Canad. Math. Bull. Vol. **57** (1), 2014 pp. 145–158
http://dx.doi.org/10.4153/CMB-2012-016-1
©Canadian Mathematical Society 2012

The Essential Spectrum of the Essentially Isometric Operator

H. S. Mustafayev

Abstract. Let T be a contraction on a complex, separable, infinite dimensional Hilbert space and let $\sigma(T)$ (resp. $\sigma_e(T)$) be its spectrum (resp. essential spectrum). We assume that T is an essentially isometric operator; that is, $I_H - T^*T$ is compact. We show that if $D\backslash\sigma(T) \neq \varnothing$, then for every f from the disc-algebra

$$\sigma_e\big(f(T)\big) = f\big(\sigma_e(T)\big),$$

where D is the open unit disc. In addition, if T lies in the class $C_{0\cdot} \cup C_{\cdot 0}$, then

$$\sigma_e\big(f(T)\big) = f\big(\sigma(T) \cap \Gamma\big),$$

where Γ is the unit circle. Some related problems are also discussed.

1 Introduction

Let H be a complex, separable, infinite dimensional Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Throughout, $\sigma(T)$ denotes the spectrum and $R_\lambda(T) = (\lambda I_H - T)^{-1}$ ($\lambda \notin \sigma(T)$) the resolvent of $T \in B(H)$. We use the notations $\sigma_l(T)$ and $\sigma_r(T)$ to denote the left and right spectra of T, respectively. The unit circle in the complex plane will be denoted by Γ, whereas D indicates the open unit disk. The disc-algebra and the algebra of all bounded analytic functions on D are denoted by $A(D)$ and $H^\infty := H^\infty(D)$, respectively.

For $T \in B(H)$, the uniform operator topology closure of all polynomials taken in T is denoted by A_T. Note that A_T is a commutative unital Banach algebra. The Gelfand space of A_T can be identified with $\sigma_{A_T}(T)$, the spectrum of T with respect to the algebra A_T. Since $\sigma(T)$ is a (closed) subset of $\sigma_{A_T}(T)$, for every $\lambda \in \sigma(T)$, there exists a multiplicative functional ϕ_λ on A_T such that $\phi_\lambda(T) = \lambda$. By $\widehat{S}$ we denote the Gelfand transform of $S \in A_T$. Here and in the sequel, instead of $\widehat{S}(\phi_\lambda)(= \phi_\lambda(S))$, where $\lambda \in \sigma(T)$, we use the notation $\widehat{S}(\lambda)$. Notice that $\lambda \mapsto \widehat{S}(\lambda)$ is a continuous function on $\sigma(T)$. It follows from the Shilov's Theorem [6, Theorem 2.3.1] that if T is a contraction, then $\sigma_{A_T}(T) \cap \Gamma = \sigma(T) \cap \Gamma$, which is the *unitary spectrum* of T.

If T is a contraction on H, then it follows from the von Neumann inequality that there exists a contractive algebra-homomorphism $h\colon A(D) \to A_T$ (with dense range) such that $h(1) = I_H$ and $h(z) = T$. We use the notation $f(T) := h(f)$, $f \in A(D)$. Thus we have $\|f(T)\| \leq \|f\|_\infty$ for all $f \in A(D)$. It is easy to check that h is an isometry if and only if $\Gamma \subset \sigma(T)$.

Received by the editors April 15, 2012; revised May 10, 2012.
Published electronically July 16, 2012.
AMS subject classification: **47A10**, 47A53, 47A60, 47B07.
Keywords: Hilbert space, contraction, essentially isometric operator, (essential) spectrum, functional calculus.

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

A contraction T on H is said to be *completely nonunitary* (c.n.u.) if it has no proper reducing subspace on which it acts as a unitary operator. If T is a c.n.u. contraction, then $f(T)$ ($f \in H^\infty$) can be defined by the Nagy–Foias functional calculus [9, Chapter III]. We put $H^\infty(T) = \{f(T) : f \in H^\infty\}$. A c.n.u. contraction T on H is called a C_0-*contraction* if there exists a nonzero function $f \in H^\infty$ such that $f(T) = 0$. A contraction T on H is said to be of class $C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^n x \to 0$ (resp. $T^{*n} x \to 0$) for every $x \in H$. We put $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$. As is known [9, Proposition II.4.2], $C_0 \subset C_{00}$. Recall that $T \in B(H)$ is said to be *essentially isometric operator* if $I_H - T^*T$ is compact.

By $K(H)$, we will denote the ideal of compact operators on H. The quotient algebra $B(H)/K(H)$ is a C^*-algebra called the *Calkin algebra*. Let $\pi\colon B(H) \to B(H)/K(H)$ be the natural map. The *essential spectrum* $\sigma_e(T)$ of $T \in B(H)$ is the spectrum of $\pi(T)$ in the Calkin algebra. As is well known, $\sigma_e(T)$ is a nonempty compact subset of $\sigma(T)$. Similarly, the *left* and *right essential spectrum* of T are defined by $\sigma_{le}(T) := \sigma_l(\pi(T))$ and $\sigma_{re}(T) := \sigma_r(\pi(T))$. Recall that T is a (left, right) *Fredholm operator*, if $\pi(T)$ is (left, right) invertible in the Calkin algebra.

Assume that a contraction T on H is from the class C_{00}. Moreover, assume that

$$\dim(I - TT^*)H = \dim(I - T^*T)H = 1.$$

According to the well-known model theorem of Nagy–Foias [9], T is unitary equivalent to its model operator $M_\varphi = P_\varphi S\,|_{K_\varphi}$ acting on the model space $K_\varphi := H^2 \ominus \varphi H^2$, where φ is an inner function, $Sf = zf$ is the shift operator on the Hardy space H^2, and P_φ is the orthogonal projection from H^2 onto K_φ. It follows that for every $f \in H^\infty$, the operator $f(T)$ is unitary equivalent to

$$f(M_\varphi) = P_\varphi f(S)\,|_{K_\varphi}\,.$$

In [11, p. 162, Corollary 1] it was proved that for every $f \in A(D)$,

$$\sigma_e\big(f(T)\big) = f\big(\sigma_e(T)\big) = f\big(\sigma(T) \cap \Gamma\big)\,.$$

On the other hand, it follows from the Lipschitz-Moeller Theorem [11, III.1] that $\sigma(T) \cap D = \varphi^{-1}(0)$ and therefore, $D \diagdown \sigma(T) \neq \varnothing$.

In this paper, we generalize this result in the following way. It is shown in Section 2 that if T is an essentially isometric contraction and $D \diagdown \sigma(T) \neq \varnothing$, then for every $f \in A(D)$, we have the essential spectral mapping equality

$$\sigma_e\big(f(T)\big) = f\big(\sigma_e(T)\big)\,.$$

The asymptotic behavior of the orbits $\{T^n S : n \geq 0\}$ are considered in Section 3. We show that if T is an essentially isometric contraction from the class $C_{0\cdot} \cup C_{\cdot 0}$, then for every $S \in A_T$,

$$\lim_{n\to\infty} \|T^n S\| = \sup_{\xi \in \sigma_{le}(T)} |\widehat{S}(\xi)| = \sup_{\xi \in \sigma_{re}(T) \cap \Gamma} |\widehat{S}(\xi)|.$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

As a corollary of this result, we obtain that if T is an essentially isometric contraction from the class $C_{0\cdot} \cup C_{\cdot 0}$, then

$$\sigma_{le}(T) = \sigma_{re}(T) \cap \Gamma = \sigma(T) \cap \Gamma.$$

In addition, if $D \diagdown \sigma(T) \neq \varnothing$, then for every $f \in A(D)$,

$$\sigma_e\big(f(T)\big) = f\big(\sigma(T) \cap \Gamma\big).$$

2 The Essential Spectral Mapping Theorem

The main result of this section is the following assertion.

Theorem 2.1 *Let T be an essentially isometric contraction such that $D \diagdown \sigma(T) \neq \varnothing$. Then for every $f \in A(D)$, we have*

$$\sigma_e\big(f(T)\big) = f\big(\sigma_e(T)\big).$$

For the proof we need some preliminary results.

Let A be a C^*-algebra with the unit element e and let S_A be the set of all pure states on A. We know [10, Corollary V.23.3] that if $a \in A$, then $\sigma_l(a)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $f \in S_A$ such that $\lambda = f(a)$ and $f(a^*a) = f(a^*)f(a)$. Assume that $a^*a = e$. If $\lambda \in \sigma_l(a)$, then we have

$$|\lambda|^2 = \overline{f(a)}f(a) = f(a^*)f(a) = f(a^*a) = f(e) = 1.$$

This shows that $\sigma_l(a) \subset \Gamma$. In particular, if a is a unitary element of A, then $\sigma_l(a) = \sigma_r(a) = \sigma(a) \subset \Gamma$.

Let T be an essentially isometric operator on H. Since $\pi(T)^*\pi(T) = \pi(I_H)$, it follows that $\sigma_{le}(T) = \sigma_l(\pi(T)) \subset \Gamma$. Moreover, if T is an *essentially unitary operator*, that is, if both $I_H - T^*T$ and $I_H - TT^*$ are compact, then $\sigma_{le}(T) = \sigma_{re}(T) = \sigma_e(T) \subset \Gamma$. Recall that an operator $T \in B(H)$ is said to be *essentially normal* if $TT^* - T^*T$ is compact. Similarly, we can see that if T is an essentially normal operator, then $\sigma_{le}(T) = \sigma_{re}(T) = \sigma_e(T)$.

Let T be an essentially isometric contraction on H and let V be the partial isometry in its polar decomposition. From the identity

$$\sqrt{1-z} = 1 + \sum_{k=1}^{\infty} a_k z^k$$

(the series is absolutely convergent on $\overline{D}$ and $a_k < 0$, $k = 1, 2, \dots$) we can write

$$|T| = (T^*T)^{\frac{1}{2}} = I_H + \sum_{k=1}^{\infty} a_k (I_H - T^*T)^k = I_H + K.$$

So, we have

$$T = V|T| = V(I_H + K) = V + VK$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

(it can be seen that $I_H - V^*V$ is of finite rank). Recall that if T is invertible, then V is unitary.

Now let T be an essentially isometric contraction on H such that $D \setminus \sigma(T) \neq \varnothing$. Then T is essentially unitary (see for example, Proposition 3.4(f)) and therefore, $\sigma_e(T) \subset \Gamma$. Notice also that T is a Fredholm operator. Let $\lambda_0 \in D \setminus \sigma(T)$. We know [3, Proposition XI.3.4] that $\operatorname{ind}(T - \lambda I_H)$ is constant on the components of $\mathbb{C} \setminus \sigma_e(T)$. It follows that

$$\operatorname{ind}(T - \lambda I_H) = \operatorname{ind}(T - \lambda_0 I_H) = 0, \quad \forall \lambda \in D.$$

In particular, we have $\operatorname{ind} T = 0$. Since T is a Fredholm operator, T has the form $T = S + K$, where S is invertible and K is compact. Notice also that S is essentially unitary. As we already noted above, $S = U + K$, where U is unitary and K is compact. Thus, we obtain that T is a compact perturbation of a unitary operator.

We call $\lambda \in \sigma(T)$ a *normal eigenvalue* of $T \in B(H)$ if it is an isolated point of $\sigma(T)$ and if the corresponding Riesz projection has finite rank. We denote by $\sigma_{np}(T)$ the set of all normal eigenvalues of T. Notice that if N is a normal operator, then $\sigma_{np}(N)$ consists of all $\lambda \in \sigma(N)$ for which λ is an isolated eigenvalue of N having finite multiplicity. Consequently, we have $\sigma_e(N) = \sigma(N) \setminus \sigma_{np}(N)$ [3, Proposition XI.2.9].

Let T be an essentially isometric contraction such that $D \setminus \sigma(T) \neq \varnothing$. As we have seen above, T is a compact perturbation of a unitary operator; $T = U + K$, where U is unitary and K is compact. By [4, Theorem I.5.3],

$$\sigma(T) \setminus \sigma_{np}(T) = \sigma(U) \setminus \sigma_{np}(U).$$

Consequently, we have

$$\sigma_e(T) = \sigma_e(U) = \sigma(U) \setminus \sigma_{np}(U) = \sigma(T) \setminus \sigma_{np}(T).$$

Let $C(\Gamma)$ be the space of all continuous functions on Γ. Notice that if U is unitary and $f \in C(\Gamma)$, then $f(U)$ is a normal operator. Moreover, the spectral mapping property $\sigma(f(U)) = f(\sigma(U))$ holds.

However, we have the following result.

Lemma 2.2 *If U is a unitary operator on H, then for every $f \in C(\Gamma)$,*

$$\sigma_e\big(f(U)\big) = f\big(\sigma_e(U)\big).$$

Proof Let $f \in C(\Gamma)$. If $\xi \in \sigma_e(U)$, then there exists a sequence $\{x_n\}$ of unit vectors in H such that $x_n \to 0$ weakly and

$$\lim_{n\to\infty} \|(U - \xi I_H)x_n\| = 0.$$

It follows that for an arbitrary trigonometric polynomial Q,

$$\lim_{n\to\infty} \|(Q(U) - Q(\xi) I_H)x_n\| = 0.$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

This shows that $Q(\xi) \in \sigma_e(Q(U))$. On the other hand, there exists a sequence of trigonometric polynomials $\{Q_n\}$ such that $Q_n \to f$ uniformly on Γ. Consequently, $Q_n(U) \to f(U)$ in the operator norm. Since $Q_n(\xi) \in \sigma_e(Q_n(U))$ and $Q_n(\xi) \to f(\xi)$, this clearly implies that $f(\xi) \in \sigma_e(f(U))$.

By E_N we will denote the spectral measure of an arbitrary normal operator N. Below, we will use the following fact [1, Proposition 2.8.1]: If N is normal, then all accumulation points of $\sigma(N)$ belong to $\sigma_e(N)$.

Now let $\lambda \in \sigma_e(f(U))$. Then either λ is an accumulation of $\sigma(f(U))$ or λ is an isolated eigenvalue of $f(U)$ having infinite multiplicity. Since $\sigma(f(U)) = f(\sigma(U))$, in the first case, there is a sequence of distinct points $\{\mu_n\}$ in $\sigma(U)$ such that $f(\mu_n) \to \lambda$. We may assume that $\mu_n \to \mu$ for some $\mu \in \sigma(U)$. Then $\mu \in \sigma_e(U)$ and $\lambda = f(\mu)$. Now assume that λ is an isolated eigenvalue of $\sigma(f(U))$ with infinite multiplicity. If the set $f^{-1}(\lambda) \cap \sigma(U)$ is infinite, then there is a sequence of distinct points $\{\mu_n\}$ in $\sigma(U)$ such that $f(\mu_n) = \lambda$. If $\mu_n \to \mu$, then $\mu \in \sigma_e(U)$ and $\lambda = f(\mu)$. Hence, we may assume that $f^{-1}(\lambda) \cap \sigma(U)$ is a finite set, say $\{\mu_1, \dots, \mu_n\}$. Assume on the contrary that $\mu_i \notin \sigma_e(U)$ $(i = 1, \dots, n)$. Then each μ_i is an isolated eigenvalue of $\sigma(U)$ of finite multiplicity. Since

$$E_{f(U)}\{\lambda\} = E_U\{\mu_1\} + \cdots + E_U\{\mu_n\},$$

it follows that λ is an eigenvalue of $f(U)$ of finite multiplicity. This is a contradiction. ■

We are now able to prove the main result of this section.

Proof of Theorem 2.1 As we have seen above, T is a compact perturbation of a unitary operator; $T = U + K$, where U is unitary and K is compact. Take $f \in A(D)$. Then there exists a sequence of polynomials $\{P_n\}$ such that $P_n \to f$ uniformly on D. Consequently,

$$P_n(T) - P_n(U) \to f(T) - f(U)$$

in the operator norm. Since $P_n(T) - P_n(U)$ $(n \in \mathbb{N})$ is compact, it follows that $f(T) - f(U)$ is compact. Now, taking into account the preceding lemma, we can write

$$\sigma_e\big(f(T)\big) = \sigma_e\big(f(U)\big) = f\big(\sigma_e(U)\big) = f\big(\sigma_e(T)\big).$$

The proof is complete. ■

3 Asymptotic Behavior of Essentially Isometric Contractions

Let T be an essentially isometric contraction and $S \in A_T$. In this section, we study the asymptotic behavior of the orbits $\{T^nS : n \geq 0\}$ in terms of the essential spectrum of T. By $\{T\}'$, we will denote the commutant of $T \in B(H)$.

The main result of this section is the following theorem.

Theorem 3.1 *If T is an essentially isometric contraction from the class $C_{0\cdot} \cup C_{\cdot 0}$, then for every $S \in A_T$,*

$$\lim_{n\to\infty} \|T^nS\| = \sup_{\xi\in\sigma_{le}(T)} |\widehat{S}(\xi)| = \sup_{\xi\in\sigma_{re}(T)\cap\Gamma} |\widehat{S}(\xi)|.$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

For the proof we need some results.

Proposition 3.2 *The following assertions hold:*

(a) *If T is a c.n.u. contraction, then for every $K \in \{T\}' \cap K(H)$, $\lim_{n\to\infty} \|T^n K\| = 0$.*
(b) *If T is in the class $C_{0\cdot}$, then for every $K \in K(H)$, $\lim_{n\to\infty} \|T^n K\| = 0$.*
(c) *If T is in the class $C_{\cdot 0}$, then for every $K \in K(H)$, $\lim_{n\to\infty} \|KT^n\| = 0$.*

Proof (a) As it is known [7, Lemma 3.3], if T is a c.n.u. contraction, then $T^n \to 0$ in the weak operator topology. If $K \in \{T\}' \cap K(H)$, then for every $x \in H$, we have

$$\lim_{n\to\infty} \|T^n Kx\| = \lim_{n\to\infty} \|KT^n x\| = 0.$$

Since the set $\{Kx : \|x\| \le 1\}$ is relatively compact, for a given $\varepsilon > 0$, it has a finite ε-mesh, say $\{Kx_1, \dots, Kx_k\}$, where $\|x_i\| \le 1$ $(i = 1, \dots, k)$. So, we have

$$\|T^n K\| \le \max_i \{\|T^n K x_i\|\} + \varepsilon \ (n \in \mathbb{N}).$$

It follows that $\lim_{n\to\infty} \|T^n K\| = 0$.

The proofs of (b) and (c) are similar. ■

It easily follows from Proposition 3.2(a) that if T is a c.n.u. contraction and if there exists a compact operator in $\{T\}'$ with zero kernel or dense range, then T is in the class $C_{0\cdot} \cup C_{\cdot 0}$. Notice also that if $H^\infty(T) \cap K(H) \ne \{0\}$, then T is in the class C_{00}. This fact can be derived from the dilation arguments of Nagy–Foias [9, p. 140].

The proof of the following proposition, being very easy, is omitted.

Proposition 3.3

(a) *If V is a nonunitary isometry on H, then $\sigma_l(V) = \Gamma$; $\sigma_r(V) = \sigma(V) = \overline{D}$.*
(b) *If V is an arbitrary isometry on H, then $\sigma_l(V) = \sigma_r(V) \cap \Gamma = \sigma(V) \cap \Gamma$.*

Let H_0 be the linear space of all weakly null sequences $\{x_n\}$ in H. Let us define a semi-inner product on H_0 by

$$\big\langle \{x_n\}, \{y_n\} \big\rangle = \text{l.i.m.} \langle x_n, y_n \rangle,$$

where l.i.m. is a Banach limit. Let

$$E = \big\{ \{x_n\} \in H_0 : \text{l.i.m.} \|x_n\|^2 = 0 \big\}.$$

Then H_0 / E becomes a pre-Hilbert space with respect to the inner product defined by

$$\big\langle \{x_n\} + E, \{y_n\} + E \big\rangle = \text{l.i.m.} \langle x_n, y_n \rangle.$$

Let $\widetilde{H}$ be the completion of H_0 / E with respect to the induced norm. Then $\widetilde{H}$ is a Hilbert space.

For a given $T \in B(H)$, define the operator $\widetilde{T}$ on H_0 / E, by

$$\widetilde{T} \colon \{x_n\} + E \mapsto \{Tx_n\} + E.$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

Then we have

$$\|\widetilde{T}(\{x_n\}+E)\| = \left(\text{l.i.m.}\|Tx_n\|^2\right)^{\frac{1}{2}} \le \|T\|\left(\text{l.i.m.}\|x_n\|^2\right)^{\frac{1}{2}} = \|T\|\|\{x_n\}+E\|.$$

Since H_0/E is dense in $\widetilde{H}$, the operator $\widetilde{T}$ can be extended to the whole $\widetilde{H}$ which we also denote by $\widetilde{T}$. Clearly, $\|\widetilde{T}\| \le \|T\|$.

The pair $(\widetilde{H}, \widetilde{T})$ (sometimes the operator $\widetilde{T}$) will be called the *limit operator associated with T*.

Proposition 3.4 *Let $T \in B(H)$ and let $(\widetilde{H}, \widetilde{T})$ be the limit operator associated with T. Then the following assertions hold:*

(a) *The mapping $T \mapsto \widetilde{T}$ is a contractive algebra-$*$-homomorphism.*
(b) *T is compact if and only if $\widetilde{T} = 0$.*
(c) *$\sigma_l(\widetilde{T}) \subset \sigma_{le}(T)$, $\sigma_r(\widetilde{T}) \subset \sigma_{re}(T)$, and $\sigma(\widetilde{T}) \subset \sigma_e(T)$.*
(d) *If T is a contraction, then $\widetilde{f(T)} = f(\widetilde{T})$, $\forall f \in A(D)$.*
(e) *T is an essentially isometric (resp. essentially unitary, essentially normal) operator if and only if $\widetilde{T}$ is an isometry (resp. unitary, normal).*
(f) *If T is an essentially isometric operator and if $\sigma_{le}(T) \ne \Gamma$ (or $\sigma_{re}(T) \ne \overline{D}$), then T is essentially unitary.*
(g) *For every $T \in B(H)$, $\|\pi(T)\| \le \|\widetilde{T}\| \le \|T\|$.*

Proof The proofs of (a), (d), and (e) are straightforward.

(b) It is obvious that if T is compact, then $\widetilde{T} = 0$. If $\widetilde{T} = 0$, then for every weakly null sequence $\{x_n\}$ in H, we have $\text{l.i.m.}\|Tx_n\|^2 = 0$. Consequently, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\overline{\lim_{n\to\infty}}\|Tx_n\|^2 = \lim_{k\to\infty}\|Tx_{n_k}\|^2 = \text{l.i.m.}\|Tx_{n_k}\|^2 = 0.$$

It follows that $\lim_{n\to\infty}\|Tx_n\| = 0$, and therefore T is compact.

(c) If $\lambda \notin \sigma_{le}(T)$, then $\lambda I_H - T$ is a left Fredholm operator. So, there exists $S \in B(H)$ such that $S(\lambda I_H - T) - I_H \in K(H)$. It follows from (a) and (b) that $\widetilde{S}(\lambda I_{\widetilde{H}} - \widetilde{T}) = I_{\widetilde{H}}$. This shows that $\lambda \notin \sigma_l(\widetilde{T})$. The proof of the second and third parts of (c) is similar.

(f) It follows from (c) that $\sigma_l(\widetilde{T}) \subset \sigma_{le}(T)$ and therefore $\sigma_l(\widetilde{T}) \ne \Gamma$. By Proposition 3.3(a), $\widetilde{T}$ is unitary and so $\widetilde{T}\widetilde{T}^* = I_{\widetilde{H}}$. This means that $I_H - TT^*$ is compact.

(g) It suffices to show that $\|\pi(T)\| \le \|\widetilde{T}\|$. We know [2, p. 94] that

$$\|\pi(T)\| = \sup\left\{\overline{\lim_{n\to\infty}}\|Tx_n\| : \|x_n\| = 1\ (n \in \mathbb{N}),\ x_n \to 0 \text{ weakly}\right\}.$$

Therefore, for a given $\varepsilon > 0$, there exists a sequence $\{x_n\}$ in H such that $\|x_n\| = 1$ $(n \in \mathbb{N})$, $x_n \to 0$ weakly, and

$$\overline{\lim_{n\to\infty}}\|Tx_n\| \ge \|\pi(T)\| - \varepsilon.$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \|Tx_{n_k}\| = \overline{\lim_{n\to\infty}} \|Tx_n\| \geq \|\pi(T)\| - \varepsilon.$$

On the other hand, we have

$$\|\widetilde{T}\| = \sup\Big\{ \big(\text{l.i.m.}\|Tx_n\|^2\big)^{\frac{1}{2}} : \text{l.i.m.}\|x_n\|^2 = 1,\ x_n \to 0 \text{ weakly}\Big\}.$$

As $\text{l.i.m.}\|x_{n_k}\|^2 = 1$ and $x_{n_k} \to 0$ weakly, from the preceding identity we can write

$$\|\widetilde{T}\| \geq \lim_{k\to\infty} \|Tx_{n_k}\| \geq \|\pi(T)\| - \varepsilon.$$

Since ε was arbitrary, we obtain the required inequality. ■

Let T be an essentially isometric contraction such that $D\setminus\sigma(T) \neq \varnothing$. In [5, Theorem 2.1] it was proved that if $f \in A(D)$ vanishes on $\sigma(T) \cap \Gamma$, then $f(T)$ is compact. Notice that this result is an immediate consequence of Proposition 3.4.

Now we provide the proof of Theorem 3.1.

Proof of Theorem 3.1 Let $S \in A_T$. For every $\xi \in \sigma_{le}(T)$, there exists a multiplicative functional ϕ_ξ on A_T such that $\phi_\xi(T) = \xi$. Then we have

$$|\widehat{S}(\xi)| = |\xi^n \widehat{S}(\xi)| = |\phi_\xi(T^n S)| \leq \|T^n S\| \ (n \in \mathbb{N}).$$

It follows that

$$\lim_{n\to\infty} \|T^n S\| \geq \sup_{\xi\in\sigma_{le}(T)} |\widehat{S}(\xi)|.$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Then there exists $f \in A(D)$ such that $\|S - f(T)\| \leq \varepsilon$. This implies

$$\|T^n S\| \leq \|T^n f(T)\| + \varepsilon \ (n \in \mathbb{N}) \tag{3.1}$$

and

$$\sup_{\xi\in\sigma_{le}(T)} |f(\xi)| \leq \sup_{\xi\in\sigma_{le}(T)} |\widehat{S}(\xi)| + \varepsilon. \tag{3.2}$$

Let $\widetilde{T}$ be the limit operator associated with T. It follows from Proposition 3.4(e), (c) and Proposition 3.3(b) that $\widetilde{T}$ is an isometry, and $\sigma(\widetilde{T}) \cap \Gamma \subset \sigma_{le}(T)$. If $\widetilde{T}$ is nonunitary, then we have $\sigma(\widetilde{T}) \cap \Gamma = \Gamma$, and so $\Gamma = \sigma_{le}(T) = \sigma(T) \cap \Gamma$. As we already mentioned above, in that case the mapping $f \mapsto f(T)$ from $A(D)$ into A_T is an isometry and therefore, there is nothing to prove. Consequently, we may assume that $\widetilde{T}$ is unitary. We also have $\sigma(\widetilde{T}) \subset \sigma_{le}(T)$. Now, taking into account Proposition 3.4(g) and (d), we can write

$$\|\pi(f(T))\| \leq \|\widetilde{f(T)}\| = \|f(\widetilde{T})\| = \sup_{\xi\in\sigma(\widetilde{T})} |f(\xi)| \leq \sup_{\xi\in\sigma_{le}(T)} |f(\xi)|.$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

Therefore, there exists $K_\varepsilon \in K(H)$ such that

$$\|f(T) + K_\varepsilon\| \le \sup_{\xi \in \sigma_{le}(T)} |f(\xi)| + \varepsilon.$$

It follows that

$$\|T^n f(T) + T^n K_\varepsilon\| \le \sup_{\xi \in \sigma_{le}(T)} |f(\xi)| + \varepsilon$$

and

$$\|T^n f(T) + K_\varepsilon T^n\| \le \sup_{\xi \in \sigma_{le}(T)} |f(\xi)| + \varepsilon \ (n \in \mathbb{N}).$$

Since $T \in C_{0\cdot} \cup C_{\cdot 0}$, by Proposition 3.2(b) and (c), either

$$\lim_{n\to\infty} \|T^n K_\varepsilon\| = 0 \text{ or } \lim_{n\to\infty} \|K_\varepsilon T^n\| = 0.$$

Letting $n \to \infty$ in the preceding inequalities, we get

$$\lim_{n\to\infty} \|T^n f(T)\| \le \sup_{\xi \in \sigma_{le}(T)} |f(\xi)| + \varepsilon. \tag{3.3}$$

Now, taking into account (3.1), (3.3), and (3.2), we can write

$$\begin{aligned}\lim_{n\to\infty} \|T^n S\| &\le \lim_{n\to\infty} \|T^n f(T)\| + \varepsilon \le \sup_{\xi \in \sigma_{le}(T)} |f(\xi)| + 2\varepsilon \\ &\le \sup_{\xi \in \sigma_{le}(T)} |\widehat{S}(\xi)| + 3\varepsilon.\end{aligned}$$

Since ε was arbitrary, we obtain that

$$\lim_{n\to\infty} \|T^n S\| \le \sup_{\xi \in \sigma_{le}(T)} |\widehat{S}(\xi)|.$$

Replacing $\sigma_{le}(T)$ by $\sigma_{re}(T) \cap \Gamma$ in the above proof, we can see that

$$\lim_{n\to\infty} \|T^n S\| = \sup_{\xi \in \sigma_{re}(T)\cap\Gamma} |\widehat{S}(\xi)|. \qquad \blacksquare$$

We have the following result as a corollary.

Corollary 3.5 *If T is an essentially isometric contraction of class $C_{0\cdot} \cup C_{\cdot 0}$, then*

$$\sigma_{le}(T) = \sigma_{re}(T) \cap \Gamma = \sigma(T) \cap \Gamma.$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

Proof In [8], it was proved that if T is an arbitrary contraction on a Hilbert space, then for every $S \in A_T$,

$$\lim_{n\to\infty} \|T^n S\| = \sup_{\xi\in\sigma(T)\cap\Gamma} |\widehat{S}(\xi)|.$$

From this and from Theorem 3.1 we can write

$$\sup_{\xi\in\sigma_{le}(T)} |\widehat{S}(\xi)| = \sup_{\xi\in\sigma_{re}(T)\cap\Gamma} |\widehat{S}(\xi)| = \sup_{\xi\in\sigma(T)\cap\Gamma} |\widehat{S}(\xi)| \ (S \in A_T).$$

It remains to show that if Q_1 and Q_2 are two closed subsets of Γ and if for every $f \in A(D)$,

$$\sup_{\xi\in Q_1} |f(\xi)| = \sup_{\xi\in Q_2} |f(\xi)|,$$

then $Q_1 = Q_2$. For this, it is enough to show that $Q_1 \subset Q_2$. Assume that there exists $\xi_0 \in Q_1$, but $\xi_0 \notin Q_2$. If we take the function $f(z) = \frac{1}{2}(1 + \overline{\xi_0} z)$, then it is easy to see that $\sup_{\xi\in Q_1} |f(\xi)| = 1$, but $\sup_{\xi\in Q_2} |f(\xi)| < 1$. ■

The next result is an immediate consequence of Theorem 2.1 and Corollary 3.5.

Corollary 3.6 *Let T be an essentially isometric contraction such that $D\setminus\sigma(T) \neq \varnothing$. If T is in the class $C_{0\cdot} \cup C_{\cdot 0}$, then for every $f \in A(D)$,*

$$\sigma_e\big(f(T)\big) = f\big(\sigma(T) \cap \Gamma\big).$$

4 C_1-contractions

Let T be an essentially normal operator. In this section, we investigate the problem when T turns out to be an essentially unitary operator in terms of some metric conditions about T. For C_1-contraction T, we provide some sufficient conditions to have the equality $\sigma_e(T) = \sigma(T)$. Recall that a contraction T on H is said to be a C_1-*contraction* if $\inf_n \|T^n x\| > 0$ for every $x \in H \setminus \{0\}$.

The following result is of independent interest.

Proposition 4.1 *Let T be a C_1-contraction on H such that $\sigma(T) \neq \overline{D}$. If T is normal, then it is unitary.*

Proof As in the proof of [9, Proposition II.5.3],

$$\lim_{n\to\infty} \langle T^n x, T^n y\rangle \ (x, y \in H)$$

(by the polarization identity, this limit exists) defines a sesquilinear form on H. Therefore, there exists $Y \in B(H)$ such that

$$\lim_{n\to\infty} \langle T^n x, T^n y\rangle = \langle Y x, y\rangle.$$

It follows that

$$\langle Y x, x\rangle \geq \inf_n \|T^n x\|^2 > 0 \ \big(x \in H \setminus \{0\}\big).$$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

If we put $X = Y^{\frac{1}{2}}$, then X is a positive operator and $\|Xx\| = \|XTx\|$. Define the operator V on XH by $VXx = XTx$. Since $\|VXx\| = \|Xx\|$ and X has dense range, V can be extended to the whole H, which we also denote by V. Then V is an isometry and $VX = XT$, where

$$\|Xx\| = (\lim_{n\to\infty} \|T^n x\|^2)^{\frac{1}{2}}.$$

Let us show that $\sigma(V) \subset \sigma(T)$. Assume that $\lambda \notin \sigma(T)$. Define the operator W_λ on XH, by $W_\lambda Xx = XR_\lambda(T)x$ $(x \in H)$. Then we have

$$\begin{aligned}\|W_\lambda Xx\| &= \|XR_\lambda(T)x\| = \big(\lim_{n\to\infty} \|T^n R_\lambda(T)x\|^2\big)^{\frac{1}{2}} = \big(\lim_{n\to\infty} \|R_\lambda(T)T^n x\|^2\big)^{\frac{1}{2}}\\ &\leq \|R_\lambda(T)\|\big(\lim_{n\to\infty} \|T^n x\|^2\big)^{\frac{1}{2}} = \|R_\lambda(T)\|\|Xx\|.\end{aligned}$$

Since X has dense range, it follows that W_λ can be extended to the whole H, which we also denote by W_λ. Thus we have $W_\lambda X = XR_\lambda(T)$. Consequently, we can write

$$(\lambda I_H - V)W_\lambda X = (\lambda I_H - V)XR_\lambda(T) = X(\lambda I_H - T)R_\lambda(T) = X,$$

which implies $(\lambda I_H - V)W_\lambda = I_H$. Similarly, one can see that $W_\lambda(\lambda I_H - V) = I_H$. Thus, $\lambda \notin \sigma(V)$.

Since $\sigma(T) \neq \overline{D}$, we have $\sigma(V) \neq \overline{D}$, and therefore V is unitary. Now let us show that $T = V$. By the Fuglede–Putnam Theorem, $V^*X = XT^*$, which implies $XV = TX$. Hence, we have

$$VX^2 = (VX)X = (XT)X = X(TX) = X(XV) = X^2V.$$

Consequently, for every polynomial P, we can write $VP(X^2) = P(X^2)V$. Further, there exists a sequence of polynomials $\{P_n\}$ such that $P_n(t) \to \sqrt{t}$ uniformly on $\big[0, \|Y\|\big]$. As $n \to \infty$, from the identities

$$VP_n(X^2) = P_n(X^2)V,$$

we get $VX = XV$. Thus, we have $XT = XV$. Since X has zero kernel, finally we obtain $T = V$. ■

Corollary 4.2 *Let T be an essentially normal contraction on H such that $\sigma_e(T) \neq \overline{D}$. If*

$$\inf_n \inf_{\{x_k\}} \varliminf_{k\to\infty} \big\{ \|T^n x_k\| : \|x_k\| = 1,\ x_k \to 0 \textit{ weakly}\big\} > 0,$$

then T is essentially unitary.

Proof Let $\widetilde{T}$ be the limit operator associated with T. By Proposition 3.4(e) and (c), $\widetilde{T}$ is normal and $\sigma(\widetilde{T}) \subset \sigma_e(T)$. So, we have $\sigma(\widetilde{T}) \neq \overline{D}$. On the other hand, the above condition shows that $\widetilde{T}$ is a C_1-contraction. Now it follows from the preceding proposition that $\widetilde{T}$ is unitary. This means that T is essentially unitary. ■

We conclude the paper with the following result.

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

Theorem 4.3 *Let T be a c.n.u. C_1-contraction on H such that T is invertible and*

$$\sum_{n=1}^{\infty} \frac{\log \|T^{-n}\|}{1+n^2} < \infty.$$

Then $\sigma(T) = \sigma_e(T)$.

For the proof we need some preliminary facts.

Let $\omega = \{\omega_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers with $\omega_n \geq 1$ and $\omega_{n+m} \leq \omega_n\omega_m$, for all $n, m \in \mathbb{Z}$. We say then that ω is a *weight* on $\mathbb{Z}$. The *Beurling algebra* $A_\omega(\Gamma)$ is the set of all $f \in C(\Gamma)$ for which

$$\|f\|_\omega = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\omega_n < \infty,$$

where $\widehat{f}(n)$ is the n-th Fourier coefficient of f. As is well known, $A_\omega(\Gamma)$ is a commutative Banach algebra with respect to the pointwise multiplication. If

$$\sum_{n\in\mathbb{Z}} \frac{\log \omega_n}{1+n^2} < \infty,$$

then ω is called a *nonquasianalytic* weight. If ω is a nonquasianalytic weight, then the structure space of $A_\omega(\Gamma)$ can be identified with Γ. Moreover, the algebra $A_\omega(\Gamma)$ is (Shilov) regular [2, Theorem XII.5.1].

Let T be an invertible operator on H. We denote by $A_{T,T^{-1}}$, the closure in the uniform operator topology of all trigonometric polynomials in T and T^{-1}. We call T an *ω-nonquasianalytic operator*if there exists a nonquasianalytic weight ω on $\mathbb{Z}$ such that

$$\|T^n\| = O(\omega_n) \ (n \in \mathbb{Z}).$$

If T is an ω-nonquasianalytic operator, then for every $f \in A_\omega(\Gamma)$, we can define $f(T) \in B(H)$ by

$$f(T) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)T^n.$$

Then the mapping $f \mapsto f(T)$ of $A_\omega(\Gamma)$ into $A_{T,T^{-1}}$ is a continuous algebra-homomorphism with its dense range. The standard Banach algebra techniques involves that the structure space of $A_{T,T^{-1}}$ can be identified with the hull of the closed ideal

$$I_T := \left\{ f \in A_\omega(\Gamma) : f(T) = 0 \right\}.$$

It follows that $\sigma(T) \subset \Gamma$. Moreover, the spectral mapping property $\sigma(f(T)) = f(\sigma(T))$ holds.

Proof of Theorem 3.1 Let $(\widetilde{H}, \widetilde{T})$ be the limit operator associated with T. In view of Proposition 3.4(c), $\sigma(\widetilde{T}) \subset \sigma_e(T) \subset \sigma(T)$. Hence, it suffices to show that $\sigma(T) \subset$

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

$\sigma(\widetilde{T})$. Since T is a c.n.u. contraction, $T^n \to 0$ in the weak operator topology. Consequently, we can define the linear operator $J\colon H \to \widetilde{H}$ by

$$Jx = \{T^n x\} + E \ (x \in H).$$

(recall that E consists of all weakly null sequences $\{x_n\}$ in H such that $\text{l.i.m.}\|x_n\|^2 = 0$), where

$$\|Jx\| = \big(\lim_{n\to\infty} \|T^n x\|^2\big)^{\frac{1}{2}}.$$

Moreover,

$$JTx = \{T^{n+1}x\} + E = \widetilde{T}\big(\{T^n x\} + E\big) = \widetilde{T}Jx.$$

So, we have

$$JT = \widetilde{T}J. \tag{4.1}$$

Notice that as T is a C_1-contraction, J is injective.

Let us define the weight $\omega = (\omega_n)_{n\in\mathbb{Z}}$ on $\mathbb{Z}$ by

$$\omega_n = \begin{cases} 1, & n \geq 0; \\ \|T^n\|, & n < 0. \end{cases}$$

Then T is an ω-nonquasianalytic operator. As the mapping $T \mapsto \widetilde{T}$ is a contractive algebra-homomorphism, $\widetilde{T}$ is also an ω-nonquasianalytic operator. Now assume on the contrary that there exists $\xi_0 \in \sigma(T)$, but $\xi_0 \notin \sigma(\widetilde{T})$. Let O be an open set such that $\sigma(\widetilde{T}) \subset O$ and $\xi_0 \notin \overline{O}$. In view of the regularity of the algebra $A_\omega(\Gamma)$, there exists $f \in A_\omega(\Gamma)$ such that f vanishes on O, but $f(\xi_0) \neq 0$. Let $g \in A_\omega(\Gamma)$ be such that $g(\xi) = 1$ for all $\xi \in \sigma(\widetilde{T})$ and $g(\xi) = 0$ outside O. Since $fg = 0$, we have $f(\widetilde{T})g(\widetilde{T}) = 0$. By the spectral mapping property, $g(\widetilde{T})$ is invertible and therefore $f(\widetilde{T}) = 0$. On the other hand, from the identity (4.1), we can write $Jf(T) = f(\widetilde{T})J$, which implies that $Jf(T) = 0$. Since J is injective, we obtain $f(T) = 0$. If $\xi \in \sigma(T)$, then there exists a multiplicative functional ϕ_ξ on $A_{T,T^{-1}}$ such that $\phi_\xi(T) = \xi$ and $\phi_\xi(T^{-1}) = \xi^{-1}$. Consequently, we have $f(\xi) = \phi_\xi(f(T)) = 0$. It follows that f vanishes on $\sigma(T)$. This contradicts $f(\xi_0) \neq 0$. ■

Acknowledgments The author is grateful to the referee for helpful remarks.

References

[1] W. Arveson, *A short course on spectral theory.* Graduate Texts in Mathematics, 209, Springer-Verlag, New York, 2002.

[2] B. Beauzamy, *Introduction to operator theory and invariant subspaces.* North-Holland Mathematical Library, 42, North Holland, Amsterdam, 1988.

[3] J. B. Conway, *A course in functional analysis.* Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1985.

[4] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear non-selfadjoint operators in Hilbert space.* (Russian), Nauka, Moscow, 1965.

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press

[5] K. Kellay and M. Zarrabi, *Compact operators that commute with a contraction.* Integral Equations Operator Theory **65**(2009), no. 4, 543–550. http://dx.doi.org/10.1007/s00020-009-1724-8
[6] R. Larsen, *Banach algebras. An introduction.* Pure and Appled Mathematics, 24, Marcel Dekker Inc., New York, 1973.
[7] P. S. Muhly, *Compact operators in the commutant of a contraction.* J. Functional Analysis **8**(1971), 197–224. http://dx.doi.org/10.1016/0022-1236(71)90010-3
[8] H. S. Mustafayev, *Asymptotic behavior of polynomially bounded operators.* C. R. Acad. Sci. Paris, Ser. I **348**(2010), no. 9–10, 517–520. http://dx.doi.org/10.1016/j.crma.2010.04.003
[9] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space.* (Russian), Mir, Moscow, 1970.
[10] M. A. Naǐmark, *Normed rings.* (Russian) second ed., Nauka, Moscow, 1968.
[11] N. K. Nikol'skiǐ, *Lectures on the shift operator.* (Russian), Nauka, Moscow, 1980.

Yuzuncu Yıl University, Faculty of Science, Department of Mathematics, 65080, VAN-TURKEY
e-mail: hsmustafayev@yahoo.com

https://doi.org/10.4153/CMB-2012-016-1 Published online by Cambridge University Press