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EFFECTIVE BOUND FOR SINGULARITIES ON TORIC FIBRATIONS

BINGYI CHEN

Abstract. It was conjectured by McKernan and Shokurov that for any Fano contraction $f: X \to Z$ of relative dimension r with X being ϵ-lc, there is a positive δ depending only on r, ϵ such that Z is δ-lc and the multiplicity of the fiber of f over a codimension one point of Z is bounded from above by $1/\delta$. Recently, this conjecture was confirmed by Birkar [9]. In this article, we give an explicit value for δ in terms of ϵ, r in the toric case, which belongs to $O(\epsilon^{2^r})$ as $\epsilon \to 0$. The order $O(\epsilon^{2^r})$ is optimal in some sense.

§1. Introduction.

We work over an algebraically closed field k of characteristic zero. Given a contraction $f: X \to Z$, that is, a projective morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Z$, a fundamental problem is to relate the singularities on X and those on Z. This problem is important as it appears frequently in inductive arguments. Assume that f is a Fano contraction, McKernan conjectured that in this case the singularities on Z are bounded in terms of those on X.

CONJECTURE 1.1 (McKernan). *Fix a positive integer r and a real number $0 < \epsilon \leq 1$. There exists $\delta > 0$ depending only on r, ϵ and satisfying the following. Assume:*

- *$f: X \to Z$ is a contraction of relative dimension r;*
- *X is ϵ-lc;*
- *$-K_X$ is ample over Z; and*
- *Z is $\mathbb{Q}$-Gorenstein.*

Then, Z is δ-lc.

Recently, this conjecture was solved by Birkar [9]. Indeed, he proved a more general conjecture—Shokurov conjecture (see Conjecture 1.7 below), which implies McKernan conjecture. Another interesting consequence of Shokurov conjecture is that under the setting of Conjecture 1.1, the multiplicity of the fiber of f over a codimension one point of Z is bounded above. For more historical results on these two conjectures, we refer to [1], [5], [6], [10], [11], [14], [19], [26], [27].

The next problem is to give an explicit value for δ in terms of r, ϵ. When $r = 1$ and $\epsilon = 1$, Han, Jiang, and Luo [19] showed that the optimal value of δ is $1/2$. When $r = 1$, in [14] the author showed that one can take $\delta = \epsilon^2/2$. The main purpose of this article is to treat the toric case for arbitrary r, ϵ. Our main result is the following.

THEOREM 1.2. *Let r be a positive integer and $0 < \epsilon \leq 1$ be a real number. Let $f: X \to Z$ be a toric contraction of relative dimension r such that $-K_X$ is ample over Z and X*

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is ϵ-lc vertically over Z, that is, $a(E,X,0) \geq \epsilon$ for any prime divisor E over X with $f(\mathrm{center}_X E) \neq Z$. Let

$$\delta = \delta(r,\epsilon) = \frac{\epsilon^{2^r}}{2^{2^r-1}\prod_{i=1}^{r} i^{2^i}}. \tag{1.1}$$

Then:

(1) *if Z is $\mathbb{Q}$-Gorenstein, then Z is δ-lc;*

(2) *for any codimension one point z of Z, the multiplicity of each component of f^*z is bounded from above by $1/\delta$.*

REMARK 1.3. (1) Comparing with Conjecture 1.1, in Theorem 1.2, we require a weaker condition that X is ϵ-lc vertically over Z instead of the original condition that X is ϵ-lc. Note that under the original condition the general fiber F of f is an ϵ-lc Fano variety, so it belongs to a bounded family by [7, 8]. However, under the new condition, the general fiber may not belong to a bounded family.

(2) For the first assertion in Theorem 1.2, when $r=2$, the order $O(\epsilon^4)$ is optimal. Indeed, Alexeev and Borisov [1, Theorem 1.5] constructed a sequence of toric Fano contractions $X \to Z$ such that $\dim X = 4$, $\dim Z = 2$, $\mathrm{mld}(X) \to 0$ and $\mathrm{mld}(Z) \approx C \cdot \mathrm{mld}(X)^4$.

(3) For the second assertion in Theorem 1.2, the order $O(\epsilon^{2^r})$ is optimal by the following example.

EXAMPLE 1.4. Let q, r be two positive integers. Let $u_{i,q}$ $(i \in \mathbb{Z}_{>0})$ be a sequence of integers defined recursively as follows:

$$u_{1,q} = q, \quad u_{k+1,q} = u_{k,q}(u_{k,q}+1) \text{ for any } k \in \mathbb{Z}_{>0}.$$

Then, $u_{r+1,q} \in O(q^{2^r})$ when $q \to +\infty$.

Let $e_1, \ldots, e_{r+1}$ be the standard basis of $\mathbb{Z}^{r+1}$ and denote $e = \sum_{i=1}^{r} e_i$. Let

$$v_i = (1+u_{i,q})e_1 - qe \text{ for } 1 \leq i \leq r,$$
$$v_{r+1} = -e, \quad v_{r+2} = (u_{r+1,q}-1)e_{r+1} - qe.$$

Let X be the toric variety associated to the fan in $\mathbb{R}^{r+1}$ whose maximal cones are generated by v_{r+2} and subsets of $\{v_1, \ldots, v_{r+1}\}$ of size r. The support of the fan of X is $\mathbb{R}^r \times \mathbb{R}_{\geq 0}$. The projection $\mathbb{Z}^{r+1} \to \mathbb{Z}$ onto the $(r+1)$th coordinate induces a toric morphism $f: X \to Z$, where $Z = \mathbb{A}^1$ with a distinguished point o. Then, $f: X \to Z$ is a toric Fano contraction of relative dimension r. Moreover,

$$f^*o = (u_{r+1,q}-1) \cdot D,$$

where D is the toric divisor on X corresponding to the ray $\mathbb{R}_{\geq 0} \cdot v_{r+2}$.

Let S be the lattice simplex in $\mathbb{R}^{r+1}$ with vertices $v_1, \ldots, v_{r+2}$. Let F be the face of S which is the intersection of S and the subspace spanned by $e_1, \ldots, e_r$. Then, X is $\frac{1}{q}$-lc if and only if

$$\mathrm{int}(\frac{1}{q}S) \cap \mathbb{Z}^{r+1} = \emptyset \quad \text{and} \quad \mathrm{relint}(\frac{1}{q}F) \cap \mathbb{Z}^{r+1} = \{\mathbf{0}\}.$$

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This condition is satisfied because S is contained in the lattice simplex S' with vertices

$$v_i \ (1 \le i \le r), \ u_{r+1,q}e_{r+1} - qe, \ -u_{r+1,q}e_{r+1} - qe$$

and $\mathrm{int}(\frac{1}{q}S') \cap \mathbb{Z}^{r+1} = \{\mathbf{0}\}$ by [30, Theorem 1.3]. Therefore, X is $\frac{1}{q}$-lc.

The following is a local and more general version of Theorem 1.2.

THEOREM 1.5. *Let r be a positive integer, $0 < \epsilon \le 1$ be a real number and $\delta = \delta(r,\epsilon)$ as in (1.1). Let $f : X \to Z$ be a toric contraction of relative dimension r and let $z \in Z$ be a codimension ≥ 1 point. Suppose there is a pair (X,B) on X such that $K_X + B \sim_{\mathbb{R}} 0/Z$ and* $\mathrm{mld}(X/Z \ni z, B) \ge \epsilon$. *Then:*

(1) *if Z is $\mathbb{Q}$-Gorenstein, then* $\mathrm{mld}(Z \ni z, 0) \ge \delta$;

(2) *if the codimension of z in Z is one, then the multiplicity of each component of f^*z is bounded from above by $1/\delta$.*

Here, we denote by $\mathrm{mld}(X/Z \ni z, B)$ (resp. $\mathrm{mld}(Z \ni z, 0)$) the infimum of the log discrepancy of E with respect to (X,B) (resp. $(Z,0)$), where E runs over all prime divisors over X (resp. Z) whose image on Z is the closure $\overline{z}$ of z (see Definition 2.3).

REMARK 1.6. Notice that the assumption "$\mathrm{mld}(X/Z \ni z, B) \ge \epsilon$" is weaker than "$X$ is ϵ-lc over some neighborhood of z" since the former does not put restriction on the log discrepancy of such prime divisor whose image on Z is not $\overline{z}$ but contains z.

As mentioned earlier, Shokurov proposed a more general conjecture which implies McKernan conjecture. In order to state Shokurov conjecture, we recall some background on adjunction for fibrations (also called the canonical bundle formula). Let $f : X \to Z$ be a contraction between normal varieties. Let (X,B) be a pair which is lc over the generic point of Z and such that $K_X + B \sim_{\mathbb{R}} 0/Z$. By the work of Kawamata [22], [23] and Ambro [2], [3], we may write

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z),$$

where B_Z is called the discriminant divisor and M_Z is called the moduli divisor. The discriminant divisor is defined by lc thresholds, more precisely, the coefficient of a prime divisor D in B_Z is set to be $1-t$, where t is the largest number such that $(X, B + tf^*D)$ is lc over the generic point of D. The moduli divisor is then automatically determined up to $\mathbb{R}$-linear equivalence.

For each birational model Z' over Z, similarly we can define $B_{Z'}, M_{Z'}$ so that their pushdowns on Z coincide with B_Z, M_Z. This defines a discriminant ***b***-divisor $\mathbf{B}$ and a moduli ***b***-divisor $\mathbf{M}$ over Z. It was shown that $\mathbf{M}$ is a ***b***-nef ***b***-divisor and we hence obtain a generalized pair $(Z, B_Z, \mathbf{M})$, which is called the generalized pair given by adjunction for $f : (X,B) \to Z$ (see §2.4 for more details). We are now ready to state Shokurov conjecture.

CONJECTURE 1.7 (Shokurov). *Fix a positive integer r and a real number $0 < \epsilon \le 1$. There exists $\delta > 0$ depending only on r, ϵ and satisfying the following. Let (X,B) be a pair and $f : X \to Z$ be a contraction such that:*

- $\dim X - \dim Z = r$;
- (X,B) *is ϵ-lc;*
- $K_X + B \sim_{\mathbb{R}} 0/Z$; *and*
- *X is of Fano type over Z, equivalently, $-K_X$ is big over Z.*

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Let $(Z, B_Z, \mathbf{M})$ *be the generalized pair given by adjunction for* $f : (X, B) \to Z$*. Then,* $(Z, B_Z, \mathbf{M})$ *is generalized* δ*-lc.*

As mentioned earlier, Shokurov conjecture was proved by Birkar [9] recently. Before this celebrated result, in [11, Theorem 1.4] Birkar and Chen showed a variant of Shokurov conjecture in the toric setting, which says that Shokurov conjecture holds after taking an average with the toric boundary. This is enough for some interesting applications. Building on ideas from their work and combining the main result in [14], we give an explicit value for δ in [11, Theorem 1.4] as follows.

THEOREM 1.8. *Let* r *be a positive integer and* $0 < \epsilon \leq 1$ *be a real number. Assume:*

- $f: X \to Z$ *is a toric contraction of relative dimension* r *with* $z \in Z$ *a codimension* ≥ 1 *point;*
- (X, B) *is a pair* (B *is not necessarily toric*) *such that* $\mathrm{mld}(X/Z \ni z, B) \geq \epsilon$*;*
- $K_X + B \sim_{\mathbb{R}} 0/Z$*; and*
- Δ *is the toric boundary divisor of* X*.*

Let

$$\Gamma^{\alpha} = \alpha B + (1-\alpha)\Delta, \qquad \text{where } \alpha = 1/r!$$

and let $(Z, \Gamma^{\alpha}_Z, \mathbf{N}^{\alpha})$ *be the generalized pair given by adjunction for* $f : (X, \Gamma^{\alpha}) \to Z$*. Then,*

$$\mathrm{mld}(Z \ni z, \Gamma^{\alpha}_Z, \mathbf{N}^{\alpha}) \geq \delta,$$

where $\delta = \delta(r, \epsilon)$ *as in (1.1).*

Theorems 1.2 and 1.5 are consequences of Theorem 1.8. Another interesting corollary is the following.

THEOREM 1.9. *Let* r *be a positive integer,* $0 < \epsilon \leq 1$ *be a real number and* $\delta = \delta(r, \epsilon)$ *as in (1.1). Let* $f : X \to Z$ *be a toric contraction of relative dimension* r *with a toric pair* (X, B) *and a codimension one point* $z \in Z$*. Suppose there is an* $\mathbb{R}$*-divisor* B^+*(not necessary toric) such that* $B^+ \geq B$*,* $K_X + B^+ \sim_{\mathbb{R}} 0/Z$*, and* $\mathrm{mld}(X/Z \ni z, B^+) \geq \epsilon$*. Then,* $(X, B + \delta f^*\overline{z})$ *is lc over some neighborhood of* $z \in Z$*.*

REMARK 1.10. After this work was completed, Ambro informed me that he also got some explicit lower bounds in the toric case. Let $f : X \to Z$ be a toric Fano contraction of relative dimension r with X being ϵ-lc. Let F be the general fiber and let γ be the α-invariant of F. There exists a sharp lower bound for γ just in terms of r and ϵ (cf. [4]). Ambro got explicit bounds for δ in terms of ϵ, r, and γ in Theorems 1.2, 1.5, and 1.9.

1.1 Idea of the proof of Theorem 1.8.

The proof is built on ideas from [11] with some modifications. In [11], by running toric minimal model program (MMP), they reduced the problem to the case for toric Mori fiber spaces. Then, they showed that after taking a finite cover and extracting a toric divisor, a $\mathbb{Q}$-factorial toric Mori fiber space can be factored as the composition of toric contractions of smaller relative dimension. Therefore, they can reduce the problem to the case for contractions of relative dimension one. However, after taking a finite cover and extracting a divisor, the pullback of $K_X + B$ may be a sub-pair rather than a pair, so it is necessary to take average $\Gamma^{\alpha} = \alpha B + (1-\alpha)\Delta$ with the toric boundary to make its pullback a pair. To guarantee that the singularities of (X,Γ^α) are not too bad, α can not be too small and hence it is important to control the order n of the finite cover and the log discrepancy of the extracted divisor. They showed the boundedness of the order n, however, it seems not easy to give an explicit bound for n, as it involves all possibilities of the fans corresponding to ϵ-lc toric Fano varieties up to the action of $GL_r(\mathbb{Z})$. In this article, we make some modifications to their method. We factor a toric Mori fiber space after extracting a toric divisor with log discrepancy $\le r$, without taking a finite cover (see Lemma 3.6). Recall that in relative dimension one, an explicit value for δ in Shokurov conjecture was given in [14]. Combining this result, we obtain an explicit value for δ in [11, Theorem 1.4].

§2. Preliminaries.

We will freely use the standard notations and definitions in [12], [24]. A *contraction* $f: X \to Z$ is a projective morphism of varieties with $f_*\mathcal{O}_X = \mathcal{O}_Z$. An *extremal contraction* is a contraction $f: X \to Z$ with the relative Picard number $\rho(X/Z) = 1$.

2.1 Fano type varieties.

Let $X \to Z$ be a contraction of normal varieties. We say X is of *Fano type* over Z if there is a klt pair (X,B) on X such that $-(K_X+B)$ is ample over Z.

We say $X \to Z$ is a *Mori fiber space* if $-K_X$ is ample over Z and the relative Picard number $\rho(X/Z) = 1$.

2.2 *b*-divisors.

Let X be a normal variety. A ***b**-divisor* $\mathbf{D}$ over X is a collection of $\mathbb{R}$-divisors $\mathbf{D}_Y$ for each birational model Y over X, such that $\sigma_*\mathbf{D}_{Y_1} = \mathbf{D}_{Y_2}$ for any birational morphism $\sigma: Y_1 \to Y_2/X$.

Let $\mathbf{D}$ be a ***b***-divisor over X and Y_0 be a birational model over X. We say $\mathbf{D}$ *descends* to Y_0 if $\mathbf{D}_{Y_0}$ is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor and $\mathbf{D}_Y = \sigma^*\mathbf{D}_{Y_0}$ for any birational morphism $\sigma: Y \to Y_0/X$.

Let $X \to U$ be a projective morphism. We say that a ***b***-divisor $\mathbf{D}$ over X is ***b**-nef*$/U$ (resp. ***b**-semiample*$/U$) if $\mathbf{D}$ descends to some birational model Y_0 and $\mathbf{D}_{Y_0}$ is nef$/U$ (resp. semiample$/U$).

We denote by $\mathbf{0}$ the ***b***-divisor $\mathbf{D}$ such that $\mathbf{D}_Y = 0$ for each birational model Y over X.

2.3 Generalized pairs.

We will follow the original definitions in [13] and adopt the notations in [20].

DEFINITION 2.1. A *generalized sub-pair* (*g-sub-pair* for short) $(X,B,\mathbf{M})/U$ consists of a normal variety X associated with a projective morphism $X \to U$, an $\mathbb{R}$-divisor B on X, and a ***b***-nef$/U$***b***-divisor $\mathbf{M}$ over X.

A g-sub-pair $(X,B,\mathbf{M})/U$ is called a *sub-pair* if $\mathbf{M} = \mathbf{0}$. In this case, we denote it by $(X,B)/U$ or (X,B).

A g-sub-pair $(X,B,\mathbf{M})/U$ is called a *generalized pair* (*g-pair* for short) if $B \ge 0$. A sub-pair (X,B) is called a *pair* if $B \ge 0$.

We may drop U when we emphasize the structures of $(X,B,\mathbf{M})$ that are independent of the choice of U, for example, the singularities of $(X,B,\mathbf{M})$.

DEFINITION 2.2. Let $(X,B,\mathbf{M})/U$ be a g-(sub-)pair and E be a prime divisor over X, that is, a prime divisor on a normal variety Y with a birational morphism $\pi: Y \to X$.

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The *center* of E on X is defined as the image of E on X under the morphism π and it is denoted by $\mathrm{center}_X E$. Write

$$K_Y + B_Y + \mathbf{M}_Y := \pi^*(K_X + B + \mathbf{M}_X).$$

Then, the *log discrepancy* of E with respect to $(X, B, \mathbf{M})$ is defined as $1 - \mathrm{mult}_E B_Y$ and it is denoted by $a(E, X, B, \mathbf{M})$, where $\mathrm{mult}_E B_Y$ is the coefficient of E in B_Y.

DEFINITION 2.3. Let $(X, B, \mathbf{M})/U$ be a g-(sub-)-pair, $f: X \to Z/U$ be a projective morphism and $z \in Z$ be a (not necessary closed) point. The *minimal log discrepancy* of $(X, B, \mathbf{M})$ over z is defined as

$$\begin{aligned}\mathrm{mld}(X/Z \ni z, B, \mathbf{M}) := \inf\{a(E, X, B, \mathbf{M}) \mid\ & E \text{ is a prime divisor over } X \\ & \text{with } f(\mathrm{center}_X(E)) = \overline{z}\}.\end{aligned}$$

In the case, that $Z = X$, $z = x$ and f is the identity morphism, we will use $\mathrm{mld}(X \ni x, B, \mathbf{M})$ instead of $\mathrm{mld}(X/Z \ni z, B, \mathbf{M})$.

DEFINITION 2.4. A g-(sub-)pair $(X, B, \mathbf{M})$ is said to be *(sub-)ϵ-glc* (resp. *(sub-)ϵ-gklt*, *(sub-)glc*, *(sub-)gklt*) if $\mathrm{mld}(X \ni x, B, \mathbf{M}) \geq \epsilon$ (resp. $> \epsilon$, ≥ 0, > 0) for any codimension ≥ 1, point $x \in X$.

If $\mathbf{M} = \mathbf{0}$ and $(X, B, \mathbf{M})$ is (sub-)ϵ-glc (resp. (sub-)ϵ-gklt, (sub-)glc, (sub-)gklt), we say that (X, B) is *(sub-)ϵ-lc* (resp. *(sub-)ϵ-klt*, *(sub-)lc*, *(sub-)klt*). In the case when $B = 0$, we also say X is *ϵ-lc* (resp. *ϵ-klt*, *lc*, *klt*).

DEFINITION 2.5. Let $(X, B, \mathbf{M})/U$ be a g-(sub-)pair and D be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. The *lc threshold* of D with respect to $(X, B, \mathbf{M})$ is defined to be

$$\mathrm{lct}(X, B, \mathbf{M}; D) := \sup\{t \geq 0 \mid (X, B + tD, \mathbf{M}) \text{ is (sub-)glc}\}.$$

DEFINITION 2.6. Let $(X, B, \mathbf{M})/U$ and $(X, \Gamma, \mathbf{N})/U$ be two g-(sub-)pairs. We say $(X, B, \mathbf{M})$ has *better singularities* than $(X, \Gamma, \mathbf{N})$ if

$$a(E, X, B, \mathbf{M}) \geq a(E, X, \Gamma, \mathbf{N})$$

for any prime divisor E over X.

LEMMA 2.7. *Let $(X, B, \mathbf{M})/U$ be a g-(sub−)-pair, $f: X \to Z/U$ be a projective morphism and $z \in Z$ be a (not necessary closed) point. Then, $\mathrm{mld}(X/Z \ni z, B, \mathbf{M}) \geq 0$ if and only if $(X, B, \mathbf{M})$ is (sub−)glc over some neighborhood of $z \in Z$.*

Proof. This is essentially [19, Lemma 2.8] where it was stated only for $\mathbf{M} = 0$. By definition, the "if" part is obvious. Next, we show the "only if" part.

Assume the contrary that $\mathrm{mld}(X/Z \ni z, B, \mathbf{M}) \geq 0$ but $(X, B, \mathbf{M})$ is not (sub-)glc over any neighborhood of $z \in Z$. Then, there is a prime divisor E over X such that $z \in f(\mathrm{center}_X E)$ and $a(E, X, B, \mathbf{M}) < 0$. Let $\pi: Y \to X$ be a resolution with $K_Y + B_Y + \mathbf{M}_Y = \pi^*(K_X + B + \mathbf{M}_X)$ such that:

- $\mathbf{M}$ descends to Y;
- E is a prime divisor on Y;
- $\overline{\pi^{-1} f^{-1}(z)}$ is a divisor on Y, say F; and
- $E + F$ is a simple normal crossing divisor on Y.

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We can find an irreducible component D of F such that $f(\pi(D\cap E))=\overline{z}$ (indeed, since $z\in f(\pi(E))$, there is a point $\eta\in E$ such that $f(\pi(\eta))=z$, then, we take D to be a component of F which contains η).

Denote $d=\operatorname{mult}_D B_Y$ and $e=\operatorname{mult}_E B_Y>1$. Blowing up $D\cap E$, we get a new resolution $\pi':Y'\to X$ with $K_{Y'}+B_{Y'}+\mathbf{M}_{Y'}=\pi'^*(K_X+B+\mathbf{M}_X)$. Denote by D' the exceptional/Y divisor on Y' and by E' the birational transformation of E on Y'. By construction, we have $f(\pi'(D'))=\overline{z}$, D' meets E' transversely, $f(\pi'(D'\cap E'))=\overline{z}$ and $\operatorname{mult}_{D'}B_{Y'}\geq d+e-1>d$.

So, by successively blowing up, we eventually obtain a prime divisor $\widetilde{D}$ over X such that $f(\operatorname{center}_X\widetilde{D})=\overline{z}$ and $a(\widetilde{D},X,B,\mathbf{M})<0$, which contradicts that $\operatorname{mld}(X/Z\ni z, B,\mathbf{M})\geq 0$. □

2.4 Adjunction for generalized fibrations.

Let $f:X\to Z$ be a contraction between normal varieties over U with $\dim Z>0$. Let $(X,B,\mathbf{M})/U$ be a g-pair which is glc over the generic point of Z and such that $K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}0/Z$. Then, $K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}f^*L$ for some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L on Z.

For any prime divisor D on Z, let t_D be the lc threshold of f^*D with respect to $(X,B,\mathbf{M})$ over the generic point of D. This make sense even if D is not $\mathbb{Q}$-Cartier because we only need the pullback of D over the generic point of D, where Z is smooth. We set $B_Z=\sum_D(1-t_D)D$, where D runs over all prime divisors on Z and set $M_Z=L-K_Z-B_Z$. The former is called the *discriminant divisor* and the latter is called the *moduli divisor*.

Let $\sigma:Z'\to Z$ be a birational morphism from a normal variety Z' and let X' be the resolution of the main component of $X\times_Z Z'$ with induced morphism $\tau:X'\to X$ and $f':X'\to Z'$. Write $K_{X'}+B'+\mathbf{M}_{X'}=\tau^*(K_X+B+\mathbf{M}_X)$, then $K_{X'}+B'+\mathbf{M}_{X'}\sim_{\mathbb{R}}f'^*\sigma^*L$. Similarly, we can define the discriminant divisor $B_{Z'}$ and the moduli divisor $M_{Z'}$ for the contraction $(X',B',\mathbf{M})\to Z'$. One can check that $\sigma_*B_{Z'}=B_Z$ and $\sigma_*M_{Z'}=M_Z$. Hence, there exist $\boldsymbol{b}$-divisors $\mathbf{B}^Z,\mathbf{M}^Z$ such that $\mathbf{B}^Z_{Z'}=B_{Z'}$ and $\mathbf{M}^Z_{Z'}=M_{Z'}$ for any birational model Z' over Z, which are called the *discriminant $\boldsymbol{b}$-divisor* and the *moduli $\boldsymbol{b}$-divisor*, respectively. By construction, we have

$$K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}f^*(K_Z+B_Z+\mathbf{M}^Z_Z).$$

It was shown that $\mathbf{M}^Z$ is a $\boldsymbol{b}$-nef/$U$$\boldsymbol{b}$-divisor over Z (see [15, Theorem 11.4.4]). Hence, we can regard $(Z,B_Z,\mathbf{M}^Z)/U$ as a g-pair. We call $(Z,B_Z,\mathbf{M}^Z)/U$ the *g-pair given by adjunction* for $f:(X,B,\mathbf{M})\to Z$. In the case, that $(X,B,\mathbf{M})$ is glc, $(Z,B_Z,\mathbf{M}^Z)$ is also a glc g-pair.

For more details about adjunction for generalized fibrations, we refer the readers to [17], [21] and [15, Section 11.4].

LEMMA 2.8 [11, Lemma 2.1]. *Assume that:*

- *$(X,B,\mathbf{M})/U$ is a g-pair which is glc over the generic point of Z;*
- *$X\xrightarrow{g}Y\xrightarrow{h}Z$ are contractions of normal varieties/U with* $\dim Z>0$*; and*
- *$K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}0/Z$.*

Let $(Y,B_Y,\mathbf{M}^Y)/U$ be the g-pair given by adjunction for $g:(X,B,\mathbf{M})\to Y$ and let $(Z,B_Z,\mathbf{M}^Z)/U$ be the g-pair given by adjunction for $h\circ g:(X,B,\mathbf{M})\to Z$. Then, $(Z,B_Z,\mathbf{M}^Z)/U$ is also the g-pair given by adjunction for $h:(Y,B_Y,\mathbf{M}^Y)\to Z$.

LEMMA 2.9. *Let $f:X\to Z$ be a contraction between normal varieties over U. Let $(X,B,\mathbf{M})/U$ and $(X,\Gamma,\mathbf{N})/U$ be two g-pairs on X which are glc over the generic point of Z.*

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Assume that $K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}0/Z$ *and* $K_X+\Gamma+\mathbf{N}_X\sim_{\mathbb{R}}0/Z$. *Let* $(Z,B_Z,\mathbf{M}^Z)/U$ *and* $(Z,\Gamma_Z,\mathbf{N}^Z)/U$ *be the g-pairs given by adjunction for* $(X,B,\mathbf{M})$ *and* $(X,\Gamma,\mathbf{N})$ *over* Z, *respectively. Suppose that* $(X,B,\mathbf{M})$ *has better singularities than* $(X,\Gamma,\mathbf{N})$, *then* $(Z,B_Z,\mathbf{M}^Z)$ *has better singularities than* $(Z,\Gamma_Z,\mathbf{N}^Z)$ *(see Definition 2.6 for "better singularities").*

Proof. Let D be a prime divisor on some high resolution $Z'\to Z$. Let $\pi:X'\to X$ be a high enough resolution such that the induced map $f':X'\dashrightarrow Z'$ is a morphism. Write $K_{X'}+B'+\mathbf{M}_{X'}$ (resp. $K_{X'}+\Gamma'+\mathbf{N}_{X'}$) for the pullback of $K_X+B+\mathbf{M}_X$ (resp. $K_X+\Gamma+\mathbf{N}_X$). Denote by t (resp. s) the lc threshold of f'^*D with respect to $(X',B',\mathbf{M})$ (resp. $(X',\Gamma',\mathbf{N})$) over the generic point of D. It suffices to show $t\geq s$.

By construction, $(X',\Gamma'+sf'^*D,\mathbf{N})$ is sub-glc over the generic point of D. Since $(X,B,\mathbf{M})$ has better singularities than $(X,\Gamma,\mathbf{N})$, $(X',B'+sf'^*D,\mathbf{M})$ also has better singularities than $(X',\Gamma'+sf'^*D,\mathbf{N})$ and it hence is sub-glc over the generic point of D. Therefore, $t\geq s$. □

LEMMA 2.10. *Let* $f:X\to Z$ *be a contraction of normal varieties over* U. *Let* $(X,B,\mathbf{M})/U$ *and* $(X,\Gamma,\mathbf{N})/U$ *be two g-pairs on* X *which are glc over the generic point of* Z. *Assume that* $K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}0/Z$ *and* $K_X+\Gamma+\mathbf{N}_X\sim_{\mathbb{R}}0/Z$. *Let* $0\leq\alpha\leq1$ *be a real number and let*

$$\Omega=\alpha B+(1-\alpha)\Gamma \quad \textit{and} \quad \mathbf{L}=\alpha\mathbf{M}+(1-\alpha)\mathbf{N}.$$

Let $(Z,B_Z,\mathbf{M}^Z)/U$, $(Z,\Gamma_Z,\mathbf{N}^Z)/U$, *and* $(Z,\Omega_Z,\mathbf{L}^Z)/U$ *be the g-pairs given by adjunction for* $(X,B,\mathbf{M})$, $(X,\Gamma,\mathbf{N})$, *and* $(X,\Omega,\mathbf{L})$ *over* Z, *respectively. Then,* $(Z,\Omega_Z,\mathbf{L}^Z)$ *has better singularities than*

$$(Z,\alpha B_Z+(1-\alpha)\Gamma_Z,\alpha\mathbf{M}^Z+(1-\alpha)\mathbf{N}^Z). \tag{2.1}$$

See Definition 2.6 for "better singularities".

Proof. Let $Z'\to Z$ be any resolution and D be a prime divisor on Z'. Take a high enough resolution $X'\to X$ such that the induced map $h':X'\dashrightarrow Z'$ is a morphism. Let t (resp. s) be the lc threshold of h'^*D with respect to $(X',B',\mathbf{M})$ (resp. $(X',\Gamma',\mathbf{N})$) over the generic point of D, where $K_{X'}+B'+\mathbf{M}_{X'}$ (resp. $K_{X'}+\Gamma'+\mathbf{N}_{X'}$) is the pullback of $K_X+B+\mathbf{M}_X$ (resp. $K_X+\Gamma+\mathbf{N}_X$). By definition, the coefficient of D in $B_{Z'}$ (resp. $\Gamma_{Z'}$) is $1-t$ (resp. $1-s$), where $K_{Z'}+B_{Z'}+\mathbf{M}^Z_{Z'}$ (resp. $K_{Z'}+\Gamma_{Z'}+\mathbf{N}^Z_{Z'}$) is the pullback of $K_Z+B_Z+\mathbf{M}^Z_Z$ (resp. $K_Z+\Gamma_Z+\mathbf{M}^Z_Z$). Hence, $a(D,Z,B_Z,\mathbf{M}^Z)=t$ and $a(D,Z,\Gamma_Z,\mathbf{N}^Z)=s$. So the log discrepancy of D with respect to the g-pair (2.1) is $\alpha t+(1-\alpha)s$.

Now,

$$(X',\alpha B'+(1-\alpha)\Gamma'+\alpha th'^*D+(1-\alpha)sh'^*D,\alpha\mathbf{M}+(1-\alpha)\mathbf{N})$$

is glc over the generic point of D. Assuming that $K_{X'}+\Omega'+\mathbf{L}_{X'}$ is the pullback of $K_X+\Omega+\mathbf{L}_X$, we have $\Omega'=\alpha B'+(1-\alpha)\Gamma'$ and $\mathbf{L}=\alpha\mathbf{M}+(1-\alpha)\mathbf{N}$. Hence, the lc threshold of h'^*D with respect to $(X',\Omega',\mathbf{L})$ over the generic point of D is as least $\alpha t+(1-\alpha)s$. By definition, the coefficient of D in $\Omega_{Z'}$ is at most $1-\alpha t-(1-\alpha)s$, where $K_{Z'}+\Omega_{Z'}+\mathbf{L}^Z_{Z'}$ is the pullback of $K_Z+\Omega_Z+\mathbf{L}^Z_Z$. So

$$a(D,Z,\Omega_Z,\mathbf{L}^Z)\geq\alpha t+(1-\alpha)s$$

and the proof is completed. □

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2.5 Toric varieties and toric morphisms.

We refer to [18], [28] or [16] for the general theory of toric varieties. Here, we collect some definitions and facts on toric varieties. All toric varieties in this article are assumed to be normal.

A *toric variety* X is given by a pair (N_X, Σ_X), where N_X is a lattice and Σ_X is a rational polyhedral fan in $N_X \otimes \mathbb{R}$. A *toric morphism* between toric varieties X and Y is given by a $\mathbb{Z}$-linear map $\phi : N_X \to N_Y$ which is compatible with the fan Σ_X and Σ_Y, that is to say, for any cone $\sigma_1 \in \Sigma_X$, there is a cone $\sigma_2 \in \Sigma_Y$ such that $\phi_{\mathbb{R}}(\sigma_1) \subset \sigma_2$, where $\phi_{\mathbb{R}} : N_X \otimes \mathbb{R} \to N_Y \otimes \mathbb{R}$ is the extension of ϕ.

A *toric divisor* on a toric variety X is a divisor which is invariant under the torus action. We say a pair (X, B) is a *toric pair* if X is a toric variety and B is a toric $\mathbb{R}$-divisor.

There is a one-to-one correspondence between the cones σ in Σ_X and the torus orbits $O(\sigma)$ in X. The dimension of the cone σ is equal to the codimension of the orbit $O(\sigma)$ in X. In particular, a one-dimensional-cone σ, called a *ray*, corresponds to a toric prime divisor $\overline{O(\sigma)}$.

If Δ is the toric boundary divisor of a toric variety X, that is, Δ is the sum of all the toric prime divisors on X, then (X, Δ) is lc and $K_X + \Delta \sim 0$. Moreover, $a(D, X, \Delta) = 0$ for any toric prime divisor D over X.

A toric variety X is $\mathbb{Q}$-factorial if and only if the fan Σ_X is simplicial, that is, every cone in Σ_X is generated by a set of $\mathbb{R}$-linear independent vectors.

If a toric morphism $f : X \to Y$ given by $\phi : N_X \to N_Y$ is a contraction, then ϕ is surjective.

If $f : X \to Z$ is a toric contraction, then X is of Fano type over Z.

Every toric varieties is a Mori dream space, that is to say, if $X \to Z$ is a toric contraction, then we can run a MMP on any $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D relatively over Z and it terminates with either a D-negative fibre space or a D-minimal model. Moreover, all the steps of the MMP are toric (see [25, Chapter 14] for the $\mathbb{Q}$-factorial case.

LEMMA 2.11 [16, p. 133]. *Let X, Z be two toric varieties given by (N_X, Σ_X), (N_Z, Σ_Z), respectively, and $f : X \to Z$ be a toric morphism given by a surjective $\mathbb{Z}$-linear map $\phi : N_X \to N_Z$. Let F be a toric varieties given by (N_0, Σ_0), where $N_0 = \ker(\phi)$ and*

$$\Sigma_0 = \{\sigma \in \Sigma_X \mid \sigma \subset (N_0)_{\mathbb{R}}\}$$

is a sub-fan of Σ_X. Then, $f^{-1}(T_Z) \simeq F \times T_Z$, where T_Z is the torus in Z.

We also need the following lemma in [11] regarding adjunction for toric pairs.

LEMMA 2.12 [11, Lemma 2.11]. *Let $f : X \to Z$ be a toric contraction between toric varieties and Δ, Δ_Z be the toric boundary divisors of X, Z respectively. Then, $(Z, \Delta_Z, \mathbf{0})$ is the g-pair given by adjunction for $f : (X, \Delta) \to Z$.*

§3. Proofs of main results.

In this section, we will prove a more general form of Theorem 1.8 for generalized pairs as follows.

THEOREM 3.1 (cf. [11, Theorem 1.7]). *Let r be a positive integer and $0 < \epsilon \leq 1$ be a real number. Assume:*

(a) *$f : X \to Z$ is a toric contraction of relative dimension r between toric varieties over U with a codimension ≥ 1 point $z \in Z$;*

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(b) $(X,B,\mathbf{M})/U$ *is a g-pair (not necessarily toric) with* $\mathrm{mld}(X/Z\ni z,B,\mathbf{M})\geq\epsilon$*;*
(c) $K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}0/Z$*; and*
(d) Δ *is the toric boundary divisor of* X*.*

Let

$$\Gamma^\alpha=\alpha B+(1-\alpha)\Delta \quad \textit{and} \quad \mathbf{N}^\alpha=\alpha\mathbf{M}, \qquad \textit{where } \alpha=1/r!$$

and let $(Z,\Gamma^\alpha_Z,\mathbf{N}^{\alpha,Z})/U$ *be the g-pair on* Z *given by adjunction for* $f:(X,\Gamma^\alpha,\mathbf{N}^\alpha)\to Z$*. Then,*

$$\mathrm{mld}(Z\ni z,\Gamma^\alpha_Z,\mathbf{N}^{\alpha,Z})\geq\delta,$$

where $\delta=\delta(r,\epsilon)$ *as in (1.1).*

We start with showing a generalized version of [14, Theorem 1.4], that is, showing that one can take $\delta=\epsilon^2/2$ in a generalized version of Shokurov conjecture in relative dimension one. Its proof is similar to that of [11, Lemma 3.1].

Lemma 3.2 (cf. [11, Lemma 3.1]). *Let* $f:X\to Z$ *be a contraction between normal varieties over* U*,* $(X,B,\mathbf{M})/U$ *be a g-pair and* $z\in Z$ *be a codimension* ≥ 1 *point such that:*

- $\dim X-\dim Z=1$*;*
- $K_X+B+\mathbf{M}_X\sim_{\mathbb{R}}0/Z$*;*
- $\mathrm{mld}(X/Z\ni z,B,\mathbf{M})\geq\epsilon$*, where* $0<\epsilon\leq 1$*; and*
- X *is of Fano type over* Z*.*

Let $(Z,B_Z,\mathbf{M}^Z)/U$ *be the g-pair given by adjunction for* $f:(X,B,\mathbf{M})\to Z$*. Then,*

$$\mathrm{mld}(Z\ni z,B_Z,\mathbf{M}^Z)\geq\delta(1,\epsilon)=\epsilon^2/2.$$

Proof. Since the singularities of $(Z,B_Z,\mathbf{M}^Z)/U$ are independent of the choice of U, we may assume that $U=Z$. Shrinking Z around z, by Lemma 2.7, we may assume that $(X,B,\mathbf{M})$ is glc. Let D be a prime divisor on some high resolution $Z'\to Z$ with $\mathrm{center}_Z D=\overline{z}$. Let $\pi:X'\to X$ be a high enough resolution such that $\mathbf{M}$ descends to X' and the induced map $f':X'\dashrightarrow Z'$ is a morphism. Write $K_{X'}+B'+\mathbf{M}_{X'}$ for the pullback of $K_X+B+\mathbf{M}_X$. Then, (X',B') is sub-lc and $\mathrm{mld}(X'/Z\ni z,B')\geq\epsilon$. Denote by t the lc threshold of f'^*D with respect to (X',B') over the generic point of D. It suffices to show that t is bounded from below by $\epsilon^2/2$.

We may assume that X is $\mathbb{Q}$-factorial. Since X is of Fano type over Z, X is klt and $-K_X$ is big over Z. So we can write

$$\pi^*(-K_X)\sim_{\mathbb{Q}}H'+C'/Z,$$

where H' is ample over Z and $C'\geq 0$. We can also write $\pi^*K_X=K_{X'}+E'$. Then, $E'\leq B'$ and (X',E') is sub-klt. For each sufficiently small real number $u>0$, let

$$B'_u=(1-u)B'+uE',$$

then we have (X',B'_u) is sub-klt and $\mathrm{mld}(X'/Z\ni z,B'_u)\geq\epsilon$. So we can find a general

$$0\leq L'\sim_{\mathbb{R}}(1-u)\mathbf{M}_{X'}+uH'/Z$$

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(note that H' is ample/Z and $\mathbf{M}_{X'}$ is nef/Z) such that letting

$$\Delta' = B'_u + uC' + L',$$

we have $\mathrm{mld}(X'/Z \in z, \Delta') \geq \epsilon'$ where $\epsilon - \epsilon' > 0$ is sufficiently small. Moreover, over Z we have

$$\begin{aligned} K_{X'} + \Delta' &\sim_{\mathbb{R}} K_{X'} + (1-u)B' + uE' + uC' + (1-u)\mathbf{M}_{X'} + uH' \\ &= (1-u)(K_{X'} + B' + \mathbf{M}_{X'}) + u(K_{X'} + E') + u(H' + C') \\ &\sim_{\mathbb{R}} (1-u)(K_{X'} + B' + \mathbf{M}_{X'}) \sim_{\mathbb{R}} 0. \end{aligned}$$

Therefore, letting $\Delta = \pi_*\Delta'$, we deduce that $K_{X'} + \Delta'$ is the pullback of $K_X + \Delta$.

Now, if we choose $u > 0$ to be sufficiently small, the lc threshold s of f'^*D with respect to (X', Δ') over the generic point of D is sufficiently close to t. Applying [14, Theorem 1.4] to $(X, \Delta) \to Z$, we deduce that $s \geq \epsilon'^2/2$, where $\epsilon - \epsilon' > 0$ is sufficiently small. Hence, $t \geq \epsilon^2/2$. ∎

To prove Theorem 3.1, we need a couple of lemmas.

LEMMA 3.3 (cf. [11, Lemma 3.2]). *Let $0 < \epsilon \leq 1$ be a real number and r, s, t be positive integers such that $r = s + t$. Suppose Theorem 3.1 holds in relative dimension s and in relative dimension t. Keep the assumptions (a),(b),(c), and (d) in Theorem 3.1. Additionally assume that $f : X \to Z$ can be factored as $X \xrightarrow{g} Y \xrightarrow{h} Z$, where h and g are toric contractions of relative dimension s and t, respectively. Let*

$$\Gamma^\beta = \beta B + (1-\beta)\Delta \quad \textit{and} \quad \mathbf{N}^\beta = \beta \mathbf{M}, \qquad \textit{where } \beta = 1/(s!t!)$$

and let $(Z, \Gamma^\beta_Z, \mathbf{N}^{\beta,Z})/U$ be the g-pair given by adjunction for $f : (X, \Gamma^\beta, \mathbf{N}^\beta) \to Z$. Then,

$$\mathrm{mld}(Z \ni z, \Gamma^\beta_Z, \mathbf{N}^{\beta,Z}) \geq \delta\big(t, \delta(s,\epsilon)\big) = \frac{\epsilon^{2^{s+t}}}{2^{2^{s+t}-1} \prod_{i=1}^{s} i^{2^{i+t}} \cdot \prod_{i=1}^{t} i^{2^i}}.$$

Proof. By assumption, Theorem 3.1 holds for both h and g in the following sense. Let

$$\Gamma^\lambda = \lambda B + (1-\lambda)\Delta \quad \text{and} \quad \mathbf{N}^\lambda = \lambda \mathbf{M}, \qquad \text{where } \lambda = 1/s!$$

and let $(Y, \Gamma^\lambda_Y, \mathbf{N}^{\lambda,Y})/U$ be the g-pair given by adjunction for $g : (X, \Gamma^\lambda, \mathbf{N}^\lambda) \to Y$. Then,

$$\mathrm{mld}(Y/Z \ni z, \Gamma^\lambda_Y, \mathbf{N}^{\lambda,Y}) \geq \delta(s, \epsilon).$$

On the other hand, let

$$\Omega^\gamma_Y = \gamma \Gamma^\lambda_Y + (1-\gamma)\Delta_Y \text{ and } \mathbf{L}^{\gamma,Y} = \gamma \mathbf{N}^{\lambda,Y}, \quad \text{where } \gamma = 1/t!$$

and Δ_Y is the toric boundary divisor of Y. Let $(Z, \Omega^\gamma_Z, \mathbf{L}^{\gamma,Z})/U$ be the g-pair given by the adjunction for $h : (Y, \Omega^\gamma_Y, \mathbf{L}^{\gamma,Y}) \to Z$. Then,

$$\mathrm{mld}(Z \ni z, \Omega^\gamma_Z, \mathbf{L}^{\gamma,Z}) \geq \delta\big(t, \delta(s,\epsilon)\big). \tag{3.1}$$

Now, let

$$\Gamma^\beta = \beta B + (1-\beta)\Delta \quad \text{and} \quad \mathbf{N}^\beta = \beta \mathbf{M}, \qquad \text{where } \beta = \lambda\gamma = 1/(s!t!).$$

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By construction, we have

$$\Gamma^\beta = \gamma\Gamma^\lambda + (1-\gamma)\Delta \text{ and } \mathbf{N}^\beta = \gamma\mathbf{N}^\lambda.$$

Let $(Y, \Gamma_Y^\beta, \mathbf{N}^{\beta,Y})/U$ be the g-pair given by adjunction for $g : (X, \Gamma^\beta, \mathbf{N}^\beta) \to Y$. and let $(Z, \Gamma_Z^\beta, \mathbf{N}^{\beta,Z})$ be the g-pair given by adjunction for $f : (X, \Gamma^\beta + \mathbf{N}^\beta) \to Z$. By Lemma 2.8, $(Z, \Gamma_Z^\beta, \mathbf{N}^{\beta,Z})$ is also the g-pair given by adjunction for $h : (Y, \Gamma_Y^\beta, \mathbf{N}^{\beta,Y}) \to Z$.

Since

$$\Gamma^\beta = \gamma\Gamma^\lambda + (1-\gamma)\Delta,\ \mathbf{N}^\beta = \gamma\mathbf{N}^\lambda$$

and

$$\Omega_Y^\gamma = \gamma\Gamma_Y^\lambda + (1-\gamma)\Delta_Y,\ \mathbf{L}^{\gamma,Y} = \gamma\mathbf{N}^{\lambda,Y},$$

by Lemmas 2.10 and 2.12, the g-pair $(Y, \Gamma_Y^\beta, \mathbf{N}^{\beta,Y})$ has better singularities than $(Y, \Omega_Y^\gamma, \mathbf{L}^{\gamma,Y})$, which implies that $(Z, \Gamma_Z^\beta, \mathbf{N}^{\beta,Z})$ has better singularities than $(Z, \Omega_Z^\gamma, \mathbf{L}^{\gamma,Z})$ by Lemma 2.9. So by (3.1), we have

$$\operatorname{mld}(Z \ni z, \Gamma_Z^\beta, \mathbf{N}^{\beta,Z}) \geq \delta\big(t, \delta(s,\epsilon)\big). \qquad \square$$

LEMMA 3.4. *Let F be a $\mathbb{Q}$-factorial complete toric variety given by (N, Σ) with $\rho(F) = 1$ and $\dim F = r \geq 2$. Then:*

(1) *its fan Σ has exactly $r+1$ rays generated by primitive elements v_i, $i = 1, \ldots, r+1$, and there exist positive integers q_i such that $\sum_{i=1}^{r+1} q_i v_i = 0$;*

(2) *let E_i, $i = 1, \ldots, r+1$, be the prime divisor over F corresponding to $-v_i$, then*

$$a(E_i, F, 0) = \frac{q_1 + \cdots + \widehat{q_i} + \cdots + q_{r+1}}{q_i},$$

where the hat indicates that we omit that term;

(3) *let $\pi : F' \to F$ be an extremal toric divisorial contraction with the exceptional divisor E_i for some i, then there exists a toric contraction $g : F' \to G$ such that $\dim G = r-1$.*

REMARK 3.5. Since $\rho(F') = 2$, $\overline{\mathrm{NE}}(F')$ has exactly two extremal rays. One corresponds to $F' \to F$ and the other corresponds to $F' \to G$.

Proof. (1) This assertion was showed in the proof of [11, Lemma 3.3]. We give another proof here. By [16, Proposition 6.4.1], for a $\mathbb{Q}$-factorial complete toric variety, the number of rays in its fan is equal to the sum of its Picard number and its dimension. As $\rho(F) = 1$ and $\dim F = r$, Σ has $r+1$ rays, say $R_1, \ldots R_{r+1}$. Let v_i be the primitive element of the ray R_i for $i = 1, \ldots, r+1$.

As F is complete, the support of Σ (denoted by $|\Sigma|$) is $N_\mathbb{R}$, so $v_1, \ldots, v_{r+1}$ span $N_\mathbb{R}$. We may assume that $v_1, \ldots, v_r$ form a basis of $N_\mathbb{Q}$. Then, there exist rational numbers $c_1, \ldots, c_r$ such that $v_{r+1} = \sum_{i=1}^r c_i v_i$. We claim that all c_i are negative. Indeed, if one of them (say c_1) is non-negative, the support $|\Sigma|$ is contained in the half space $\{\sum_{i=1}^r a_i v_i \mid a_1 \geq 0\}$, which leads to a contradiction. We can find a positive integer q such that all qc_i are integers. Let $q_i = -qc_i$ for $i = 1, \ldots r$ and let $q_{i+1} = q$. Then, $\sum_{i=1}^{r+1} q_i v_i = 0$.

(2) Without loss of generality, we may suppose that $i = r+1$. Let Δ_F be the toric boundary divisor of F, then $a(D, F, \Delta_F) = 0$ for any toric prime divisor D over F, which implies that $a(D, F, 0)$ is equal to the coefficient of D in the pullback of Δ_F (denoted by $\operatorname{mult}_D \Delta_F$).

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Since $-v_{r+1}$ is in the interior of the cone σ generated by $v_1,\dots,v_r$, the center of E_{r+1} is contained in the affine chart U_σ. On the chart U_σ, Δ_F is determined by $m \in N^* \otimes \mathbb{Q}$ with $\langle m, v_i\rangle = 1$ for $i = 1,\dots,r$. Then,

$$\operatorname{mult}_{E_{r+1}} \Delta_F = \langle m, -v_{r+1}\rangle = \frac{\langle m, q_1v_1 + \cdots + q_rv_r\rangle}{q_{r+1}} = \frac{q_1 + \cdots + q_r}{q_{r+1}}.$$

(3) Without loss of generality, we may suppose that $i = r+1$. The toric variety F' is given by (N, Σ'), where Σ' is the star subdivision of Σ along $-v_{r+1}$, more precisely,

$$\Sigma' = (\Sigma \setminus \{\sigma\}) \cup \Sigma^*(\sigma),$$

where σ is the cone generated by $v_1,\dots,v_r$ and $\Sigma^*(\sigma)$ is the set of all cones generated by subsets of $\{-v_{r+1}, v_1,\dots,v_r\}$ not containing $\{v_1,\dots,v_r\}$.

Let $\phi : N \to N/(\mathbb{Z}v_{r+1}) := N_G$ be the quotient map and let

$$\Sigma_G = \{\phi_{\mathbb{R}}(\tau) \subset (N_G)_{\mathbb{R}} \mid \tau \in \Sigma \text{ and } -v_{r+1} \in \tau\}.$$

Then, Σ_G is a fan in $(N_G)_{\mathbb{R}}$ ([16, Exercise 3.2.7]). Let G be the toric variety given by (N_G, Σ_G). We claim that ϕ is compatible with Σ and Σ_G, that is, for any $\tau \in \Sigma$, $\phi_{\mathbb{R}}(\tau)$ is contained in some cone in Σ_G. Indeed, the claim holds obviously when $-v_{r+1} \in \tau$, so we may assume that $-v_{r+1} \notin \tau$. Then, τ is generated by a subset S of $\{v_1,\dots,v_{r+1}\}$ not containing $\{v_1,\dots,v_r\}$. Let τ' be the cone generated by S', where

$$S' = \begin{cases} S \cup \{-v_{r+1}\}, & \text{if } v_{r+1} \notin S, \\ (S \setminus \{v_{r+1}\}) \cup \{-v_{r+1}\}, & \text{if } v_{r+1} \in S. \end{cases}$$

Then, $\phi_{\mathbb{R}}(\tau') \in \Sigma_G$ and $\phi_{\mathbb{R}}(\tau) = \phi_{\mathbb{R}}(\tau')$. Therefore, the claim holds and then $\phi : N \to N_G$ determines a toric contraction from F to G. ∎

LEMMA 3.6 (cf. [11, Lemma 3.4]). *Let $f : X \to Z$ be a toric Mori fiber space of relative dimension $r \geq 2$, where X is $\mathbb{Q}$-factorial. Then, there is a commutative diagram*

$$\begin{array}{ccccc} W & \xrightarrow{g} & Y & \xrightarrow{h} & Z \\ \downarrow{\scriptstyle \pi} & & & \nearrow{\scriptstyle f} & \\ X & & & & \end{array}$$

such that:

- *π, h, g are toric contractions;*
- *$\pi : W \to X$ is an extremal toric divisorial contraction with the exceptional divisor E satisfying $a(E, X, 0) \leq r$; and*
- *$\dim W - 1 = \dim Y > \dim Z$.*

Proof. By Lemma 2.11, over the torus T_Z in Z, $f^{-1}(T_Z)$ is isomorphic to $F \times T_Z$, where F is a general fiber of f. Since $f : X \to Z$ is a Mori fiber space, F is a Fano variety with $\rho(F) = 1$. Moreover, F is $\mathbb{Q}$-factorial, as by Lemma 2.11 its fan Σ_F is a sub-fan of the fan Σ_X of X which is simplicial. By Lemma 3.4 (1), the fan Σ_F has exactly $r+1$ rays generated by primitive elements v_i, $i = 1,\dots,r+1$, and there exist positive integers q_i such that $\sum_{i=1}^{r+1} q_iv_i = 0$. Pick e such that $q_e \geq q_i$ for any $i = 1,\dots,r+1$ and denote by E_F the toric prime divisor over F corresponding to $-v_e$. Extracting E_F gives an extremal contraction $F' \to F$. By Lemma 3.4 (2), there is a toric contraction $F' \to G$ with $\dim G = r-1$.

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The closure E of the exceptional divisor $E_F \times T_Z$ of $F' \times T_Z \to F \times T_Z$ is a toric divisor over X, so it determines an extremal toric divisorial contraction $\pi : W \to X$ with the exceptional divisor E. Then, $\rho(W/Z) = 2$. Over T_Z, the two contractions $W \to X$ and $F' \times T_Z \to F \times T_Z$ coincides. By Lemma 3.4 (2), we have

$$a(E,X,0) = a(E_F \times T_Z, F \times T_Z, 0) \leq r.$$

Let $g : W \to Y$ be a $(-E)$-negative toric extremal contraction over Z. Then, $\rho(Y/Z) = 1$. Over T_Z, the restriction $g|_{T_Z} : F' \times T_Z \to Y|_{T_Z}$ is either an isomorphism or a $(-E_F \times T_Z)$-negative toric extremal contraction over T_Z. But the former case is impossible because $\rho(F') = 2$ and $\rho(Y/Z) = 1$. Note that $\overline{\mathrm{NE}}(F' \times T_Z/T_Z)$ has exactly two extremal rays. One corresponds to $F' \times T_Z \to F \times T_Z$, and the other corresponds to $F' \times T_Z \to G \times T_Z$. So $g|_{T_Z}$ coincides with one of them. It is impossible that $g|_{T_Z}$ coincides with $F' \times T_Z \to F \times T_Z$ because $-E_F \times T_Z$ is ample over $F \times T_Z$. So $g|_{T_Z}$ coincides with $F' \times T_Z \to G \times T_Z$, which implies that $\dim Y = \dim W - 1$. ∎

Lemma 3.7 (cf. [11, Lemma 3.4]). *Assume that Theorem 3.1 holds in relative dimension* $\leq r-1$. *Then, Theorem 3.1 holds in relative dimension r when $f : X \to Z$ is a toric Mori fiber space and X is $\mathbb{Q}$-factorial.*

Proof. By Lemma 3.2, we may suppose that the relative dimension $r \geq 2$. By taking a toric $\mathbb{Q}$-factorialisation, we can assume X is $\mathbb{Q}$-factorial. By Lemma 3.6, there is a commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{g} & Y & \xrightarrow{h} & Z \\ \downarrow{\scriptstyle \pi} & & & \nearrow{\scriptstyle f} & \\ X & & & & \end{array}$$

satisfying the properties listed in that lemma. Let Δ_W be the the toric boundary divisor of W, then $K_W + \Delta_W = \pi^*(K_X + \Delta)$. Write $K_W + B_W + \mathbf{M}_W = \pi^*(K_X + B + \mathbf{M}_X)$. Let

$$\Gamma_W^\theta = \theta B_W + (1-\theta)\Delta_W \quad \text{and} \quad \mathbf{N}^\theta = \theta\mathbf{M}, \qquad \text{where } \theta = 1/r.$$

Since $a(E,X,0) \leq r$, the coefficient of E in B_W is bounded below by $1-r$. Then, $\Gamma_W^\theta \geq 0$ since the coefficient of E in Δ_W is 1.

By construction, $\mathrm{mld}(W/Z \ni z, \Gamma_W^\theta, \mathbf{N}^\theta) \geq \frac{\epsilon}{r}$. Applying Lemma 3.3 to $(W, \Gamma_W^\theta, \mathbf{N}^\theta)$ over Z (taking $s = 1$ and $t = r-1$ in the lemma), we deduce that if we let

$$\Omega_W^\beta = \beta\Gamma_W^\theta + (1-\beta)\Delta_W \quad \text{and} \quad \mathbf{L}^\beta = \beta\mathbf{N}^\theta, \qquad \text{where } \beta = 1/(r-1)!,$$

and $(Z, \Omega_Z^\beta, \mathbf{L}^{\beta,Z})/U$ be the g-pair given by adjunction for $h \circ g : (W, \Omega_W^\beta, \mathbf{L}^\beta) \to Z$, then

$$\mathrm{mld}(Z \ni z, \Omega_Z^\beta, \mathbf{L}^{\beta,Z}) \geq \frac{(\epsilon/r)^{2^r}}{2^{2^r-1} \cdot \prod_{i=1}^{r-1} i^{2^i}} = \delta(r,\epsilon).$$

It is easy check that

$$\Omega_W^\beta = \alpha B_W + (1-\alpha)\Delta_W \quad \text{and} \quad \mathbf{L}^\beta = \alpha\mathbf{M}, \qquad \text{where } \alpha = \theta\beta = 1/r!.$$

Hence,

$$K_W + \Omega_W^\beta + \mathbf{L}_W^\beta = \pi^*(K_X + \Gamma^\alpha + \mathbf{N}_X^\alpha),$$

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where

$$\Gamma^\alpha = \alpha B + (1-\alpha)\Delta \quad \text{and} \quad \mathbf{N}^\alpha = \alpha \mathbf{M}.$$

Therefore, $(Z, \Omega_Z^\beta, \mathbf{L}^{\beta,Z})/U$ is also the g-pair given by adjunction for $f : (X, \Gamma^\alpha, \mathbf{N}^\alpha) \to Z$. This finishes the proof of the lemma. □

Proof of Theorem 3.1. By induction on relative dimension, we may assume that the theorem holds in relative dimension $\leq r-1$. Taking a toric $\mathbb{Q}$-factorization of X and running an MMP on K_X over Z, we may assume that X is $\mathbb{Q}$-factorial and it has a toric Mori fiber space structure $X \to Y/Z$.

If $Y \to Z$ is birational, we can replace Z by Y, then we are done by Lemma 3.7. Otherwise $\dim Y > \dim Z$. Denote $s = \dim X - \dim Y$ and $t = \dim Y - \dim Z$, then $r = s+t$. Applying Lemma 3.3, we deduce that if we let

$$\Gamma^\beta = \beta B + (1-\beta)\Delta \quad \text{and} \quad \mathbf{N}^\beta = \beta \mathbf{M}, \qquad \text{where } \beta = 1/(s!t!)$$

and $(Z, \Gamma_Z^\beta, \mathbf{N}^{\beta,Z})/U$ be the g-pair given by adjunction for $f : (X, \Gamma^\beta, \mathbf{N}^\beta) \to Z$, then

$$\operatorname{mld}(Z \ni z, \Gamma_Z^\beta, \mathbf{N}^{\beta,Z}) \geq \frac{\epsilon^{2^r}}{2^{2^r-1} \prod_{i=1}^{s} i^{2^{i+t}} \cdot \prod_{i=1}^{t} i^{2^i}}. \tag{3.2}$$

Let

$$\Gamma^\alpha = \alpha B + (1-\alpha)\Delta \quad \text{and} \quad \mathbf{N}^\alpha = \alpha \mathbf{M}, \qquad \text{where } \alpha = 1/r!.$$

Then, we have

$$\Gamma^\alpha = \theta \Gamma^\beta + (1-\theta)\Delta \quad \text{and} \quad \mathbf{N}^\alpha = \theta \mathbf{N}^\beta, \qquad \text{where } \theta = \alpha/\beta = (s!t!)/r!.$$

Let $(Z, \Gamma_Z^\alpha, \mathbf{N}^{\alpha,Z})$ be the g-pair given by adjunction for $f : (X, \Gamma^\alpha, \mathbf{N}^\alpha) \to Z$. By Lemmas 2.10 and 2.12, $(Z, \Gamma_Z^\alpha, \mathbf{N}^{\alpha,Z})$ has better singularities than

$$(Z, \theta \Gamma_Z^\beta + (1-\theta)\Delta_Z, \theta \mathbf{N}^{\beta,Z}).$$

Hence, by (3.2), we have

$$\begin{aligned} \operatorname{mld}(Z \ni z, \Gamma_Z^\alpha, \mathbf{N}^{\alpha,Z}) &\geq \frac{s!t!}{r!} \cdot \frac{\epsilon^{2^r}}{2^{2^r-1} \prod_{i=1}^{s} i^{2^{i+t}} \cdot \prod_{i=1}^{t} i^{2^i}} \\ &\geq \frac{\epsilon^{2^r}}{2^{2^r-1} \prod_{i=1}^{r} i^{2^i}} = \delta(r, \epsilon). \end{aligned}$$

□

Proof of Theorem 1.8. It is a special case of Theorem 3.1. □

Proof of Theorem 1.9. By Lemma 2.7, shrinking Z around z, we may suppose that (X, B) is lc. Since (X, B) is a toric lc pair, we have $B \leq \Delta$, where Δ is the toric boundary divisor of X. Let

$$\Gamma^\alpha = \alpha B^+ + (1-\alpha)\Delta, \qquad \text{where } \alpha = 1/r!.$$

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Then, $\Gamma^\alpha \geq B$. Let $(Z,\Gamma^\alpha_Z,\mathbf{N}^{\alpha,Z})$ be the g-pair given by adjunction for $f:(X,\Gamma^\alpha)\to Z$. By Theorem 1.8, $\mathrm{mld}(Z\ni z,\Gamma^\alpha_Z,\mathbf{N}^{\alpha,Z})\geq \delta=\delta(r,\epsilon)$. Denote the prime divisor $\overline{z}$ by D. Then, the coefficient of D in Γ^α_Z is bounded from above by $1-\delta$. This means that $(X,\Gamma^\alpha+\delta f^*D)$ is lc over the generic point of D. Since $\Gamma^\alpha\geq B$, we deduce that $(X,B+\delta f^*D)$ is lc over the generic point of D. ∎

Proof of Theorem 1.5. Let

$$\Gamma^\alpha=\alpha B+(1-\alpha)\Delta, \quad \text{where } \alpha=1/r!$$

and let $(Z,\Gamma^\alpha_Z,\mathbf{N}^{\alpha,Z})$ be the g-pair given by adjunction for $f:(X,\Gamma^\alpha)\to Z$. By Theorem 1.8, $\mathrm{mld}(Z\ni z,\Gamma^\alpha_Z,\mathbf{N}^{\alpha,Z})\geq\delta(r,\epsilon)$. Hence, $\mathrm{mld}(Z\ni z,0)\geq\delta(r,\epsilon)$ and the first assertion holds.

The second assertion is an immediate consequence of Theorem 1.9 (taking $B=0$ in Theorem 1.9). ∎

Proof of Theorem 1.2. This is a direct consequence of Theorem 1.5. ∎

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References

[1] V. Alexeev and A. Borisov, *On the log discrepancies in toric Mori contractions*, Proc. Amer. Math. Soc. **12** (2014), no. 11, 3687–3694.
[2] F. Ambro, *The adjunction conjecture and its applications*, preprint, 1999, arXiv: math/9903060v3.
[3] F. Ambro, *The moduli b-divisor of an lc-trivial fibration*, Compos. Math. **141** (2005), no. 2, 385–403.
[4] F. Ambro, *Variation of log canonical thresholds in linear systems*, Int. Math. Res. Notices **2016** (2016), no. 14, 4418–4448.
[5] C. Birkar, *Singularities on the base of a Fano type fibration*, J. Reine Angew. Math. **715** (2016), 125–142.
[6] C. Birkar, *Log Calabi-Yau fibrations*, preprint, 2018, arXiv:1811.10709v2.
[7] C. Birkar, *Anti-pluricanonical systems on Fano varieties*, Ann. Math. **190** (2019), no .2, 345–463.
[8] C. Birkar, *Singularities of linear systems and boundedness of Fano varieties*, Ann. Math. **193** (2021), 347–405.
[9] C. Birkar, *Singularities on Fano fibrations and beyond*, preprint, 2023, arXiv:2305.18770.
[10] C. Birkar, *Boundedness of Fano type fibrations*, Ann. Sci. Éc. Norm. Supér. **57** (2024), no. 3, 787–840.
[11] C. Birkar and Y. Chen, *Singularities on toric fibrations*, Sb. Math. **212** (2021), no. 3, 20–38.
[12] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468.
[13] C. Birkar and D.-Q. Zhang, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Publ. Math. IHES. **123** (2016), 283–331.
[14] B. Chen, *Optimal bound for singularities on Fano type fibrations of relative dimension one*, preprint, 2024, arXiv:2210.08469v3.
[15] G. Chen, J. Han, J. Liu and L. Xie, *Minimal model program for algebraically integrable foliations and generalized pairs*, preprint, 2023, arXiv:2309.15823v2.
[16] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, 124, American Mathematical Society, Providence, RI, 2011.
[17] S. Filipazzi, *On a generalized canonical bundle formula and generalized adjunction*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **21** (2020), 1187–1221.
[18] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131, Princeton University Press, Princeton, NJ, 1993.
[19] H. Han, C. Jiang and Y. Luo, *Shokurov's conjecture on conic bundles with canonical singularities*, Forum Math. Sigma **10** (2022), e38.

[20] C. D. Hacon and J. Liu, *Existence of flips for generalized lc pairs*, Camb. J. Math. **11** (2023), no. 4, 795–828.
[21] J. Jiao, J. Liu and L. Xie, *On generalized lc pairs with b-log abundant nef part*, preprint, 2022, arXiv:2202.11256v2.
[22] Y. Kawamata, *Subadjunction of log canonical divisors for a variety of codimension 2*, Contemp. Math. **207** (1997), 79–88.
[23] Y. Kawamata, *Subadjuntion of log canonical divisors. II*, Amer. J. Math. **120** (1998), 893–899.
[24] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, 134, Cambridge University Press, Cambridge, 1998.
[25] K. Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002.
[26] S. Mori and Y. G. Prokhorov, *On $\mathbb{Q}$-conic bundles*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 315–369.
[27] S. Mori and Y. G. Prokhorov, *Multiple fibers of del Pezzo fibrations*, Proc. Steklov Inst. Math. **264** (2009), 131–145.
[28] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3., 15, Springer-Verlag, Berlin, 1988. Transl. from the Japan.
[29] Y. G. Prokhorov and V. V. Shokurov, *Towards the second main theorem on complements*, J. Algebraic Geom. **18** (2009), no. 1, 151–199.
[30] Y. Zou, *Optimal upper bounds for anti-canonical volumes of singular toric Fano varieties*, preprint, 2024, arXiv:2407.19870v2.

Bingyi Chen
Department of Mathematics
Sun Yat-sen University
Guangzhou
P. R. China
chenby253@mail.sysu.edu.cn

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