Canad. Math. Bull. Vol. 18 (2), 1975

# ON RING PROPERTIES OF INJECTIVE HULLS

BY
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Let $R$ be an associative ring and denote by $\hat{R}$ the injective hull of the right module $R_R$. If $\hat{R}$ can be endowed with a ring multiplication which extends the existing module multiplication, we say that $\hat{R}$ is a ring and the statement that $R$ is a ring will always mean in this sense

It is known that $\hat{R}$ is a regular ring (in the sense of von Neumann) if and only if the singular ideal of $R$ is zero. In this case, $\hat{R}=Q$, the complete ring of quotients. The fact that $\hat{R}$ can be a ring properly containing $Q$ was first shown by an example in [5] and at the end of this paper, a class of rings is given with the same property.

The present paper is in two sections. In §1, we assume that $\hat{R}$ is a ring and determine some properties that the multiplication must possess. In particular, although this multiplication is not necessarily unique, we show that it is determined modulo the singular submodule and also up to the ring of quotients $Q$, which is always a subring of $\hat{R}$. The singular submodule is moreover the Jacobson radical of $\hat{R}$ and necessary and sufficient conditions are found for $\hat{R}$ to be a local ring.

In §2 some negative results are obtained in the case where $R$ is commutative and a complete answer to the question (of when $\hat{R}$ is a ring) is given in the case where either $R$ or its quotient ring is Artinian. Some of these results are generalizations of results obtained by Harui in [1]. In particular, Harui's condition that $R$ be in the centre of $\hat{R}$ is dropped.

Definitions and preliminaries. Unless otherwise stated, all modules considered are right unital modules over a ring $R$ with identity. A submodule $A$ of an $R$-module $M$ is said to be *essential* or *large* if, for any non-zero submodule $B$ of $M$ we have $A \cap B \neq (0)$. The *singular submodule* of an $R$-module $M$ is defined as $Z_R(M)=\{m \in M \mid mI=0$ for some essential right ideal $I$ of $R\}$. In particular $Z_R(R)$ is a two-sided ideal of $R$.

The following notation and remarks are as in Lambek, [2]. Let $H=\mathrm{Hom}_R(\hat{R}, \hat{R})$. Then the map $h \to h(1)$ of $H$ onto $\hat{R}$ is surjective and if $N=\{h \in H \mid h(I)=0$ for some essential right ideal of $I$ of $R\}$ we have an additive isomorphism $H/N \cong \hat{R}/Z_R(\hat{R})$. Since $H/N$ is a regular ring, it follows that $\hat{R}/Z_R(\hat{R})$ can be made a ring with the same property. The complete ring of quotients $Q$ can be defined as $Q=\mathrm{Hom}_R(\hat{R}, \hat{R})$

Received by the editors November 12, 1971 and, in revised form, April 5, 1972.

Part of this research was supported by N.R.C. Grant. The rest is taken from the author's dissertation written at Tulane University under the direction of Laszlo Fuchs.

The elements of $Q$ are usually written on the right and we have the bimodule ${}_H\hat{R}_Q$. The map $q \to 1q$ is an embedding of $Q$ into $\hat{R}$ as $R$-modules.

Finally, it has been shown by Osofsky in [5] that if $\hat{R}$ is a ring, then 1 is the two-sided identity of $\hat{R}$.

§1. In this section we assume that $\hat{R}$ is a ring and denote the multiplication by $\circ$. Thus if $\hat{r} \in \hat{R}$ and $r \in R$, we have $\hat{r} \circ r = \hat{r}r$. The first two propositions give some information about $\circ$.

Proposition 1. *If $\hat{r}, \hat{s} \in \hat{R}$ are represented respectively as $h(1)$, $h'(1)$ where $h, h' \in H$ then $\hat{r} \circ \hat{s} \equiv h[h'(1)] \bmod Z_R(\hat{R})$.*

**Proof.** Let $J = \{r \in R \mid h'(r) \in R\}$. Since $R$ is essential in $\hat{R}$, it follows that $J$ is an essential right ideal of $R$ and for $x \in J$, we have $\{h(1) \circ h'(1) - h[h'(1)]\}x = h(1) \circ h'(x) - h[h'(x)] = h(1)h'(x) - h(1)h'(x) = 0$. This shows that $\hat{r} \circ \hat{s} - h[h'(1)]$ annihilates $J$ and is therefore in $Z_R(\hat{R})$.

Proposition 2. *$Z_R(\hat{R})$ is an ideal in $\hat{R}$.*

**Proof.** Take $z \in Z_R(\hat{R})$. Then $zI = 0$ for some essential right ideal $I$ of $R$ and if $\hat{r} \in \hat{R}$, $(\hat{r} \circ z)I = 0$, so that $\hat{r} \circ z \in Z_R(\hat{R})$. Now let $J = \{r \in R \mid \hat{r}r \in I\}$. Then $J$ is essential and $(z \circ \hat{r})J = z \circ \hat{r}J \subseteq zI = 0$.

It follows that although the multiplication on $\hat{R}$ is not necessarily unique, the quotient structure $\hat{R}/Z_R(\hat{R})$ always carries the same multiplication as that inherited from $H/N$ by the aforementioned isomorphism.

The next observation shows that multiplication on $\hat{R}$ is further restricted by the fact that $1Q$ is a subring.

Proposition 3. The multiplication on $\hat{R}$ extends module multiplication by elements of $Q$.

Proposition 3. *The multiplication on $\hat{R}$ extends module multiplication by elements of $Q$.*

**Proof.** Take $\hat{r} \in \hat{R}$ and $q \in Q$. The module product $\hat{r}q$ is just the image of $\hat{r}$ under $q$. Let $h$ denote the element of $H$ given by left multiplication by $\hat{r}$. Then $\hat{r} \circ (1q) = h[(1q)] = [h(1)]q = \hat{r}q$.

We now prove a result which increases further the importance of the singular submodule.

Theorem 1. *$Z_R(\hat{R})$ is the Jacobson radical of $\hat{R}$.*

**Proof.** Rad $\hat{R} \subseteq Z_R(\hat{R})$ follows from the above results which show that $\hat{R}/Z_R(\hat{R})$ is regular.

Now take $z \in Z_R(\hat{R})$. Define $z^1 = \{r \in R \mid zr = 0\}$. This is an essential right ideal of $R$. Then $(1-z)^1$ is a right ideal of $R$ and $x \in z_n^1 (1-z)^1 \Rightarrow 0 = zx = x - zx \Rightarrow x = 0$. It follows that $(1-z)^1 = 0$ since $z^1$ is essential and since $R$ is an essential submodule of $\hat{R}$, the right annihilator of $(1-z)$ in $\hat{R}$ must be zero. Then the map $\hat{r} \to (1-z) \circ \hat{r}$

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is an $R$-isomorphism of $\hat{R}$ onto $(1-z)\circ\hat{R}$ so that the latter is an injective submodule and therefore a direct summand. But for $x\in z^1$, we have $x=(1-z)x$ so that $z^1\subseteq(1-z)\circ\hat{R}$ which is therefore essential as an $R$-submodule of $\hat{R}$. This means that $(I-z)\circ\hat{R}=\hat{R}$ and hence there exists $\hat{r}\in\hat{R}$ such that $(1-z)\circ\hat{r}=1$. It follows that $z$ is right quasi-regular (with inverse $(1-\hat{r})$) and so $Z_R(\hat{R})$, as a right quasi-regular ideal, must be contained in Rad $\hat{R}$.

Theorem 1, together with an observation by Hans Storrer, reveals another special property of $\hat{R}$.

PROPOSITION 4. *The Jacobson radical of $\hat{R}$ is contained in its singular ideal.*

**Proof.** Take $z\in Z_R(\hat{R})$. Then $zI=0$ for some essential right ideal $I$ of $R$. It follows that $z\circ(I\circ\hat{R})=(zI)\circ\hat{R}=0$. But $I\circ\hat{R}$ is clearly essential as a right ideal of $\hat{R}$ so that $z\in Z_{\hat{R}}(\hat{R})$.

The last result of this section is a typical consequence of the above propositions and is used later.

THEOREM 2. *$\hat{R}$ is local if and only if every non-trivial right ideal of $R$ is essential.*

**Proof.** Every right ideal of $R$ is essential if and only if $R$ is indecomposable as an $R$-module and it is shown in Matlis [3] that the latter is equivalent to $H$ being a local ring. But $N$ is the Jacobson radical of $H$ so that $H$ is local if and only if $H/N$ is a division ring, which is equivalent by above remarks to $\hat{R}/Z_R(\hat{R})$ being a division ring. The desired equivalence follows using Theorem 1.

§2. In this section, unless otherwise stated, $R$ is a commutative ring. In this case, there is no distinction between the left and right injective hulls and the module multiplication is commutative. That is, for $\hat{r}\in\hat{R}$ and $r\in R$ we have $\hat{r}r=r\hat{r}$. But we do not assume that $r\circ\hat{r}=r\hat{r}$. As can be seen from [1], this assumption would greatly simplify some of the following proofs, but, although the writer knows of no examples of commutative rings $R$ where $\hat{R}$ is a ring and does not contain $R$ in the centre, nor does he know of a proof that it must be so. We therefore avoid the assumption.

Before proving the next result, we remark that for $r\in R$ and $\hat{r}\in\hat{R}$, the difference $r\circ\hat{r}-\hat{r}\circ r$ annihilates the essential ideal $X=\{x\in R \mid \hat{r}x\in R\}$, so that $r\circ\hat{r}-\hat{r}\circ r\in Z(\hat{R})$.

LEMMA 1. *If the socle of $R$ is essential and $\hat{R}$ is a ring, then $S$ is the right socle of $\hat{R}$ and is contained in its left socle.*

**Proof.** For $x\in\hat{R}$, the map $\bar{x}:R\to\hat{R}$ defined by $r\to xr$ is an $R$-homomorphism and since $S$ is essential as a right $R$-submodule of $\hat{R}$, the ideal $x^{-1}S=\{r\in R \mid xr\in S\}$ is essential in $R$ and therefore contains $S$. This shows that $S$ is a left ideal of $\hat{R}$.

Now let $x$ be an element of some homogeneous component $H$ of $R$ and suppose that there exists $\hat{r}\in\hat{R}$ such that $\hat{r}x=y\notin H$. Then left multiplication by $\hat{r}$ is an $R$-isomorphism of $xR$ onto $yR$, a contradiction. It follows that each homogeneous

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component is a left ideal of $\hat{R}$. Now take $x, y \in H$. The map: $xR \to yR$ defined by $x \to y$ is an $R$-homomorphism from an ideal of $R$ into $\hat{R}$ and is therefore induced by an element of $\hat{R}$. That is, there exists $\hat{r} \in \hat{R}$ such that $\hat{r}x = y$. This shows that each homogeneous component is a minimal left ideal in $\hat{R}$ and $S$ is contained in the left socle of $R$.

[We cannot conclude that $S$=left socle of $\hat{R}$ since not assuming $r \circ \hat{r} = r\hat{r}$ leaves open the possibility that a left $\hat{R}$-ideal may fail to be a left $R$-module and hence may fail to intersect $R$.]

Now take $x \in S$ such that $xR$ is a minimal ideal of $R$. Then $S = xR \oplus A$ where $A$ is an ideal in $R$ and we can define a map: $S \to S \subseteq \hat{R}$ by $x \to x$ and $a \to 0$ for $a \in A$. As before, this map is induced by an element $n$ of $\hat{R}$. Let $\hat{r} \in \hat{R}$ such that $x \circ \hat{r} \in S$. Then $x \circ \hat{r} = (nx) \circ \hat{r} = n \circ x \circ \hat{r} = n(x \circ \hat{r})$, so that $x \circ \hat{r}$ is an element of $S$ not moved by $n$. It follows that $x \circ \hat{r} \in xR$, which means that $x \circ \hat{R} \cap S = xR$. Now take $0 \neq l \in x \circ \hat{R}$. We have $0 \neq l \circ \hat{R} \cap S \subseteq x \circ \hat{R} \cap S = xR$ which is minimal. Then $l \circ \hat{R} \cap S = xR$ which implies that $x \in l \circ \hat{R}$ and it follows that $x \circ \hat{R}$ is a minimal right $\hat{R}$-ideal. In particular $S \subseteq$ right socle of $\hat{R}$. We now prove that $x \circ \hat{R} = xR$ for $x$ as above, giving the result that $S$ is the right socle of $\hat{R}$. Suppose that $x$ is in the homogeneous component $H$. Then $S = H \oplus C$ and this is left ideal decomposition. Let $B$ be a maximal sum of minimal left $\hat{R}$ ideals which are isomorphic to $H$ as left $\hat{R}$-modules and which fails to intersect $S$. Then $B \oplus H \oplus C$ is a direct sum of left $\hat{R}$-ideals. Now for $\hat{r} \in \hat{R}$, $x \circ \hat{r}$ is in the left homogeneous component containing $H$, for this is a right $\hat{R}$-ideal. Then $x \circ \hat{r} = b + h + c$ in obvious notation. Suppose that $b = x \circ \hat{r} - (h + c)$ is not zero. Then there exists $r \in R$ such that $0 \neq br = s \in S$ and $r \circ b = r \circ x \circ \hat{r} - r \circ (h + c) = x \circ r \circ \hat{r} - (h + c)r = x \circ (\hat{r} \circ r + z) - (h + c)r$ where $z \in Z(\hat{R})$. Then $r \circ b = [x \circ \hat{r} - (h + c]r = br = s$ since $x \circ z = 0$ by Theorem 1 and the fact that $x$ is in the right socle of $\hat{R}$. But the left ideal generated by $b$ intersects $S$ in zero. This contradiction means that $b = 0$ and therefore that $x \circ \hat{r} \in S$. It follows by the above that $x \circ \hat{r} \in xR$.

It is quite easy to see that no two homogeneous components of $S$ can be isomorphic as left $\hat{R}$-modules, so that $c$ in the above proof is zero.

COROLLARY 1. *For $R$ as in Lemma 1 we have, for $\hat{r} \in \hat{R}$, $\hat{r}S = 0 \Rightarrow S \circ \hat{r} = 0$.*

**Proof.** $\hat{r}S = 0 \Rightarrow \hat{r} \in Z_R(\hat{R}) = \text{Rad } \hat{R}$ by Theorem 1 then $S \circ \hat{r} = 0$ by Lemma 1.

We are now able to find sufficient conditions on a commutative ring that its injective hull cannot be made a ring.

THEOREM 3. *Let $R$ be a ring with the following properties:*

(i) *The socle $S$ of $R$ is large*
(ii) *No homogeneous component of $S$ is simple*
(iii) *$S$ is the sum of a finite number of minimal ideals.*

*Then $\hat{R}$ cannot be made into a ring.*

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**Proof.** Suppose that $\hat{R}$ is a ring. Let $S=\bigoplus_{k=1}^{n} H_k$ where the $H_k$ are homogeneous components and $H_k=\oplus x_i^k R$, a finite sum of at least two minimal ideals. Then the maps: $S\to S$ defined by $x_j^k\to x_i^k$ and $x_l^t\to 0$ for $l\neq j$ or $t\neq k$ are induced as before by elements $u_{ij}^k$ of $\hat{R}$. That is $u_{ij}^k x_j^k=x_i^k$ and $u_{ij}^k x_l^t=0$ for $l\neq j$ or $t\neq k$, and it is clear that the difference $u_{jj}^k-u_{jt}^k\circ u_{tj}$ annihilates $S$ on the left. By Corollary 1 it follows that $s\circ u_{jj}^k=s\circ u_{jt}^k\circ u_{tj}^k$ for all $s\in S$. We express this fact as relation:

$$\alpha u_{jj}^k \equiv u_{jt}^k\circ u_{tj}^k \bmod Z_R(\hat{R})$$

In the same way we obtain:

$$\beta u_{jt}^k\circ u_{jt}^k \equiv 0 \bmod Z_R(\hat{R}) \quad \text{for} \quad j\neq t.$$

Now write $x_l^l=x$ and consider the element $x\circ u_{jt}^k$ for any $j$, $t$, $k$ with $j\neq t$. We know from the proof of Lemma 1 that $x\circ u_{jt}^k=xr=rx$ for some $r\notin R$ and, using $\beta$ we get $0=x\circ u_{jt}^k\circ u_{jt}^k=(rx)\circ u_{jt}^k=r\circ(x\circ u_{jt}^k)=r^2x$. Since the annihilator of a minimal ideal is prime, we get $0=xr=x\circ u_{jt}^k$. Now $\alpha$ gives $x\circ u_{jj}^k=x\circ u_{jt}^k\circ u_{tj}^k=0$ for any $j$, $k$. But $\sum_{j,k} u_{jj}^k\equiv 1 \bmod Z_R(\hat{R})$ which gives a contradiction. It follows that $\hat{R}$ cannot be made a ring.

It will be seen later that commutativity is a necessary condition in Theorem 3.

Suppose now that $R=A\oplus B$, where $A$, $B$ are ideals in $R$, which need not be commutative for the moment. Consider $\hat{A}_R$. Let $X=\{x\in\hat{A}_R \mid xB=0\}$ and $Y=\{y\in\hat{A}_R \mid yA=0\}$. Then $\hat{A}_R=X\oplus Y$ and $A\subseteq X$. But $A$ is essential as an $R$-submodule of $\hat{A}_R$ so that $Y=0$. That is $\hat{A}_R$ is annihilated by $B$. Now consider $\hat{A}_R$ as an $A$-module $(\hat{A}_R)_A$ and consider the diagram:

$$\begin{array}{l} C_A \xrightarrow{f} (\hat{A}_R)_A \\ \cap| \\ D_A \end{array}$$

where $C_A$, $D_A$ are $A$-modules and $f$ is an $A$-homomorphism. Now make $C_A$ and $D_A$ into $R$-modules by letting $B$ annihilate them and do the same with $(\hat{A}_R)_A$. Without specifying all the identifications involved, we can now say that $f$ has become an $R$-homomorphism into $\hat{A}_R$ which therefore lifts giving finally an $A$-homomorphism from $D_A$ into $(\hat{A}_R)_A$. Since $A$ is essential in $(\hat{A}_R)_A$, it follows that $(\hat{A}_R)_A=\hat{A}_A$. Furthermore, $\hat{A}_A$ is a ring (extending multiplication by elements of $A$) if and only if $\hat{A}_R$ is a ring (extending multiplication by $R$). Therefore, if $\hat{A}_A$ and $\hat{B}_B$ are rings, then so is $\hat{R}=\hat{A}_R\oplus\hat{B}_R$. Now suppose that $\hat{R}$ is a ring. Take $\hat{a}_1$, $\hat{a}_2\in\hat{A}_R$. Then $\hat{a}_1\circ\hat{a}_2=\hat{a}+\hat{b}$ where $\hat{a}\in\hat{A}_R$ and $\hat{b}\in\hat{B}_R$. It follows that $\hat{b}B=0$. But $B$ is a ring with identity $b_0$ and $\hat{b}b_0=\hat{b}$. It follows that $\hat{b}=0$ and so $\hat{A}_R$ is a subring of $\hat{R}$. This makes it clear that $\hat{A}_A$ is a ring.

These remarks justify the following lemma:

LEMMA 2. *If $R=\bigoplus_{i=1}^{n} A_i$, ring decomposition, then R is a ring if and only if each $\hat{A}_{iA}$ is a ring.*

We can now give an answer to the general question in the case of commutative Artinian rings.



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THEOREM 4. *If $R$ is a commutative Artinian ring, then the following statements are equivalent.*

(i) *$\hat{R}$ is a ring*
(ii) *$R$ is self-injective*
(iii) *Every homogeneous component of $S$ is simple.*

**Proof.** Write $1=e_1+\cdots+e_n$ where $\{e_1, \cdots, e_n\}$ is an orthogonal set of primitive idempotents. Write $e_iR=A_i$ for $i=0, \ldots, n$. Then $R=\bigoplus_{i=1}^n A_i$ and the $A_i$ are Artinian, local rings with identities $e_i$. Clearly, the socle of $A_i$ is a homogeneous component of $R$ for any $i$ and if any such socle is non-simple, then $\hat{A}_i$ is not a ring by Theorem 3 and therefore $\hat{R}$ is not a ring by Lemma 2. If, on the other hand, each $A_i$ has a simple socle, then each $A_i$ is self-injective by [3]. These remarks give the result.

COROLLARY 2. *Let $R$ be a commutative ring whose complete quotient ring $Q$ is Artinian. Then $\hat{R}$ is a ring if and only if $\hat{R}=Q$.*

**Proof.** By Proposition 3, $\hat{R}$ extends module multiplication by elements of $Q$ if it is a ring and, as can be seen in [2], $\hat{R}_Q=\hat{Q}_Q$. Since $Q$ is commutative and Artinian, $\hat{Q}$ is a ring if and only if $\hat{Q}=Q$, by Theorem 4. Then $\hat{R}$ is a ring if and only if $\hat{R}=Q$.

The following example is due to Vlastimil Dlab and Claus M. Ringel. It shows a class of rings $R$ with the property that $\hat{R}$ is a ring properly containing $Q$.

EXAMPLE 1. Let $A$ be a ring with identity $e$ and the following properties:

(i) $A$ is local with Jacobson radical $W$.
(ii) $A$ is an algebra of finite dimension $n$ over a field $K$.
(iii) $W$ is a vector space direct summand of $A$ and $A_k=eK\oplus W$.
(iv) The socle of $A$ is simple.

For a positive integer $m$, let $R$ be the subring of $A_m$ consisting of all matrices which are sums of matrices of the following two forms:

(a) scalar matrices with diagonal elements $ke$ for some fixed $k$ and zeros elsewhere.
(b) matrices with zeros everywhere except in the last column, where the first $m-1$ entries are arbitrary elements of $A$ and the $(m, m)$th entry is an element of $W$.

An arbitrary element of $R$ is therefore of the following form:

$$\begin{bmatrix} ke & & & & a_1 \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & \cdot & a_{m-1} \\ & & & & \cdot \quad ke+w \end{bmatrix}$$

where $k \in K$, $w \in W$ and $a_1, \ldots, a_{m-1} \in A$.

Now let $M_R = \operatorname{Hom}_K(R_K, K)$. This is an indecomposable right $R$-module and since ${}_RR$ is projective, $M_R$ is injective. But $R$ is a local ring with radical say $W^1$, so that every indecomposable injective module is isomorphic to $E(R/W^1)$, the injective hull of the right $R$-module $R/W^1$. Since the right socle of $R$ consists of $m$ copies of $R/W^1$, we have $E(R) \cong \bigoplus_{i=1}^m E_i(R/W^1)$ where $E_i(R/W^1) \cong M$ for $i=1, \ldots, m$. This makes it clear that $E(R)$ has dimension $m^2n$ as a vector space over $K$ and this is the dimension of $A_m$ over $K$. But $R$ is essential as an $R$-submodule of $A_m$. This shows that $A_m = E(R)$. Since $R$ is an Artinian local ring, however, it is equal to its complete ring of quotients. This example shows the necessity of commutativity in Theorem 3.

EXAMPLE 2. Let $Z_{(p)} = \operatorname{Hom}_Z(Z_{p^\infty}, Z_{p^\infty})$, where $Z_{p^\infty}$ is the Prüfer group for some prime $p$. Let $R$ be the ring whose additive group is $Z_{(p)} \oplus Z_{p^\infty} \oplus Z_{p^\infty}$ with multiplication defined by: $(z, m_1, m_2)(z', m_1', m_2') = (zz', zm_1' + z'm_1, zm_2' + z'm_2)$ in the obvious notation. As in [6] it can be seen that $R$ is a commutative ring with a large socle which is the sum of two minimal ideals, and that $R$ is not Artinian. Since $R$ satisfies the conditions of theorem 3, $\hat{R}$ is not a ring.

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