta) = u\big(\hat{b}(\zeta)\big) \text{ for } \zeta \in M(H^\infty) \text{ and } u \in L^\infty(\partial\Delta). \tag{4}$$

By (3), $\hat{b}\big(\hat{L}_x(\eta)\big) = \eta$ for $\eta \in M\big(H^\infty(\Delta)\big)$. Therefore $\hat{b} = \hat{L}_x^{-1}$ on $\overline{P(x)}$. We use this fact frequently.

LEMMA 5. *Let $\zeta \in [M(H^\infty_{\operatorname{supp}\mu_x})\backslash\overline{P(x)}] \cup \partial$. Then $\hat{L}_x\big(\hat{b}(\operatorname{supp}\mu_\zeta)\big) = \hat{L}_x\big(\hat{b}(\zeta)\big) \in \partial$. If $u \in L^\infty$ and $\xi \in \partial$, then $(u\circ\hat{L}_x)\circ b = u(\xi)$ on* $\operatorname{supp}\mu_\xi$.

PROOF. By Theorem 2, $|\phi(\zeta)| = 1$ for $\phi \in I$. If J is inner on Δ then $J\circ b \in I$. Hence $\big|J\big(\hat{b}(\zeta)\big)\big| = 1$. By [8, p. 179], $\hat{b}(\zeta) \in M\big(L^\infty(\partial\Delta)\big)$, so that $\hat{L}_x\big(\hat{b}(\zeta)\big) \in \partial$. Since inner functions separate the points in $M\big(L^\infty(\partial\Delta)\big)$ [4, p. 192], $J\big(\hat{b}(\zeta)\big) = \int_{M(L^\infty)} J\circ b\,d\mu_\zeta$ implies $\hat{b}(\operatorname{supp}\mu_\zeta) = \hat{b}(\zeta)$.

Suppose that $\xi \in \partial$. Then by (4), $[(u\circ\hat{L}_x)\circ b](\operatorname{supp}\mu_\xi) = u\Big(\hat{L}_x\big(\hat{b}(\xi)\big)\Big)$. Since $\hat{b} = \hat{L}_x^{-1}$ on $\overline{P(x)}$, $u\Big(\hat{L}_x\big(\hat{b}(\xi)\big)\Big) = u(\xi)$, so that $(u\circ\hat{L}_x)\circ b = u(\xi)$ on $\operatorname{supp}\mu_\xi$.

LEMMA 6. $\operatorname{supp}\mu_x = \operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in \partial\}]$.

PROOF. Suppose not. Then there is an open and closed subset W of $M(L^\infty)$ such that $\operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in \partial\}] \subset W$ and $\operatorname{supp}\mu_x \not\subset W$. Then $\mu_x(W) < 1$. We denote by χ_W the characteristic function for W. Since $\chi_W(\xi) = \int_{M(L^\infty)} \chi_W\,d\mu_\xi = 1$ for $\zeta \in \partial$ by (1) and (2)

$$1 = \int_\partial \chi_W\,d\lambda_x = \int_{M(L^\infty)} \chi_W\,d\mu_x,$$

so that $\mu_x(W) = 1$. This is a contradiction.

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COROLLARY 1. *Let $u \in L^\infty$. If u is constant on* $\operatorname{supp}\mu_\xi$ *for every* $\xi \in \partial$, *then* $u = (u \circ \hat{L}_x) \circ b$ *on* $\operatorname{supp}\mu_x$.

PROOF. By Lemma 5, $(u \circ \hat{L}_x) \circ b = u$ on $\operatorname{supp}\mu_\xi$ for every $\xi \in \partial$. By Lemma 6, $(u \circ \hat{L}_x) \circ b = u$ on $\operatorname{supp}\mu_x$.

For an open and closed subset U of ∂, put

$$\tilde{U} = \{\zeta \in M(L^\infty) \,;\, \hat{L}_x\big(\hat{b}(\zeta)\big) \in U\} = \{\zeta \in M(L^\infty) \,;\, \hat{b}(\zeta) \in \hat{L}_x^{-1}(U)\}.$$

By the proof of Lemma 5, $\hat{b}\big(M(L^\infty)\big) \subset M\big(L^\infty(\partial\Delta)\big)$, so that $\tilde{U}$ is an open and closed subset of $M(L^\infty)$. Also $\tilde{\partial} = M(L^\infty)$ and $(U \cap V)^\sim = \tilde{U} \cap \tilde{V}$ for open and closed subsets U and V. In this paper, $\tilde{U}$ plays the essential part.

LEMMA 7. *(i)* $\chi_{\tilde{U}} = 0$ *or* 1 *on* $[M(H^\infty_{\operatorname{supp}\mu_x})\backslash\overline{P(x)}] \cup \partial$.

(ii) For $\zeta \in [M(H^\infty_{\operatorname{supp}\mu_x})\backslash\overline{P(x)}] \cup \partial$, $\chi_{\tilde{U}}(\zeta) = 1$ *if and only if* $\hat{L}_x\big(\hat{b}(\zeta)\big) \in U$.

(iii) $\chi_{\tilde{U}} = \chi_U$ *on* ∂, *that is,* $\operatorname{supp}\mu_\xi \subset \tilde{U}$ *if and only if* $\xi \in U$ *for* $\xi \in \partial$.

(iv) For $\zeta \in \overline{P(x)}$, $\mu_\zeta(\tilde{U}) = \lambda_\zeta(U)$.

PROOF. Let $\zeta \in [M(H^\infty_{\operatorname{supp}\mu_x})\backslash\overline{P(x)}] \cup \partial$. By Lemma 5, $\hat{L}_x\big(\hat{b}(\operatorname{supp}\mu_\zeta)\big) = \hat{L}_x\big(\hat{b}(\zeta)\big) \in \partial$. If $\hat{L}_x\big(\hat{b}(\zeta)\big) \in U$, then $\operatorname{supp}\mu_\zeta \subset \tilde{U}$ and $\chi_{\tilde{U}}(\zeta) = 1$. If $\hat{L}_x\big(\hat{b}(\zeta)\big) \notin U$, then $\operatorname{supp}\mu_\zeta \cap \tilde{U} = \emptyset$ and $\chi_{\tilde{U}} = 0$. Hence we get (i) and (ii).

Since $\hat{b} = \hat{L}_x^{-1}$ on $\overline{P(x)}$, $\hat{L}_x\big(\hat{b}(\xi)\big) = \xi$ for $\xi \in \partial$. By (i) and (ii), we have (iii). Let $\zeta \in \overline{P(x)}$. By (iii), (1) and (2),

$$\lambda_\zeta(U) = \int_\partial \chi_U \, d\lambda_\zeta = \int_\partial \chi_{\tilde{U}} \, d\lambda_\zeta = \int_{M(L^\infty)} \chi_{\tilde{U}} \, d\mu_\zeta = \mu_\zeta(\tilde{U}).$$

The following proposition will be used several times in the rest.

PROPOSITION 1. *Let U be an open and closed subset of ∂. If E is a dense subset of U, then* $\tilde{U} \cap \operatorname{supp}\mu_x = \operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi \,;\, \xi \in E\}]$.

PROOF. By Lemma 7 (iii), $\bigcup\{\operatorname{supp}\mu_\xi \,;\, \xi \in U\} \subset \tilde{U}$ and $\bigcup\{\operatorname{supp}\mu_\xi \,;\, \xi \in \partial\backslash U\} \subset \operatorname{supp}\mu_x\backslash\tilde{U}$. By Lemma 6, $\operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi \,;\, \xi \in U\}] = \tilde{U} \cap \operatorname{supp}\mu_x$. For each point ξ_0 in U, there is a net $\{\xi_\alpha\}_\alpha$ in E such that $\xi_\alpha \longrightarrow \xi_0$. Since $\int_{M(L^\infty)} f \, d\mu_{\xi_\alpha} \longrightarrow \int_{M(L^\infty)} f \, d\mu_{\xi_0}$ for $f \in L^\infty$,

$$\operatorname{supp}\mu_{\xi_0} \subset \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_{\xi_\alpha} \,;\, \alpha\}\Big] \subset \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi \,;\, \xi \in E\}\Big].$$

Therefore $\operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi \,;\, \xi \in U\}] = \operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi \,;\, \xi \in E\}]$.

The following theorem gives the relation between $\operatorname{supp}\mu_\zeta$ and $\operatorname{supp}\lambda_\zeta$.

THEOREM 4. *Let* $\zeta \in \overline{P(x)}$. *Then*

(i) $\operatorname{supp}\mu_\zeta = \operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi \,;\, \xi \in \operatorname{supp}\lambda_\zeta\}]$;

(ii) $\operatorname{supp}\lambda_\zeta = \{\xi \in \partial \,;\, \operatorname{supp}\mu_\xi \subset \operatorname{supp}\mu_\zeta\}$.

PROOF. Let $\xi \in \operatorname{supp}\lambda_\zeta$. To prove $\operatorname{supp}\mu_\xi \subset \operatorname{supp}\mu_\zeta$, suppose not. Since $\operatorname{supp}\mu_\zeta$ is a weak peak set for H^∞ [8, p. 207], there is a function h in H^∞ such that $\|h\|_\infty = 1$,

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$h = 1$ on $\operatorname{supp}\mu_\zeta$ and $|h(\xi)| < 1$. Since $1 = h(\zeta) = \int_\partial h\,d\lambda_\zeta$, $h = 1$ on $\operatorname{supp}\lambda_\zeta$, so that $h(\xi) = 1$. This is a contradiction. Hence we have

$$\operatorname{supp}\mu_\zeta \supset \operatorname{cl}\big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in \operatorname{supp}\lambda_\zeta\}\big] \text{ and}$$
$$\operatorname{supp}\lambda_\zeta \subset \{\xi \in \partial\ ;\ \operatorname{supp}\mu_\xi \subset \operatorname{supp}\mu_\zeta\}.$$

(i) Let W be an arbitrary open and closed subset of $M(L^\infty)$ such that

$$W \supset \operatorname{cl}\big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in \operatorname{supp}\lambda_\zeta\}\big].$$

Since $\chi_W(\xi) = \int_{M(L^\infty)} \chi_W\,d\mu_\xi = 1$ for $\xi \in \operatorname{supp}\lambda_\zeta$, by (1) and (2) we have

$$\mu_\zeta(W) = \int_{M(L^\infty)} \chi_W\,d\mu_\zeta = \int_\partial \chi_W\,d\lambda_\zeta = 1.$$

Hence $\operatorname{supp}\mu_\zeta \subset W$, so that we get (i).

(ii) Let $\xi \in \partial$ such that $\operatorname{supp}\mu_\xi \subset \operatorname{supp}\mu_\zeta$. Let U be an arbitrary open and closed subset of ∂ such that $\operatorname{supp}\lambda_\zeta \subset U$. By (i) and Lemma 7 (iii), $\operatorname{supp}\mu_\zeta \subset \tilde{U}$. Hence $\operatorname{supp}\mu_\xi \subset \tilde{U}$. By Lemma 7 (iii) again, $\xi \in U$. Consequently, $\xi \in \operatorname{supp}\lambda_\zeta$.

COROLLARY 2. *For* $\zeta \in \overline{P(x)}$, $\hat{L}_x\big(\hat{b}(\operatorname{supp}\mu_\zeta)\big) = \operatorname{supp}\lambda_\zeta$.

PROOF. Since $\hat{b} = \hat{L}_x^{-1}$ on $\overline{P(x)}$, $\hat{L}_x\big(\hat{b}(\xi)\big) = \xi$ for $\xi \in \partial$. Then

$$\begin{aligned} \hat{L}_x\big(\hat{b}(\operatorname{supp}\mu_\zeta)\big) &= \operatorname{cl}\big[\bigcup\{\hat{L}_x\big(\hat{b}(\operatorname{supp}\mu_\xi)\big)\ ;\ \xi \in \operatorname{supp}\lambda_\zeta\}\big] \quad \text{by Theorem 4(i)} \\ &= \operatorname{cl}\big[\bigcup\{\hat{L}_x\big(\hat{b}(\xi)\big)\ ;\ \xi \in \operatorname{supp}\lambda_\zeta\}\big] \quad \text{by Lemma 5} \\ &= \operatorname{supp}\lambda_\zeta. \end{aligned}$$

3. **The Douglas algebra** $B_2 = [H^\infty_{\operatorname{supp}\mu_x}, \bar{\phi}\ ;\ \phi \in I]$**.** Put $B_0 = H^\infty_{\operatorname{supp}\mu_x}$, $B_1 = [H^\infty_{\operatorname{supp}\mu_x}, \bar{b}]$ and $B_2 = [H^\infty_{\operatorname{supp}\mu_x}, \bar{\phi}\ ;\ \phi \in I]$. By the Chang and Marshall theorem [3, 13], for every Douglas algebra B,

$$M(B) = \{\zeta \in M(H^\infty)\ ;\ |J(\zeta)| = 1 \text{ for every inner } J \text{ with } \bar{J} \in B\}.$$

By Lemma 1, $M(B_1) = M(B_0)\backslash P(x)$, and by Theorem 2, $M(B_2) = [M(B_0)\backslash\overline{P(x)}] \cup \partial$. Let $\mathrm{QC}_B = B \cap \bar{B}$ and let C_B be the C^*-algebra generated by inner functions J with $\bar{J} \in B$. Then

$$\mathrm{QC}_B = \{f \in B\ ;\ f \text{ is constant on } \operatorname{supp}\mu_\zeta \text{ for each } \zeta \in M(B)\}.$$

We denote by $\mathrm{QC}(\Delta)$ the QC-functions on Δ. In this section, we study B_2 mainly.

PROPOSITION 2. $\mathrm{QC}_{B_1} = \{f \in B_1\ ;\ f = q \circ b$ *on* $\operatorname{supp}\mu_x$ *for some* $q \in \mathrm{QC}(\Delta)\}$.

PROOF. Let $f \in B_1$ such that $f = q \circ b$ on $\operatorname{supp}\mu_x$ for some $q \in \mathrm{QC}(\Delta)$. Let $\zeta \in M(B_1)$. Then $\zeta \in [M(B_0)\backslash\overline{P(x)}]$ or $\zeta \in \overline{P(x)}\backslash P(x)$. If $\zeta \in M(B_0)\backslash\overline{P(x)}$, then by Lemma 5 $q \circ b(\operatorname{supp}\mu_\zeta) = q\big(\hat{b}(\zeta)\big)$, so that $q \circ b$ is constant on $\operatorname{supp}\mu_\zeta$. If $\zeta \in \overline{P(x)}\backslash P(x)$, there is a point η in $M\big(H^\infty(\Delta)\big)\backslash\Delta$ with $\zeta = \hat{L}_x(\eta)$. By Corollary 2, $\operatorname{supp}\sigma_\eta = \hat{L}_x^{-1}(\operatorname{supp}\lambda_\zeta) = \hat{b}(\operatorname{supp}\mu_\zeta)$. Since q is constant on $\operatorname{supp}\sigma_\eta$, $q \circ b$ is constant on $\operatorname{supp}\mu_\zeta$. Therefore $f \in \mathrm{QC}_{B_1}$.

Let $g \in \mathrm{QC}_{B_1}$. Then g is constant on $\operatorname{supp}\mu_y$ for each $y \in M(B_1)$. Since $\partial \subset M(B_1)$, by Corollary 1, $g = (g \circ \hat{L}_x) \circ b$ on $\operatorname{supp}\mu_x$. To prove $g \circ \hat{L}_x \in \mathrm{QC}(\Delta)$, let $\eta \in M\big(H^\infty(\Delta)\big)\backslash\Delta$. Put $\zeta = \hat{L}_x(\eta)$. Since g is constant on $\operatorname{supp}\mu_\zeta$, $g \circ \hat{L}_x$ is constant on $\hat{b}(\operatorname{supp}\mu_\zeta)$. Since $\operatorname{supp}\sigma_\eta = \hat{b}(\operatorname{supp}\mu_\zeta)$, $g \circ \hat{L}_x$ is constant on $\operatorname{supp}\sigma_\eta$.

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PROPOSITION 3. *(i)* $\mathrm{QC}_{B_2} = \{f \in B_2\ ; f = h \circ b$ *on* $\operatorname{supp} \mu_x$ *for some* $h \in L^\infty(\partial\Delta)\}$.
(ii) $C_{B_2} = \mathrm{QC}_{B_2}$.

PROOF. In the same way as the proof of Proposition 2, we can get (i). By [4, p. 192], $L^\infty(\partial\Delta)$ is the C^*-algebra generated by inner functions on Δ. Since $J \circ b \in I \subset C_{B_2}$ for every inner function J, by (i) we can get (ii).

REMARK 1. In the same way, we have

$$C_{B_1} = \{f \in B_1\ ; f = h \circ b \text{ on } \operatorname{supp} \mu_x \text{ for some } h \in C(\partial\Delta)\}.$$

And this is a restatement of the result in [7, Section 3].

For $\xi \in \partial$, there is QC_{B_2}-level set R_ξ such that $\operatorname{supp} \mu_\xi \subset R_\xi$. By Lemma 5, $\hat{b}(\operatorname{supp} \mu_\xi) = \hat{L}_x^{-1}(\xi)$. Hence by Proposition 3,

$$R_\xi = \{\zeta \in \operatorname{supp} \mu_x\ ; \hat{L}_x\big(\hat{b}(\zeta)\big) = \xi\},$$

and $\{R_\xi\ ;\ \xi \in \partial\}$ is the partition of $\operatorname{supp} \mu_x$ by QC_{B_2}-level sets. Of course $R_{\xi_1} \neq R_{\xi_2}$ if $\xi_1 \neq \xi_2$. In Section 4, we shall prove that $\operatorname{supp} \mu_\xi \subsetneq R_\xi$ for every $\xi \in \partial$ (Corollary 5).
For an inner function I, we put

$$U_I = \mathrm{cl}\{\xi \in \partial\ ; |I(\xi)| < 1\}.$$

Then $U_I = \hat{L}_x(\mathrm{cl}\{\eta \in M\big(L^\infty(\partial\Delta)\big); |I \circ \hat{L}_x(\eta)| < 1\})$. Since $\mathrm{cl}\{\eta \in M\big(L^\infty(\partial\Delta)\big); |I \circ \hat{L}_x(\eta)| < 1\}$ is an open and closed subset of $M\big(L^\infty(\partial\Delta)\big)$, U_I is an open and closed subset of ∂.

THEOREM 5. *Let I be an inner function with $\bar{I} \notin B_2$. Then*
(i) $N_{B_2}(\bar{I}) = \tilde{U}_I \cap \operatorname{supp} \mu_x$*;*
(ii) for $\xi \in \partial$, $R_\xi \subset N_{B_2}(\bar{I})$ *or* $R_\xi \cap N_{B_2}(\bar{I}) = \emptyset$.

We need the following lemma which will be used also in Section 4.

LEMMA 8. *Let I be an interpolating Blashcke product and let U be an open and closed subset of ∂. Then there is a factorization $I = I_1 I_2$ such that*
(i) if $\zeta \in M(B_2)$ *and* $|I_1(\zeta)| < 1$ *then* $\operatorname{supp} \mu_\zeta \subset \tilde{U}$*;*
(ii) if $\zeta \in M(B_2)$ *and* $|I_2(\zeta)| < 1$ *then* $\operatorname{supp} \mu_\zeta \subset \operatorname{supp} \mu_x \backslash \tilde{U}$*;*
(iii) $|I_1| = 1$ *on* $\partial \backslash U$ *and* $|I_2| = 1$ *on* U*;*
(iv) $|I_1| = |I|$ *on* U *and* $|I_2| = |I|$ *on* $\partial \backslash U$.

PROOF. By Lemma 7, $\chi_{\tilde{U}}$ takes 0 or 1 on $M(B_2)$. Since $Z(I)$ is a totally disconnected set, there is an open and closed subset W of $Z(I)$ such that

$$W \cap M(B_2) = Z(I) \cap \{\zeta \in M(B_2)\ ; \chi_{\tilde{U}}(\zeta) = 1\}.$$

Let I_1 be a subproduct of I with the zero sequence $W \cap D \cap Z(I)$. Then $Z(I_1) = W$ (see [10]). Put $I_2 = I/I_1$. Then $Z(I_2) = Z(I) \backslash W$.

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(i) Let $\zeta \in M(B_2)$ and $|I_1(\zeta)| < 1$. Then there is a point ζ_0 in $Z(I_1)$ such that $\operatorname{supp}\mu_{\zeta_0} \subset \operatorname{supp}\mu_\zeta$. Here we have $\zeta_0 \in M(B_2)$, so that $\chi_{\tilde{U}}(\zeta_0) = 1$ and $\chi_{\tilde{U}}(\zeta) > 0$. Therefore $\chi_{\tilde{U}}(\zeta) = 1$, and $\operatorname{supp}\mu_\zeta \subset \tilde{U}$.

(ii) Let $\zeta \in M(B_2)$ and $|I_2(\zeta)| < 1$. Suppose that $\operatorname{supp}\mu_\zeta \not\subset \operatorname{supp}\mu_x\backslash\tilde{U}$, that is, $\operatorname{supp}\mu_\zeta \cap \tilde{U} \neq \emptyset$. Since $\chi_{\tilde{U}}(\zeta) = 0$ or 1, $\chi_{\tilde{U}}(\zeta) = 1$. Since $|I_2(\zeta)| < 1$, there is a point ζ_0 in $Z(I_2)$ such that $\operatorname{supp}\mu_{\zeta_0} \subset \operatorname{supp}\mu_\zeta$. Since $\chi_{\tilde{U}}(\zeta) = 1$, $\chi_{\tilde{U}}(\zeta_0) = 1$. Therefore $\zeta_0 \in W$. Since $Z(I_2) = Z(I)\backslash W$, we have a contradiction.

(iii) Suppose that $|I_1(\xi)| < 1$ for some $\xi \in \partial\backslash U$. By (i), $\chi_{\tilde{U}}(\xi) = 1$, so that by Lemma 7 we have $\xi \in U$. But this is a contradiction. Thus we get $|I_1| = 1$ on $\partial\backslash U$. Next suppose that $|I_2(\xi)| < 1$ for some $\xi \in U$. By (ii), $\chi_{\tilde{U}}(\xi) = 0$. Since $\xi \in U$, by Lemma 7 we have $\chi_{\tilde{U}}(\xi) = 1$. This contradiction shows that $|I_2| = 1$ on U.

(iv) By (iii), we have $|I| = |I_1|\,|I_2| = |I_1|$ on U and $|I| = |I_1|\,|I_2| = |I_2|$ on $\partial\backslash U$.

PROOF OF THEOREM 5. (i) By Lemma 4, we may assume that I is an interpolating Blaschke product. Since $\bar{I} \notin B_2$, $I \notin I$, so that $|I|$ is not identically 1 on ∂. Since $\{\xi \in \partial\,;\,|I(\xi)| < 1\}$ is a dense subset of U_I, by Proposition 1 we have

$$\tilde{U}_I \cap \operatorname{supp}\mu_x = \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi\,;\,\xi \in \partial, |I(\xi)| < 1\}\Big].$$

Hence $\tilde{U}_I \cap \operatorname{supp}\mu_x \subset N_{B_2}(\bar{I})$. Let $I = I_1 I_2$ be a factorization in Lemma 8 for the open and closed subset U_I. Then $|I_2| = 1$ on U_I and $|I_2| = |I|$ on $\partial\backslash U_I$. Therefore $|I_2| = 1$ on ∂ and $I_2 \in I$. By Theorem 2, $|I_2| = 1$ on $M(B_2)$. By Lemma 8 (i), $N_{B_2}(\bar{I}) = N_{B_2}(\bar{I}_1) \subset \tilde{U}_I$. Since $N_{B_2}(\bar{I}) \subset \operatorname{supp}\mu_x$, we get (i).

(ii) Let $\xi \in \partial$. Then $R_\xi = \{\zeta \in \operatorname{supp}\mu_x\,;\,\hat{L}_x\big(\hat{b}(\zeta)\big) = \xi\}$. By the definition of $\tilde{U}_I$, $\tilde{U}_I \cap \operatorname{supp}\mu_x = \{\zeta \in \operatorname{supp}\mu_x\,;\,\hat{L}_x\big(\hat{b}(\zeta)\big) \in U_I\}$. Hence if $\xi \notin U_I$ then $R_\xi \cap \tilde{U}_I = \emptyset$ and if $\xi \in U_I$ then $R_\xi \subset \tilde{U}_I$.

REMARK 2. By the above proof, for every open and closed subset U of ∂ and $\xi \in \partial$, $R_\xi \subset \tilde{U}$ or $R_\xi \cap \tilde{U} = \emptyset$.

The following is the main theorem in this section.

THEOREM 6. *Let $f, g \in L^\infty$ such that $f|_{\operatorname{supp}\mu_\zeta} \in H^\infty|_{\operatorname{supp}\mu_\zeta}$ or $g|_{\operatorname{supp}\mu_\zeta} \in H^\infty|_{\operatorname{supp}\mu_\zeta}$ for every $\zeta \in M(B_2)$. Then*

(i) for every $\xi \in \partial$, $R_\xi \subset N_{B_2}(f)$ or $R_\xi \cap N_{B_2}(f) = \emptyset$;

(ii) $N_{B_2}(f) \cap N_{B_2}(g) = \emptyset$.

PROOF. By [12, Lemma 2.2], there are sequences of inner functions $\{I_n\}_n$ and $\{J_k\}_k$ such that

$$[H^\infty, f] = [H^\infty, \bar{I}_n\,;\,n = 1, 2, \ldots] \text{ and } [H^\infty, g] = [H^\infty, \bar{J}_k\,;\,k = 1, 2, \ldots].$$

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Then we have

$$\begin{aligned} N_{B_2}(f) &= \mathrm{cl}\Big[\bigcup_{n=1}^{\infty} N_{B_2}(\bar{I}_n)\Big] \\ &= \mathrm{cl}\Big[\bigcup_{n=1}^{\infty} \tilde{U}_{I_n} \cap \operatorname{supp}\mu_x\Big] \quad \text{by Theorem 5} \\ &= \mathrm{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in \bigcup_{n=1}^{\infty} U_{I_n}\}\Big] \quad \text{by Proposition 1} \\ &= [\mathrm{cl} \bigcup_{n=1}^{\infty} U_{I_n}]^{\sim} \cap \operatorname{supp}\mu_x \quad \text{by Proposition 1.} \end{aligned}$$

Since $\mathrm{cl}\bigcup_{n=1}^{\infty} U_{I_n}$ is an open and closed subset of ∂, by Remark 2 we get (i).

By our assumption, for n and k, $|I_n(\zeta)| = 1$ or $|J_k(\zeta)| = 1$ for every $\zeta \in M(B_2)$. Then

$$\{\xi \in \partial\ ;\ |I_n(\xi)| < 1\} \cap \{\xi \in \partial\ ;\ |J_k(\xi)| < 1\} = \emptyset.$$

Since $\partial = \hat{L}_x\big(M(L^\infty(\partial\Delta))\big)$ is a Stonian space, $U_{I_n} \cap U_{I_k} = \emptyset$, so that $\mathrm{cl}[\bigcup_{n=1}^{\infty} U_{I_n}] \cap \mathrm{cl}[\bigcup_{k=1}^{\infty} U_{J_k}] = \emptyset$. Hence

$$\begin{aligned} N_{B_2}(f) \cap N_{B_2}(g) &= \Big[\mathrm{cl} \bigcup_{n=1}^{\infty} U_{I_n}\Big]^{\sim} \cap \Big[\mathrm{cl} \bigcup_{k=1}^{\infty} U_{J_k}\Big]^{\sim} \cap \operatorname{supp}\mu_x \\ &= \Big[\Big(\mathrm{cl} \bigcup_{n=1}^{\infty} U_{I_n}\Big) \cap \Big(\mathrm{cl} \bigcup_{k=1}^{\infty} U_{J_k}\Big)\Big]^{\sim} \cap \operatorname{supp}\mu_x \\ &= \emptyset. \end{aligned}$$

REMARK 3. Let I and J be inner functions. In [11, Corollary 5], the author proved that $[H^\infty + C, \bar{I}] = [H^\infty + C, \bar{J}]$ if and only if $N(\bar{I}) = N(\bar{J})$. Here we note that this fact is not true for the Douglas algebra B_2. It is not difficult to see that if $[B_2, \bar{I}] = [B_2, \bar{J}]$, then $N_{B_2}(\bar{I}) = N_{B_2}(\bar{J})$. But the converse is not true. For, take a Blaschke product I such that $I = 0$ on $\overline{P(x)}$ (see Theorem 1). There is a Blaschke product J such that $J = 0$ on $\{\zeta \in M(H^\infty + C)\ ;\ |I(\zeta)| < 1\}$. Then $[B_2, \bar{I}] \subsetneq [B_2, \bar{J}]$. Since $I = J = 0$ on ∂, we have $U_I = U_J = \partial$. Since $\tilde{\partial} = M(L^\infty)$, by Theorem 5 we have $N_{B_2}(\bar{I}) = N_{B_2}(\bar{J}) = \operatorname{supp}\mu_x$.

4. **The Douglas algebra $B_1 = [H^\infty_{\operatorname{supp}\mu_x}, \bar{b}]$.** In this section, we shall study the Douglas algebra $B_1 = [H^\infty_{\operatorname{supp}\mu_x}, \bar{b}]$. For $f \in L^\infty$ with $\|f\|_\infty \le 1$, put

$$M(f) = \mathrm{cl}\Big[\bigcup\{\operatorname{supp}\mu_\zeta\ ;\ \zeta \in M(H^\infty + C), |f(\zeta)| \neq 1\}\Big].$$

PROPOSITION 4. *Let $f \in L^\infty$ with $\|f\|_\infty \le 1$. Put $W = \mathrm{cl}\{\xi \in M(L^\infty)\ ;\ |f(\xi)| < 1\}$. Then $M(f) = W \cup N(f) \cup N(\bar{f})$.*

PROOF. $M(f) \supset W$ is trivial. Let $\zeta \in M(H^\infty + C)$ such that $f|_{\operatorname{supp}\mu_\zeta} \notin H^\infty|_{\operatorname{supp}\mu_\zeta}$. Then $|f(\zeta)| < 1$, so that $\operatorname{supp}\mu_\zeta \subset M(f)$. Hence $N(f) \subset M(f)$. Also we have $N(\bar{f}) \subset M(f)$.

To prove the converse inclusion, let $\xi \in M(H^\infty + C)$ such that $|f(\xi)| < 1$. If $f|_{\operatorname{supp}\mu_\xi} \notin H^\infty|_{\operatorname{supp}\mu_\xi}$ or $\bar{f}|_{\operatorname{supp}\mu_\xi} \notin H^\infty|_{\operatorname{supp}\mu_\xi}$ then $\operatorname{supp}\mu_\xi \subset N(f) \cup N(\bar{f})$. If $f|_{\operatorname{supp}\mu_\xi} \in H^\infty|_{\operatorname{supp}\mu_\xi}$

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and $\bar{f}|_{\operatorname{supp}\mu_\xi} \in H^\infty|_{\operatorname{supp}\mu_\xi}, f = c$ on $\operatorname{supp}\mu_\xi$ for some constant c, because $\operatorname{supp}\mu_\xi$ is an antisymmetric set for H^∞ ([15, p. 463]). Since $|f(\xi)| < 1$, $|c| < 1$, so that $\operatorname{supp}\mu_\xi \subset W$. Consequently $M(f) \subset W \cup N(f) \cup N(\bar{f})$.

REMARK 4. There are a function g in L^∞ and a QC-level set Q such that $\|g\|_\infty = 1$, $Q \not\subset M(g)$ and $Q \cap M(g) \neq \emptyset$.

PROOF. By [8, p. 80], there is a continuous function g on $D \cup \partial D$ such that g is analytic in D, $|g| < 1$ on some proper open arc J in ∂D and $|g| = 1$ on $\partial D \backslash J$. By Proposition 4, $M(g) = \{\zeta \in M(L^\infty)\ ;\ \chi_J(\zeta) = 1\}$. Since $\chi_J \notin$ QC, there is a QC-level set Q such that $Q \not\subset M(g)$ and $Q \cap M(g) \neq \emptyset$. For $f \in L^\infty$ with $\|f\|_\infty \leq 1$, we put

$$M_\partial(f) = \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\lambda_\zeta\ ;\ \zeta \in \overline{P(x)}\backslash P(x), |f(\zeta)| < 1\}\Big].$$

It is easy to see that $M_\partial(f) = \hat{L}_x\big(M(f \circ \hat{L}_x)\big)$.

THEOREM 7. *Let I be an inner function. Then* $N_{B_1}(\bar{I}) = \operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in M_\partial(I)\}]$.

PROOF. Let $\zeta \in \overline{P(x)}\backslash P(x)$ with $|I(\zeta)| < 1$. By Theorem 4 (i),

$$\operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in \operatorname{supp}\lambda_\zeta\}\Big] = \operatorname{supp}\mu_\zeta \subset N_{B_1}(\bar{I}).$$

Consequently we have

$$\operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in M_\partial(I)\}\Big] \subseteq N_{B_1}(\bar{I}).$$

Next we shall prove the converse inclusion. We note that

$$N_{B_1}(\bar{I}) = N_{B_2}(\bar{I}) \cup \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\zeta\ ;\ \zeta \in \overline{P(x)}\backslash P(x), |I(\zeta)| < 1\}\Big].$$

Since $U_I \subset M_\partial(I)$, we have

$$\begin{aligned} N_{B_2}(\bar{I}) &= \tilde{U}_I \cap \operatorname{supp}\mu_x \quad \text{by Theorem 5} \\ &= \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in U_I\}\Big] \quad \text{by Proposition 1} \\ &\subset \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in M_\partial(I)\}\Big]. \end{aligned}$$

If $\zeta \in \overline{P(x)}\backslash P(x)$ with $|I(\zeta)| < 1$, then

$$\begin{aligned} \operatorname{supp}\mu_\zeta &= \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in \operatorname{supp}\lambda_\zeta\}\Big] \quad \text{by Theorem 4} \\ &\subset \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in M_\partial(I)\}\Big]. \end{aligned}$$

Therefore we get $N_{B_1}(\bar{I}) \subset \operatorname{cl}[\bigcup\{\operatorname{supp}\mu_\xi\ ;\ \xi \in M_\partial(I)\}]$.

For $f \in L^\infty$, put

$$N_\partial(f) = \operatorname{cl}\Big[\bigcup\{\operatorname{supp}\lambda_\zeta\ ; f|_{\operatorname{supp}\lambda_\zeta} \notin H^\infty|_{\operatorname{supp}\lambda_\zeta}, \zeta \in \overline{P(x)}\}\Big].$$

Then it is easy to see that $N_\partial(f) = \hat{L}_x\big(N(f \circ \hat{L}_x)\big)$.

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COROLLARY 3. *Let I be an inner function. Then* $N_{B_1}(\bar{I}) = N_{B_2}(\bar{I}) \cup \mathrm{cl}[\bigcup\{\mathrm{supp}\,\mu_\xi\ ;\ \xi \in N_\partial(\bar{I})\}]$.

PROOF. Put $W = \mathrm{cl}\{\eta \in M\big(L^\infty(\partial\Delta)\big)\ ;\ |(I \circ \hat{L}_x)(\eta)| < 1\}$. Then

$$\begin{aligned} M_\partial(I) &= \hat{L}_x\big(M(I \circ \hat{L}_x)\big) \\ &= \hat{L}_x\big(W \cup N(I \circ \hat{L}_x) \cup N(\overline{I \circ \hat{L}_x})\big) \quad \text{by Proposition 4} \\ &= U_I \cup N_\partial(\bar{I}). \end{aligned}$$

By Proposition 1, Theorems 5 and 7, we get our assertion.

COROLLARY 4. *(i) If $I \in I$, then* $\mu_x\big(N_{B_1}(\bar{I})\big) = 0$.

(ii) If I is inner and $I \notin I$, then $\mu_x\big(N_{B_1}(\bar{I})\big) = \mu_x\big(N_{B_2}(\bar{I})\big) > 0$.

PROOF. By [11, Theorem 1], $\sigma_0\big(N(\overline{I \circ \hat{L}_x})\big) = 0$. Then $\lambda_x\big(N_\partial(\bar{I})\big) = \sigma_0\big(N(\overline{I \circ \hat{L}_x})\big) = 0$. Let $\{U_n\}_n$ be a sequence of open and closed subsets of ∂ such that $U_n \supset N_\partial(\bar{I})$ and $\lambda_x(U_n) \longrightarrow 0$. By Proposition 1, $\tilde{U}_n \supset \mathrm{cl}[\bigcup\{\mathrm{supp}\,\mu_\xi\ ;\ \xi \in N_\partial(\bar{I})\}]$. By Lemma 7 (iv), $\mu_x(\tilde{U}_n) = \lambda_x(U_n) \longrightarrow 0$, hence $\mu_x(\mathrm{cl}[\bigcup\{\mathrm{supp}\,\mu_\xi\ ;\ \xi \in N_\partial(\bar{I})\}]) = 0$. If $I \in I$, then $N_{B_2}(\bar{I}) = \emptyset$. By Corollary 3, we get (i).

Next let I be an inner function with $I \notin I$, then

$$\begin{aligned} \mu_x\big(N_{B_1}(\bar{I})\big) &= \mu_x\big(N_{B_2}(\bar{I})\big) \quad \text{by Corollary 3} \\ &= \mu_x(\tilde{U}_I) \quad \text{by Theorem 5} \\ &= \lambda_x(U_I) \quad \text{by Lemma 7} \\ &> 0. \end{aligned}$$

PROPOSITION 5. *Let $f, g \in L^\infty$ with $\|f\|_\infty \leq 1$ and $\|g\|_\infty \leq 1$. Suppose that for each point ζ in $M(H^\infty + C)$, $|f(\zeta)| = 1$ or $|g(\zeta)| = 1$. Then for each* QC*-level set Q, $f|_Q = c$ or $g|_Q = c$ for some constant c, depending on Q, with $|c| = 1$.*

PROOF. By our assumption and [12, Theorem 2.1],

$$[N(f) \cup N(\bar{f})] \cap [N(g) \cup N(\bar{g})] = \emptyset.$$

Since $N(f)$ consists of QC-level sets [12, Corollary 2.1],

$$Q \cap [N(f) \cup N(\bar{f})] = \emptyset \text{ or } Q \cap [N(g) \cup N(\bar{g})] = \emptyset.$$

Here we may assume that $Q \cap [N(g) \cup N(\bar{g})] = \emptyset$. There is a function q_1 in QC such that $q_1|_Q = 1$ and $q_1 = 0$ on $N(g) \cup N(\bar{g})$. Then $gq_1 \in$ QC, so that $g|_Q = c_1$ for some constant c_1. If $|c_1| = 1$, this is our conclusion, so that we assume $|c_1| < 1$. Then there is an open subset V of $M(L^\infty)$ such that $|g| < 1$ on V and $Q \subset V$. Let $q_2 \in$ QC such that $q_2|_Q = 1$ and $q_2 = 0$ on $M(L^\infty)\backslash V$. If $fq_2 \notin$ QC, there is a point ξ in $M(H^\infty + C)$ such that $fq_2|_{\mathrm{supp}\,\mu_\xi}$ is not constant. Then $\mathrm{supp}\,\mu_\xi \subset V$ and f is not constant on $\mathrm{supp}\,\mu_\xi$. Therefore $|g(\xi)| < 1$ and $|f(\xi)| < 1$; this contradicts our assumption. Hence $fq_2 \in$ QC, so that $f|_Q = c_2$. Since $|g|_Q| < 1$, by our assumption we have $|c_2| = 1$.

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LEMMA 9. *Let I be an inner function, B be a Douglas algebra and let Q be a* QC_B-*level set. Then*

(i) if $I|_Q$ is constant, then $Q \cap N_B(\bar{I}) = \emptyset$;

(ii) if $I|_Q$ is not constant, then there is a point ζ in $M(B)$ such that $\operatorname{supp}\mu_\zeta \subset Q$ *and* $I(\zeta) = 0$.

PROOF. Let $\pi_B: M(B) \longrightarrow M(\mathrm{QC}_B)$ be a natural continuous map such that $\pi_B^{-1}(\zeta)$ is a QC_B-level set for $\zeta \in M(\mathrm{QC}_B)$. Then it is not difficult to see that $N_B(\bar{I}) \subset \pi_B^{-1}\big(\pi_B\big(Z(I) \cap M(B)\big)\big)$. If q is a QC_B-function with $q = 0$ on $Z(I) \cap M(B)$, then $Iq \in \mathrm{QC}_B$. This means that $\pi_B(Q) \not\in \pi_B\big(Z(I) \cap M(B)\big)$ if and only if $I|_Q$ is constant. This implies our assertions.

For $\xi \in \partial$, there is a QC_{B_1}-level set Q_ξ such that $\operatorname{supp}\mu_\xi \subset R_\xi \subset Q_\xi$. By Lemma 5, $\hat{b}(\operatorname{supp}\mu_\xi) = \hat{L}_x^{-1}(\xi)$. Let $Q_{\Delta,\xi}$ be a $\mathrm{QC}(\Delta)$-level set containing the point $\hat{L}_x^{-1}(\xi)$. By Proposition 2, we have

$$Q_\xi = \{\zeta \in \operatorname{supp}\mu_x \,;\, \hat{b}(\zeta) \in Q_{\Delta,\xi}\}.$$

The following is a counterpart of Theorem 9.

THEOREM 8. *Let I and J be inner functions such that $I \in I$ and for every $\zeta \in M(B_1)$, $|(I(\zeta)| = 1$ or $|J(\zeta)| = 1$. Suppose that $\{\xi \in \partial \,;\, |J(\xi)| < 1\}$ is an open and closed subset of ∂. Then*

(i) $N_{B_1}(\bar{I}) \cap N_{B_1}(\bar{J}) = \emptyset$,

(ii) for every QC_{B_1}-level set Q, $I|_Q$ or $J|_Q$ is constant.

PROOF. (i) By our assumption, $I \circ \hat{L}_x$ is inner and $|(I \circ \hat{L}_x)(\eta)| = 1$ or $|(J \circ \hat{L}_x)(\eta)| = 1$ for $\eta \in M\big((H^\infty + C)(\Delta)\big)$. Put $W = \big\{\eta \in M\big(L^\infty(\partial\Delta)\big) \,;\, |(J \circ \hat{L}_x)(\eta)| < 1\big\}$, then W is open and closed. By Proposition 4, $M(I \circ \hat{L}_x) = N(\overline{I \circ \hat{L}_x})$ and $M(J \circ \hat{L}_x) = W \cup N(\overline{J \circ \hat{L}_x})$. By [12, Theorem 2.1], $N(\overline{I \circ \hat{L}_x}) \cap N(\overline{J \circ \hat{L}_x}) = \emptyset$. By Proposition 5, for every $\mathrm{QC}(\Delta)$-level set Q_Δ, $I \circ \hat{L}_x|_{Q_\Delta} = c$ or $J \circ \hat{L}_x|_{Q_\Delta} = c$ for some constant c with $|c| = 1$. Hence by Lemma 9, $Q_\Delta \cap N(\overline{I \circ \hat{L}_x}) = \emptyset$ or $Q_\Delta \cap W = \emptyset$. Since $N(\overline{I \circ \hat{L}_x})$ consists of $\mathrm{QC}(\Delta)$-level sets, $N(\overline{I \circ \hat{L}_x}) \cap W = \emptyset$. Consequently, $M(I \circ \hat{L}_x) \cap M(J \circ \hat{L}_x) = \emptyset$ and $M_\partial(I) \cap M_\partial(J) = \emptyset$. Take an open and closed subset U of ∂ such that $M_\partial(I) \subset U$ and $U \cap M_\partial(J) = \emptyset$. Then by Lemma 7 and Theorem 7, $N_{B_1}(\bar{I}) \subset \tilde{U}$ and $N_{B_1}(\bar{J}) \subset \operatorname{supp}\mu_x \setminus \tilde{U}$. Thus we get (i).

(ii) Suppose that there is a QC_{B_1}-level set Q such that both $I|_Q$ and $J|_Q$ are not constant. Since $I \in I$, by Corollary 1, $(I \circ \hat{L}_x) \circ b = I$ on $\operatorname{supp}\mu_x$. Hence $I \circ \hat{L}_x$ is not constant on $\hat{b}(Q)$; here $\hat{b}(Q)$ is a $\mathrm{QC}(\Delta)$-level set. By Lemma 9, there is a point ζ in $M(B_1)$ such that $\operatorname{supp}\mu_\zeta \subset Q$ and $J(\zeta) = 0$. If $\zeta \not\in \overline{P(x)}$, by Theorem 5, $\operatorname{supp}\mu_\zeta \subset \tilde{U}_J = \hat{b}^{-1}(W)$. Then $\hat{b}(Q) \cap W \neq \emptyset$ and $|J \circ \hat{L}_x| \neq 1$ on $\hat{b}(Q)$. If $\zeta \in \overline{P(x)}$, then $J|_{\operatorname{supp}\lambda_\zeta}$ is not constant, and so is $J \circ \hat{L}_x|_{\hat{L}_x^{-1}(\operatorname{supp}\lambda_\zeta)}$. By Corollary 2, $\hat{b}(\operatorname{supp}\mu_\zeta) = \hat{L}_x^{-1}(\operatorname{supp}\lambda_\zeta)$, and $J \circ \hat{L}_x|_{\hat{b}(Q)}$ is not constant. But this contradicts Proposition 5.

The following is the main theorem of the paper.

THEOREM 9. *There are inner functions I and J, and a QC_{B_1}-level set Q such that*

(i) $Q \not\subset N_{B_1}(\bar{I})$ and $Q \cap N_{B_1}(\bar{I}) \neq \emptyset$;

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(ii) *either* $|I(\zeta)| = 1$ *or* $|J(\zeta)| = 1$ *for every* $\zeta \in M(B_1)$;
(iii) $N_{B1}(\bar{I}) \cap N_{B1}(\bar{J}) \neq \emptyset$;
(iv) *both* $I|_Q$ *and* $J|_Q$ *are not constant;*
(v) $\mathrm{cl}\big[\bigcup\{\mathrm{supp}\,\mu_\xi\ ;\ \xi \in \hat{L}_x\big(\hat{b}(Q)\big)\}\big] \subsetneq Q$.

PROOF. STEP 1. First let ψ_1 be an interpolating Blaschke product such that

$$\sup\{|\psi_1(\xi)|\ ;\ \xi \in \partial\} < 1. \tag{5}$$

The existence of ψ_1 follows Theorem 1 and Lemma 4. If $Z(\psi_1)\cap\overline{P(x)} \neq \emptyset$, by Theorem 3 there is an open and closed subset W of $Z(\psi_1)$ such that $Z(\psi_1) \cap [M(B_0)\backslash\overline{P(x)}] = W \cap M(B_0)$. Then there is a subproduct ψ_1' of ψ_1 such that $Z(\psi_1') = W$. Since ψ_1/ψ_1' does not vanish on $M(B_2) = [M(B_0)\backslash\overline{P(x)}] \cup \partial$, $|\psi_1/\psi_1'| = 1$ on ∂. Hence $Z(\psi_1') \cap \overline{P(x)} = \emptyset$ and $\sup\{|\psi_1'(\xi)|\ ;\ \xi \in \partial\} < 1$. Therefore we may assume that

$$Z(\psi_1) \cap \overline{P(x)} = \emptyset. \tag{6}$$

We shall prove the existence of a sequence of interpolating Blaschke products $\{\psi_n\}_n$ such that ψ_n is a subproduct of ψ_{n-1} and

$$1 - 1/n \leq \inf\{|\psi_n(\xi)|\ ;\ \xi \in \partial\} \leq \sup\{|\psi_n(\xi)|\ ;\ \xi \in \partial\} < 1. \tag{7}$$

It is sufficient to prove that there is a subproduct ψ_n of ψ_1 satisfying (7).

For $\xi \in \partial$, by (5) there is a point ζ_ξ in $M(B_1) \cap Z(\psi_1)$ such that $\mathrm{supp}\,\mu_{\zeta_\xi} \subset \mathrm{supp}\,\mu_\xi$. Let δ be a positive number such that $r(\delta) > 1 - 1/n$ in Lemma 2. By [9, p. 82], there is a subproduct ψ' of ψ_1 such that

$$\delta(\psi') \geq \delta \text{ and } \psi'(\zeta_\xi) = 0.$$

Then $|\psi'(\xi)| < 1$. Since $P(\xi) = \{\xi\}$, $\rho\big(\xi, Z(\psi')\big) = 1$. Hence by Lemma 2,

$$1 - 1/n < r(\delta) \leq |\psi'(\xi)| < 1.$$

Take an open and closed subset U_ξ of ∂ such that $\xi \in U_\xi$ and

$$1 - 1/n < \inf\{|\psi'(\xi')|\ ;\ \xi' \in U_\xi\} \leq \sup\{|\psi'(\xi')|\ ;\ \xi' \in U_\xi\} < 1. \tag{8}$$

Applying Lemma 8 to ψ' and $U_{\xi'}$, there is a subproduct ψ_ξ of ψ' such that

$$|\psi_\xi| = |\psi'| \text{ on } U_\xi \text{ and } |\psi_\xi| = 1 \text{ on } \partial\backslash U_\xi. \tag{9}$$

By (8) and (9), we have

$$1 - 1/n < \inf\{|\psi_\xi(\xi')|\ ;\ \xi' \in U_\xi\} \leq \sup\{|\psi_\xi(\xi')|\ ;\ \xi' \in U_\xi\} < 1. \tag{10}$$

Since ∂ is compact, there is a finite sequence of points $\xi_1, \xi_2, \ldots, \xi_k$ in ∂ such that $\partial = \bigcup_{j=1}^k U_{\xi_j}$. Put $\phi_1 = \psi_{\xi_1}$. By Lemma 8, take a subproduct ϕ_2 of ψ_{ξ_2} such that $|\phi_2| = |\psi_{\xi_2}|$ on $U_{\xi_2}\backslash U_{\xi_1}$ and $|\phi_2| = 1$ on $\partial\backslash(U_{\xi_2}\backslash U_{\xi_1})$. By induction, we can take a subproduct ϕ_j of

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ψ_{ξ_j} such that $|\phi_j| = |\psi_{\xi_j}|$ on $U_{\xi_j}\backslash(U_{\xi_1}\cup\cdots\cup U_{\xi_{j-1}})$ and $|\phi_j| = 1$ on $\partial\backslash[U_{\xi_j}\backslash(U_{\xi_1}\cup\cdots\cup U_{\xi_{j-1}})]$. By our construction and Lemma 8, $Z(\phi_i)\cap Z(\phi_j)\cap M(B_1) = \emptyset$ for $i \neq j$, so that we may assume that ϕ_i and ϕ_j have disjoint zero sequences. Put $\psi_n = \prod_{j=1}^{k}\phi_j$. Then ψ_n is a subproduct of ψ_1 and by (10) we get (7).

STEP 2. Put $U_n = \mathrm{cl}\{\xi \in \partial\ ;\ 1/(n+1) < \mathrm{Re}\,\hat{b}(\xi) < 1/n\}$ for $n = 1, 2, \ldots$. Applying Lemma 8 for each ψ_n and U_n, we have a subproduct I_n of ψ_n such that

(11) $$|I_n| = |\psi_n| \text{ on } U_n \text{ and } |I_n| = 1 \text{ on } \partial\backslash U_n.$$

Since $U_n\cap U_k = \emptyset$ for $n \neq k$, $Z(I_n)\cap Z(I_k)\cap M(B_1) = \emptyset$, so that we may assume moreover that I_n and I_k have disjoint zero sequences. Since ψ_n is a subproduct of ψ_{n-1}, for each k

(12) $$\prod_{n=k}^{\infty} I_n \text{ is a subproduct of } \psi_k.$$

Put $I = \prod_{n=1}^{\infty} I_n$, then I is an interpolating Blaschke subproduct of ψ_1, so that by (6)

(13) $$Z(I)\cap\overline{P(x)} = \emptyset.$$

By (7), (11) and (12), we have the following inequalities on U_k

$$|I| = \Big|\prod_{n=k}^{\infty} I_n\Big||I_k|\Big|\prod_{n=1}^{k-1} I_n\Big| \geq |\psi_k|^2 > (1-1/k)^2\ ;\ \text{ and } |I| \leq |I_k| = |\psi_k| < 1.$$

Hence

(14) $$|I| < 1 \text{ on } \textstyle\bigcup_{k=1}^{\infty} U_k \text{ and } \lim_{k\to\infty} \sup\{|I(\xi)|\ ;\ \xi \in U_k\} \longrightarrow 1.$$

Also we have

$$|I| = \Big|\prod_{n=k}^{\infty} I_n\Big|\Big|\prod_{n=1}^{k-1} I_n\Big| \geq |\psi_k| \geq 1 - 1/k \text{ on } \partial\backslash\Big(\bigcup_{k=1}^{\infty} U_k\Big)\ ;$$

therefore

(15) $$|I| = 1 \text{ on } \partial\backslash\Big(\bigcup_{k=1}^{\infty} U_k\Big).$$

Hence $U_I = \mathrm{cl}(\bigcup_{k=1}^{\infty} U_k)$.

STEP 3. First we study the function $I\circ\hat{L}_x$ on $M\big((H^\infty + C)(\Delta)\big)$. Since $\hat{L}_x^{-1}(U_k) = \mathrm{cl}\{\eta \in M\big(L^\infty(\partial\Delta)\big)\ ;\ 1/(k+1) < \mathrm{Re}\,\hat{z}(\eta) < 1/k\}$, by (14) and (15) we have $|I\circ\hat{L}_x| < 1$ on $\big\{\eta \in M\big(L^\infty(\partial\Delta)\big)\ ;\ \mathrm{Re}\,\hat{z}(\eta) > 0\big\} = \bigcup_{k=1}^{\infty}\hat{L}_x^{-1}(U_k)$; $|I\circ\hat{L}_x| = 1$ on $\big\{\eta \in M\big(L^\infty(\partial\Delta)\big)\ ;\ \mathrm{Re}\,\hat{z}(\eta) \leq 0\big\}$; and $|I\circ L_x|$ on $\partial\Delta$ is continuous at every point $\eta \in \partial\Delta$ with $\mathrm{Re}\,\hat{z}(\eta) \leq 0$.

By (13), $I\circ\hat{L}_x$ is an outer function on Δ. Hence for every sequence $\{w_n\}_n$ in Δ such that $|w_n - \alpha| \longrightarrow 0$ for some α with $|\alpha| = 1$ and $\mathrm{Re}\,\alpha \leq 0$, we have $|(I\circ\hat{L}_x)(w_n)| \longrightarrow 1$. This means that

(16) $$|I\circ\hat{L}_x(\eta)| = 1 \text{ for every } \eta \in M\big(H^\infty(\Delta)\big)\backslash\Delta \text{ with } \mathrm{Re}\,\hat{z}(\eta) \leq 0.$$

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Put

$$V = \hat{L}_x^{-1}(U_I) = \mathrm{cl}\{\eta \in M(L^\infty(\partial\Delta))\ ;\ \mathrm{Re}\,\hat{z}(\eta) > 0\}.$$

Then U_I and V are open and closed subsets of ∂ and $M(L^\infty(\partial\Delta))$ respectively. By (16), $\overline{N(I \circ \hat{L}_x)} \subset V$, so that by Proposition 4, $M(I\circ\hat{L}_x) = V$. Since $M_\partial(I) = \hat{L}_x(M(I\circ\hat{L}_x)) = U_I$, by Proposition 1 and Theorem 7,

$$N_{B_1}(\bar{I}) = \tilde{U}_I \cap \mathrm{supp}\,\mu_x, \tag{17}$$

Since $\chi_V \notin \mathrm{QC}(\Delta)$, there is a QC($\Delta$)-level set Q_Δ such that $Q_\Delta \not\subset V$ and $Q_\Delta \cap V \neq \emptyset$. Put

$$Q = \{\zeta \in \mathrm{supp}\,\mu_x\ ;\ \hat{b}(\zeta) \in Q_\Delta\},$$

then Q is a QC_{B_1}-level set. Since $\tilde{U}_I \cap \mathrm{supp}\,\mu_x = \hat{b}^{-1}(V)$, $Q \not\subset \tilde{U}_I$ and $Q \cap \tilde{U}_I \neq \emptyset$. By (17) we get (i).

By Marshall (see [4, p. 392]), there is an inner function q such that $[H^\infty(\Delta), \chi_V] = [H^\infty(\Delta), \bar{q}]$, that is, for $\eta \in M(H^\infty(\Delta))\backslash\Delta$, $|\chi_V(\eta)| = 1$ if and only if $|q(\eta)| = 1$. If $|\chi_V(\eta)| < 1$ for $\eta \in M(H^\infty(\Delta))\backslash\Delta$, then $\mathrm{Re}\,\hat{z}(\eta) = 0$. Hence by (16),

$$|q(\eta)| = 1 \text{ or } |(I \circ \hat{L}_x)(\eta)| = 1 \text{ for } \eta \in M(H^\infty(\Delta))\backslash\Delta.$$

Put

$$J = q \circ b \in H^\infty.$$

Then $J \in I$ and $J \circ \hat{L}_x = q$ on $M(H^\infty(\Delta))$. Hence by Theorem 2, $|J| = 1$ on $M(B_1)\backslash\overline{P(x)}$, and

$$|J(\zeta)| = 1 \text{ or } |I(\zeta)| = 1 \text{ for } \zeta \in \overline{P(x)}.$$

Thus we get (ii).

Since $\chi_V|_{Q_\Delta}$ is not constant, $q|_{Q_\Delta}$ is not constant. By Lemma 9, $Q_\Delta \subset N(\bar{q})$. Since $M_\partial(J) = M_\partial(\bar{J}) = N_\partial(\bar{J})$,

$$\begin{aligned} N_{B_1}(\bar{J}) &= \mathrm{cl}\Big[\bigcup\{\mathrm{supp}\,\mu_\xi\ ;\ \xi \in N_\partial(\bar{J})\}\Big] \quad \text{by Theorem 7} \\ &= \mathrm{cl}\Big[\bigcup\big\{\mathrm{supp}\,\mu_\xi\ ;\ \xi \in \hat{L}_x\big(\overline{N(J \circ \hat{L}_x)}\big)\big\}\Big] \\ &= \Big[\bigcup\big\{\mathrm{supp}\,\mu_\xi\ ;\ \xi \in \hat{L}_x\big(N(\bar{q})\big)\big\}\Big] \\ &\supset \mathrm{cl}\Big[\bigcup\{\mathrm{supp}\,\mu_\xi\ ;\ \xi \in \hat{L}_x(Q_\Delta)\}\Big]. \end{aligned}$$

Since $Q_\Delta \cap V \neq \emptyset$, $\hat{L}_x(Q_\Delta) \cap U_I \neq \emptyset$. Since $\mathrm{supp}\,\mu_\xi \subset \tilde{U}_I$ for $\xi \in U_I$, by (17) $N_{B_1}(\bar{J}) \cap N_{B_1}(\bar{I}) = N_{B_1}(\bar{J}) \cap \tilde{U}_I \cap \mathrm{supp}\,\mu_x \neq \emptyset$. Thus we get (iii).

We have $\hat{b}(Q) = Q_\Delta$ and $J = q \circ b$. Since $q|_{Q_\Delta}$ is not constant, J is not constant on Q. We already proved $Q \cap \tilde{U}_I \neq \emptyset$. We have

$$\begin{aligned} Q &= \hat{b}^{-1}(Q_\Delta) \cap \mathrm{supp}\,\mu_x \\ &= \bigcup\{\hat{b}^{-1}(\eta)\ ;\ \eta \in Q_\Delta\} \cap \mathrm{supp}\,\mu_x \\ &= \bigcup\{R_\xi\ ;\ \xi \in \hat{L}_x(Q_\Delta)\}. \end{aligned}$$

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By Theorem 5, $Q \cap N_{B_2}(\bar{I}) = Q \cap \tilde{U}_I \neq \emptyset$, so that $R_\xi \cap N_{B_2}(\bar{I}) \neq \emptyset$ for some $\xi \in \hat{L}_x(Q_\Delta)$. By Lemma 9, there is a point ζ in $M(B_2)$ such that $I(\zeta) = 0$ and $\operatorname{supp} \mu_\zeta \subset R_\xi$. Hence $I|_{R_\xi}$ is not constant, and $I|_Q$ is not constant. Thus we get (iv).

Since $\operatorname{supp} \mu_\xi \subset R_\xi$, we have

$$\operatorname{cl}\Big[\bigcup\{\operatorname{supp} \mu_\xi \,;\, \xi \in \hat{L}_x(Q_\Delta)\}\Big] \subset Q.$$

By our construction, $|q(\eta)| = 1$ or $|(I \circ \hat{L}_x)(\eta)| = 1$ for $\eta \in M\big(H^\infty(\Delta)\big) \backslash \Delta$. Since $q|_{Q_\Delta}$ is not constant, by Proposition 5, $I \circ \hat{L}_x|_{Q_\Delta}$ is constant and $|I \circ \hat{L}_x|_{Q_\Delta}| = 1$. Hence I is constant on $\operatorname{cl}[\bigcup\{\operatorname{supp} \mu_\xi \,;\, \xi \in \hat{L}_x(Q_\Delta)\}]$. Therefore we get (v).

COROLLARY 5. *For every* $\xi \in \partial$, $\operatorname{supp} \mu_\xi \subsetneq R_\xi$.

PROOF. By the same way as the construction of I in Theorem 9, we can find an interpolating Blaschke product ψ such that $|\psi(\xi)| = 1$ and $\operatorname{supp} \mu_\xi \subset \tilde{U}_\psi$. By Theorem 5, $R_\xi \cap N_{B_2}(\bar{\psi}) \neq \emptyset$. By Lemma 9, there is a point ζ in $M(B_2)$ such that $\operatorname{supp} \mu_\zeta \subset R_\xi$ and $\psi(\zeta) = 0$. Then $\psi|_{\operatorname{supp} \mu_\zeta}$ is constant and $\psi|_{R_\xi}$ is not constant.

By Theorem 9, we cannot expect to have fruitful properties of the Douglas algebra B_1 as in [12].

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REFERENCES

1. S. Axler and A. Shields, *Extensions of harmonic and analytic functions*, Pacific J. Math. **145**(1990), 1–15.
2. P. Budde, *Support Sets and Gleason Parts of* $M(H^\infty)$, Thesis, University of California, Berkeley (1982).
3. S.-Y. Chang, *A characterization of Douglas subalgebras*, Acta Math. **137**(1976), 81–89.
4. J. Garnett, *Bounded analytic functions*, Academic Press, New York/London, 1981.
5. P. Gorkin, *Gleason parts and* COP, J. Funct. Anal. **83**(1989), 44-49.
6. P. Gorkin, H.-M. Lingenberg and R. Mortini, *Homeomorphic disks in the spectrum of* H^∞, Indiana Univ. Math. J. **39**(1990), 961–982.
7. C. Guillory, K. Izuchi and D. Sarason, *Interpolating Blaschke products and division in Douglas algebras*, Proc. Royal Irish Acad. **84A**(1984), 1–7.
8. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962.
9. ———, *Bounded analytic functions and Gleason parts*, Ann. of Math. **86**(1967), 74–111.
10. K. Izuchi, *Zero sets of interpolating Blaschke products*, Pacific J. Math, **119**(1985), 337–342.
11. ———, QC-*level sets and quotients of Douglas algebras*, J. Funct. Anal. **65**(1986), 293–308.
12. ———, *Countably generated Douglas algebras*, Trans. Amer. Math. Soc. **299**(1987), 171–192.
13. D. Marshall, *Subalgebras of* L^∞ *containing* H^∞, Acta Math. **137**(1976), 91–98.
14. D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207**(1975), 391–405.

https://doi.org/10.4153/CJM-1992-048-3 Published online by Cambridge University Press

15. ———, *The Shilov and Bishop decompositions of* $H^\infty + C$, Conference on harmonic analysis in honor of A. Zygmund, in Wadsworth Math. Series, California (1981), 461–474.

16. R. Younis, *Division in Douglas algebras and some applications*, Arch. Math. (Basel) **45**(1985), 550–560.

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