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Multiple Zeta-Functions Associated with Linear Recurrence Sequences and the Vectorial Sum Formula

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Abstract. We prove the holomorphic continuation of certain multi-variable multiple zeta-functions whose coefficients satisfy a suitable recurrence condition. In fact, we introduce more general vectorial zeta-functions and prove their holomorphic continuation. Moreover, we show a vectorial sum formula among those vectorial zeta-functions from which some generalizations of the classical sum formula can be deduced.

1 Introduction

Let $\mathbb{N}, \mathbb{N}_0, \mathbb{R}, \mathbb{R}_+, \mathbb{C}$ be the set of positive integers, non-negative integers, real numbers, positive numbers, and complex numbers, respectively.

Recently, various multiple zeta-functions have been studied very actively. One of the most fundamental multiple zeta-functions is the Euler–Zagier n-fold sum defined by

$$\zeta_{EZ,n}(\mathbf{s}) = \sum_{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{1}{(m_1+\cdots+m_i)^{s_i}}, \tag{1.1}$$

where $\mathbf{s} = (s_1,\ldots,s_n) \in \mathbb{C}^n$. When $n = 1$, $\zeta_{EZ,1}(\mathbf{s})$ is nothing but the Riemann zeta-function $\zeta(s)$. The values of $\zeta_{EZ,n}(\mathbf{s})$ at positive integers are called multiple zeta values and were originally studied by Hoffman [20] and Zagier [32] independently. The above series (1.1) is absolutely convergent in

$$V_n := \{\mathbf{s} = (s_1,\ldots,s_n) \in \mathbb{C}^n \mid \Re(s_i+\cdots+s_n) > n+1-i,\ \forall i = 1,\ldots,n\} \tag{1.2}$$

(see [23, Theorem 3]), and can be continued meromorphically to the whole space $\mathbb{C}^n$. Various proofs of this meromorphic continuation have been published ([1,3,24,33]).

On the other hand, the problem of meromorphic continuation of multiple zeta-functions of one variable has a much longer history. It was first studied by Barnes and Mellin at the beginning of the twentieth century. The most general result so far published is due to the first-named author [14], who considered the multiple series

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of the form

$$\sum_{\mathbf{m}=(m_1,\dots,m_r)\in\mathbb{N}^r} \frac{1}{P(m_1,\dots,m_r)^s},$$

where $P(\cdot)$ is a polynomial of complex coefficients. He proved the meromorphic continuation of this series under a rather weak condition. In [14], only the one-variable case was discussed; however, already in his unpublished thesis [13], the first-named author mentioned that his method can be generalized to the multi-variable situation

$$\sum_{\mathbf{m}=(m_1,\dots,m_r)\in\mathbb{N}^r} \frac{1}{P_1(m_1,\dots,m_r)^{s_1}\cdots P_n(m_1,\dots,m_r)^{s_n}}, \tag{1.3}$$

where $P_1,\dots,P_n$ are polynomials. In particular, the meromorphic continuation of the Euler–Zagier n-fold sum can be proved by his method.

The analytic continuation of a twisted variant of (1.3) was obtained by M. de Crisenoy in [7].

The method in [13, 14] is not the only method that can treat multiple series of the form (1.3). In [22], B. Lichtin used the theory of $\mathcal{D}$-module to prove (under a condition stronger than that in [14]) the meromorphic continuation of (1.3). In [25], the second-named author showed that the meromorphic continuation of (1.3) (but also under a condition stronger than that in [14]) can be proved by using Mellin–Barnes integrals. Another method is the "decalage" argument, introduced by the first-named author in [15] and further developed in [8], which is a method of proving the continuation without using integral expressions.

On the right-hand side of (1.1), or even (1.3), there is no non-trivial factor on the numerators. However, it is sometimes important to treat multiple series with some (mainly algebraic or arithmetic) coefficients on the numerators.

If the coefficients are purely periodic, then the series can be written as a linear combination of multiple series of trivial numerators, and hence the problem of meromorphic continuation is reduced to the case of trivial numerators. Typical examples are multiple series with Dirichlet characters in the numerators; see [2, 4].

How should we treat the case of non-periodic coefficients? There are at least two natural ways of adding non-trivial coefficients to the numerators on the right-hand side of (1.1), that is

$$\zeta_n^*(\mathbf{a};\mathbf{s}) := \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{a_i(m_i)}{(m_1+\cdots+m_i)^{s_i}}$$

and

$$\zeta_n^{ш}(\mathbf{a};\mathbf{s}) := \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{a_i(m_1+\cdots+m_i)}{(m_1+\cdots+m_i)^{s_i}},$$

where $\mathbf{a}=(a_1,\dots,a_n)$ with $a_i\colon \mathbb{N}\to\mathbb{C}$ $(1\le i\le n)$. (Here, the notations $*$ and ш come from $*$-products and ш-products in the theory of multiple polylogarithms.)

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More generally one can consider multiple zeta-functions defined by

$$\zeta_n^*(\mathbf{a};\mathbb{P};\mathbf{s}) := \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n} \prod_{i=1}^n \frac{a_i(m_i)}{P_i(m_1,\dots,m_i)^{s_i}} \tag{1.4}$$

and

$$\zeta_n^{ш}(\mathbf{a};\mathbb{P};\mathbf{s}) := \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n} \prod_{i=1}^n \frac{a_i(m_1+\cdots+m_i)}{P_i(m_1,\dots,m_i)^{s_i}}, \tag{1.5}$$

where $\mathbf{a} = (a_1,\dots,a_n)$ with $a_i\colon \mathbb{N}\to\mathbb{C}$ $(1\le i\le n)$, and $\mathbb{P} = (P_i)_{i=1,\dots,n}$ is a suitable family of polynomials.

As for (1.4), we can apply the method of Mellin–Barnes integrals to reduce the problem of continuation to the analytic properties of single-sum zeta-functions

$$\sum_{m_i=1}^{\infty} a_i(m_i)m_i^{-s_i} \quad (1\le i\le n) \qquad ([16,26]).$$

However, it is more difficult to treat (1.5). It is not known how to treat this type of multiple sums in general. The first purpose of the present paper is to show that if the polynomials P_i are elliptic, the *holomorphic* continuation with moderate growth of (1.5) can be proved (Theorem 2.1, Corollary 2.2) if we assume a certain recurrence condition on the coefficients a_i $(1\le i\le n)$.

To prove the continuation, we introduce a *vectorial zeta-function*

$$Z_n(\mathbf{F};\mathbb{P};\mathbf{s}) := \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n} \frac{\mathbf{F}(m_1,\dots,m_n)}{\prod_{i=1}^n P_i(m_1,\dots,m_i)^{s_i}}, \tag{1.6}$$

where $\mathbf{F}\colon \mathbb{N}^n \to \mathbb{C}^q$ is a vectorial function and where $\mathbb{P} = (P_i)_{i=1,\dots,n}$ is a family of polynomials such that $P_i \in \mathbb{R}[X_1,\dots,X_i]$ for all i. Our idea is to consider $\zeta_n^{ш}(\mathbf{a};\mathbb{P};\mathbf{s})$ as a coordinate of $Z_n(\mathbf{F};\mathbb{P};\mathbf{s})$. We will prove that $Z_n(\mathbf{F};\mathbb{P};\mathbf{s})$ can be continued holomorphically under some suitable conditions on $\mathbf{F}$ and $\mathbb{P} = (P_i)_{i=1,\dots,n}$ (Theorem 2.3).

The vectorial zeta-function $Z_n(\mathbf{F};\mathbb{P};\mathbf{s})$ itself is also an interesting object. For example, we can prove a *vectorial* sum formula in the case $n = 2$ (Theorem 2.4), from which some generalizations of the classical sum formula can be deduced. It is the second purpose of this paper to report such fascinating properties of the vectorial zeta-function.

In what follows, for any elements $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n)$ of $\mathbb{C}^n$ we write $\|\mathbf{x}\| = \sqrt{|x_1|^2+\cdots+|x_n|^2}$, $|\mathbf{x}| = |x_1|+\cdots+|x_n|$, and $\langle \mathbf{x},\mathbf{y}\rangle = x_1y_1+\cdots+x_ny_n$. We denote the canonical basis of $\mathbb{R}^n$ by $(\mathbf{e}_1,\dots,\mathbf{e}_n)$. We denote a vector in $\mathbb{C}^n$ by $\mathbf{s} = (s_1,\dots,s_n)$ and write $\mathbf{s} = \sigma + i\tau$, where $\sigma = (\sigma_1,\dots,\sigma_n)$ and $\tau = (\tau_1,\dots,\tau_n)$ are the real respectively imaginary components of $\mathbf{s}$ (*i.e.*, $\sigma_i = \Re(s_i)$ and $\tau_i = \Im(s_i)$ for all i).

The expression $f(\lambda,\mathbf{y},\mathbf{x}) \ll_{\mathbf{y}} g(\mathbf{x})$ (uniformly in $\mathbf{x}\in X$ and $\lambda\in\Lambda$) means that there exists $A = A(\mathbf{y}) > 0$, which depends neither on $\mathbf{x}$ nor λ, but could depend on the parameter vector $\mathbf{y}$, such that $|f(\lambda,\mathbf{y},\mathbf{x})| \le Ag(\mathbf{x})$ for any $\mathbf{x}\in X$ and any $\lambda\in\Lambda$.

2 Statement of Results

We fix in the sequel a family of polynomials $\mathbb{P} = (P_i)_{1\le i\le n}$, where, for all i, $P_i \in \mathbb{R}[X_1,\dots,X_i]$ is a polynomial of degree $d_i \ge 0$ such that $P_i(m_1,\dots,m_i) > 0$ for all $\mathbf{m} \in \mathbb{N}^i$. Denote by $d_i = \deg(P_i)$ the degree of the polynomial P_i. We assume the following:

(i) $d_n \ge 1$.

(ii) Each P_i $(1 \le i \le n)$ is elliptic on $[0,\infty)^i$; that is

$$P_{i\,d_i}(x_1,\dots,x_i) > 0 \quad \text{for any} \quad (x_1\dots,x_i) \in \mathbb{R}_+^i \setminus \{(0,\dots,0)\},$$

where $P_{i\,d_i}$ is the homogeneous part of P_i of degree d_i.

(iii) There exist $D \ge 0$ and $C > 0$ such that

$$\Bigl|\prod_{i=1}^n a_i(m_1+\cdots+m_i)\Bigr| \le C(m_1+\cdots+m_n)^D$$

for all $\mathbf{m} = (m_1,\dots,m_n) \in \mathbb{N}^n$.

(iv) The coefficients satisfy a recurrence condition, that is, there exist $r \in \mathbb{N}$ and constants $\lambda_{ji} \in \mathbb{C}$ $(1 \le i \le n, 0 \le j \le r-1)$ such that

$$a_i(m+r) = \sum_{j=0}^{r-1} \lambda_{ji} a_i(m+j) \quad \text{for any} \quad m \in \mathbb{N}. \tag{2.1}$$

In order to extend meromorphically the zeta-function $\zeta_n^{ш}(\mathbf{a};\mathbb{P};\mathbf{s})$ beyond its domain of convergence, the fundamental idea in this paper is to consider the zeta-function $\zeta_n^{ш}(\mathbf{a};\mathbb{P};\mathbf{s})$ as a coordinate of a *vectorial zeta-function*

$$Z_n(\mathbf{A};\mathbb{P};\mathbf{s}) := \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n} \mathbf{A}(m_1,\dots,m_n) \prod_{i=1}^n P(m_1,\dots,m_i)^{-s_i},$$

where $\mathbf{A} = (A_1,\dots,A_q)$ is a suitable vector-valued function with some "similarity properties". More precisely, set $q = r^n$, denote by $\eta_1,\dots,\eta_q$ the family of all the maps between $\{1,\dots,n\}$ and $\{0,\dots,r-1\}$, and define the function $\mathbf{A}\colon \mathbb{N}^n \to \mathbb{C}^q$ by $\mathbf{A}(\mathbf{m}) := (A_1(\mathbf{m}),\dots,A_q(\mathbf{m}))$, where $\mathbf{m} = (m_1,\dots,m_n) \in \mathbb{N}^n$ and

$$A_l(\mathbf{m}) = \prod_{i=1}^n a_i(m_1+\cdots+m_i+\eta_l(i)) \tag{2.2}$$

for all $l \in \{1,\dots,q\}$. It follows from (2.1) that for all $k = 1,\dots,n$ there exists a matrix $T_k \in \mathcal{M}_{q\times q}(\mathbb{C})$ such that

$$A(\mathbf{m}+r\mathbf{e}_k) = A(m_1,\dots,m_{k-1},m_k+r,m_{k+1},\dots,m_n) = T_k\,A(\mathbf{m}) \tag{2.3}$$

for any $\mathbf{m} \in \mathbb{N}^n$.

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Let $\mathbf{d} = (d_1, \dots, d_n)$, where $d_i = \deg(P_i)$ as above. For any $R \in \mathbb{R}$ set

$$V_n(\mathbf{d}; R) := \Big\{\mathbf{s} = \sigma + \sqrt{-1}\tau \in \mathbb{C}^n \;\Big|\; \sum_{j=i}^{n} d_i\sigma_i > R + n + 1 - i \quad (1 \le i \le n)\Big\},$$

and

$$B_n(\sigma; \mathbf{d}; R) := \sup_{1 \le i \le n} \big(n + 1 + R - i - (d_i\sigma_i + \cdots + d_n\sigma_n)\big).$$

The first main result of this paper is the following.

Theorem 2.1 *Besides the above* (i)—(iv), *assume that* 1 *is not an eigenvalue of any of the matrices* $T_1, \dots, T_n$. *Then the function* $\zeta_n^{ш}(\mathbf{a}; \mathbb{P}; \mathbf{s})$ *converges absolutely in the set* $V_n(\mathbf{d}; D)$, *has a holomorphic continuation to the whole complex space* $\mathbb{C}^n$, *and for all* $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$ *we have*

$$\zeta_n^{ш}(\mathbf{a}; \mathbb{P}; \mathbf{s}) = \zeta_n^{ш}(\mathbf{a}; \mathbb{P}; \sigma + \sqrt{-1}\tau) \ll_{\mathbf{a}, \mathbb{P}, \sigma} 1 + (1 + |\tau|)^{1 + B_n(\sigma; \mathbf{d}; D)}$$

uniformly in $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$.

Remark If $\mathbb{P} = (X_1 + \cdots + X_i)_{1 \le i \le n}$, then

$$\zeta_n^{ш}(\mathbf{a}; \mathbb{P}; \mathbf{s}) = \zeta_n^{ш}(\mathbf{a}; \mathbf{s}) = \sum_{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n} \prod_{i=1}^{n} \frac{a_i(m_1 + \cdots + m_i)}{(m_1 + \cdots + m_i)^{s_i}}.$$

Therefore our Theorem 2.1 can be applied to the classical multiple zeta-functions $\zeta_n^{ш}(\mathbf{a}; \mathbf{s})$. In this particular case one can write $\zeta_n^{ш}(\mathbf{a}; \mathbf{s})$ as a combination of twisted Euler–Zagier sums

$$\zeta_{EZ,n}(\mathbf{z}; \mathbf{s}) = \sum_{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n} \frac{z_1^{m_1} \dots z_n^{m_n}}{\prod_{i=1}^{n} (m_1 + \cdots + m_i)^{s_i}},$$

where $\mathbf{z} = (z_1, \dots, z_n)$. Several methods can then be used in this case (see [1, 3, 9–12, 19, 23, 25, 28, 29, 33], etc.) to prove meromorphic continuation of twisted Euler–Zagier sums $\zeta_{EZ,n}(\mathbf{z}; \mathbf{s})$. Therefore, these methods can give the meromorphic continuation of $\zeta_n^{ш}(\mathbf{a}; \mathbf{s})$ in some cases. However, these methods can give only the meromorphic continuation, and if one wants to prove *holomorphic* continuation, one needs in addition to verify that any divisor of (twisted) Euler–Zagier sums that appears in the sum vanishes after summation. This is generally not an easy task!

Corollary 2.2 *Assume that there exist* $r \in \mathbb{N}$ *and* $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ *satisfying*

$$\Big|\prod_{j=i}^{n} \lambda_j\Big| \le 1 \text{ and } \prod_{j=i}^{n} \lambda_j \ne 1, \tag{2.4}$$

$$a_i(m + r) = \lambda_i\, a_i(m) \tag{2.5}$$

for all $i = 1, \dots, n$ *and all* $m \in \mathbb{N}$. *Then* $\zeta_n^{ш}(\mathbf{a}; \mathbb{P}; \mathbf{s})$ *converges absolutely in* $V_n(\mathbf{d}; 0)$, *has a holomorphic continuation to the whole complex space* $\mathbb{C}^n$, *and for any* $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$ *the estimate*

$$\zeta_n^{ш}(\mathbf{a}; \mathbb{P}; \mathbf{s}) = \zeta_n^{ш}(\mathbf{a}; \mathbb{P}; \sigma + \sqrt{-1}\tau) \ll_{\mathbf{a},\mathbb{P},\sigma} 1 + \big(1 + |\tau|\big)^{1+B_n(\sigma;\mathbf{d};0)}$$

holds uniformly in $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$.

Theorem 2.1 follows easily from the following general result on the vectorial zeta-function $Z_n(\mathbf{F}; \mathbb{P}; \mathbf{s})$, defined by (1.6).

Theorem 2.3 *Assume that there exist* $r \in \mathbb{N}$ *and* n *matrices* $T_1, \dots, T_n \in \mathcal{M}_{q\times q}(\mathbb{C})$ *such that for all* $k = 1, \dots, n$ *and for all* $\mathbf{m} \in \mathbb{N}^n$

$$\mathbf{F}(\mathbf{m} + r\mathbf{e}_k) = \mathbf{F}(m_1, \dots, m_{k-1}, m_k + r, m_{k+1}, \dots, m_n) = T_k\, \mathbf{F}(\mathbf{m}). \tag{2.6}$$

Further assume that there exist $D \geq 0$ *and* $C > 0$ *such that for all*

$$\|\mathbf{F}(m_1, \dots, m_n)\| \leq C(m_1 + \dots + m_n)^D \tag{2.7}$$

for all $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$, *and also that* 1 *is not an eigenvalue of any of the matrices* $T_1, \dots, T_n$. *Then*

(i) $\mathbf{s} \mapsto Z_n(\mathbf{F}; \mathbb{P}; \mathbf{s})$ *converges absolutely in the set* $V_n(\mathbf{d}; D)$ *and has a holomorphic continuation to the whole complex space* $\mathbb{C}^n$;

(ii) *For all* $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$ *we have*

$$Z_n(\mathbf{F}; \mathbb{P}; \mathbf{s}) = Z_n(\mathbf{F}; \mathbb{P}; \sigma + \sqrt{-1}\tau) \ll_{\mathbf{F},\mathbb{P},\sigma} 1 + \big(1 + |\tau|\big)^{1+B_n(\sigma;\mathbf{d};D)}$$

uniformly in $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n$.

In the next two sections we will describe the proof of Theorem 2.3.[1] Then in Section 5, we will deduce Theorem 2.1 and Corollary 2.2 from Theorem 2.3.

The vectorial zeta-function $Z_n(\mathbf{F}; \mathbb{P}; \mathbf{s})$ is not just an auxiliary function, but an interesting object itself. A typical example of the vector $\mathbf{F}$ can be constructed by using Fibonacci numbers. We will discuss the properties of the associated zeta-function in Section 6.

Another interesting fact on the vectorial zeta-function is the *vectorial sum formula*. Recall the Euler double zeta-function

$$\zeta_{EZ,2}(s_1, s_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^{s_1}(m+n)^{s_2}},$$

which is the case $n = 2$ of (1.1) and was originally studied by Euler. This is one of the most well-known multiple zeta-functions (see [17, 32]). A famous formula for $\zeta_{EZ,2}(s_1, s_2)$ is

$$\sum_{h=2}^{k+1} \zeta_{EZ,2}(k+2-h, h) = \zeta(k+2) \quad (k \in \mathbb{N}), \tag{2.8}$$

[1] For technical reasons, we are going to prove Theorem 4.1, which is slightly more general.

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which is called the "sum formula" and was essentially proved by Euler. In particular, when $k = 1$, we have $\zeta_{EZ,2}(1, 2) = \zeta(3)$. (For a more detailed discussion, see [5].)

Here we consider the cases $n = 1, 2$. Let $\mathbf{F} = (f_1, \dots, f_q) \colon \mathbb{N}^1 \to \mathbb{C}^q$ be a function. For a fixed $k \in \mathbb{N}$, we assume that $f_j(m) = O(m^{k-\varepsilon})$ $(1 \le j \le q)$, where O implies the usual O-symbol and ε implies a sufficiently small positive number. Moreover, we define $\mathbf{F}_\nu \colon \mathbb{N}^2 \to \mathbb{C}^q$ $(1 \le \nu \le 3)$ by $\mathbf{F}_1(m, n) = \mathbf{F}(m)$, $\mathbf{F}_2(m, n) = \mathbf{F}(n)$ and $\mathbf{F}_3(m, n) = \mathbf{F}(m + n)$, and, as special cases of (1.6), consider

$$(2.9) \quad Z_1(\mathbf{F}; s) = \sum_{m=1}^{\infty} \frac{\mathbf{F}(m)}{m^s}, \quad Z_2(\mathbf{F}_\nu; s_1, s_2) = \sum_{m,n=1}^{\infty} \frac{\mathbf{F}_\nu(m, n)}{m^{s_1}(m + n)^{s_2}} \quad (1 \le \nu \le 3).$$

The *vectorial* analogue of the sum formula is as follows. Note that we need no assumption with respect to eigenvalues of T_k given in the statement of Theorem 2.3, because this result concerns the values in the convergent area.

Theorem 2.4 *Let* $\mathbf{F}$, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, *and* k *be defined as above. Then the formula*

$$(2.10) \quad \sum_{h=2}^{k+1} Z_2(\mathbf{F}_1; k + 2 - h, h) + Z_2(\mathbf{F}_2; 1, k + 1) - Z_2(\mathbf{F}_3; 1, k + 1) = Z_1(\mathbf{F}; k + 2)$$

holds. In particular when $k = 1$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mathbf{F}(m) + \mathbf{F}(n) - \mathbf{F}(m + n)}{m(m + n)^2} = \sum_{m=1}^{\infty} \frac{\mathbf{F}(m)}{m^3}.$$

The proof of this theorem will be given in Section 7. Further generalization of this theorem will be discussed in the last section, where a conjecture on a possible vectorial sum formula for multiple series will be proposed.

Here we mention several remarkable consequences of this theorem. By applying Theorem 2.4 in the case $\mathbf{F}(m) = M^m$ $(m \in \mathbb{N})$ for $M \in \mathcal{M}_{l\times l}(\mathbb{C})$, we obtain the following.

Corollary 2.5 *Let* $k \in \mathbb{N}$ *and* $M \in \mathcal{M}_{l\times l}(\mathbb{C})$ *with the assumption that each entry of* M^m *is* $O\big(m^{k-\varepsilon}\big)$ $(m \to \infty)$. *Then*

$$(2.11) \quad \sum_{h=2}^{k+1} \left\{ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{M^m}{m^{k+2-h}(m + n)^h} \right\} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{M^n}{m(m + n)^{k+1}} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{M^{m+n}}{m(m + n)^{k+1}} = \sum_{m=1}^{\infty} \frac{M^m}{m^{k+2}}.$$

Example 2.6 Consider the case when $l = 1$, that is, $M \in \mathcal{M}_{1\times 1}(\mathbb{C}) = \mathbb{C}$. It is clear that if $M = 1$, then (2.11) coincides with the ordinary sum formula (2.8).

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Moreover, if $M = x \in \mathbb{C}$ ($|x| \le 1$), then we obtain the sum formula for the double polylogarithms:

$$\sum_{h=2}^{k+1}\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^m}{m^{k+2-h}(m+n)^h}\Big\}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^n}{m(m+n)^{k+1}} \\ -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x^{m+n}}{m(m+n)^{k+1}}=\sum_{m=1}^{\infty}\frac{x^m}{m^{k+2}}\quad (k\in\mathbb{N}). \tag{2.12}$$

This is implicitly included in [27]. In fact, as can be seen from the proof of Theorem 2.4 in the last section, we can easily derive (2.12) from [27, Theorem 2.5].

Example 2.7 Let $M \in M_{l\times l}(\mathbb{C})$, which satisfies that each entry of M^m is $O(m^{k-\varepsilon})$. For any fixed i, j, we denote the (i, j)-entry of M^m by $c(m) = c_{ij}(m)$ $(m \in \mathbb{N})$. Needless to say, the explicit expression of $c(m)$ in terms of entries of M is rather complicated. From (2.11), we find that the formula

$$\sum_{h=2}^{k+1}\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{c(m)}{m^{k+2-h}(m+n)^h}\Big\}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{c(n)}{m(m+n)^{k+1}} \\ -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{c(m+n)}{m(m+n)^{k+1}}=\sum_{m=1}^{\infty}\frac{c(m)}{m^{k+2}} \tag{2.13}$$

holds. It seems not easy to find such a formula without using our vectorial zeta-function.

3 Three Elementary Lemmas

Now we start the proof of Theorem 2.3. First of all, we state the following elementary but useful lemmas.

Lemma 3.1 *For $\nu \in \mathbb{N}$ and $N \in \mathbb{N}_0$ we define the function $G_{\nu,N}\colon \mathbb{C}^\nu \times (-1,\infty)^\nu \to \mathbb{C}$ by*

$$G_{\nu,N}(\mathbf{s};\mathbf{x}) := \prod_{i=1}^{\nu}(1+x_i)^{-s_i} - \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^\nu \\ |\mathbf{k}|\le N}}\binom{-\mathbf{s}}{\mathbf{k}}\mathbf{x}^{\mathbf{k}},$$

where $\binom{-\mathbf{s}}{\mathbf{k}} = \prod_{i=1}^{\nu}\binom{-s_i}{k_i}$ and $\mathbf{x}^{\mathbf{k}} = \prod_{i=1}^{\nu} x_i^{k_i}$ for $\mathbf{s} = (s_1,\dots,s_\nu)$, $\mathbf{x} = (x_1,\dots,x_\nu)$, $\mathbf{k} = (k_1,\dots,k_\nu)$. Then,

(i) *for any $\mathbf{x} \in (-1,\infty)^\nu$, $\mathbf{s} \mapsto G_{\nu,N}(\mathbf{x};\mathbf{s})$ is holomorphic in $\mathbb{C}^\nu$;*

(ii) *for any $\delta,\gamma \in \mathbb{R}$ such that $-1 < \delta \le \gamma$, we have*

$$|G_{\nu,N}(\sigma + i\tau;\mathbf{x})| \ll_{\delta,\gamma,N,\nu,\sigma} \left(1 + (1+|\tau|)^{N+1}\right)|\mathbf{x}|^{N+1},$$

uniformly in $\mathbf{x} \in [\delta,\gamma]^\nu$ and $\tau \in \mathbb{R}^\nu$.

Proof Fix $\mathbf{s} \in \mathbb{C}^\nu$ and $\mathbf{x} \in (-1, \infty)^\nu$. Define the function $\varphi\colon [0,1] \to \mathbb{C}$ by

$$\varphi(t) := \prod_{i=1}^{\nu}(1+tx_i)^{-s_i} \quad (0 \le t \le 1). \tag{3.1}$$

We prove that, for any $q \in \mathbb{N}_0$, the derivative of φ of order q is given by

$$\varphi^{(q)}(t) = q! \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^\nu \\ |\mathbf{k}|=k_1+\cdots+k_\nu=q}} \binom{-\mathbf{s}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \prod_{i=1}^{\nu}(1+tx_i)^{-s_i-k_i} \quad (0 \le t \le 1). \tag{3.2}$$

The proof is by induction on q. If $q = 0$, then (3.2) is clearly verified. Now assume that (3.2) is true for q and prove that it remains true for $q+1$. Differentiating both sides of (3.2), we obtain

$$\begin{aligned}
&\varphi^{(q+1)}(t)\\
&= q! \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^\nu \\ |\mathbf{k}|=k_1+\cdots+k_\nu=q}} \binom{-\mathbf{s}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \sum_{i=1}^{\nu}(-s_i-k_i)x_i \Big(\prod_{\substack{j=1\\ j\ne i}}^{\nu}(1+tx_j)^{-s_j-k_j}\Big)(1+tx_i)^{-s_i-k_i-1}\\
&= q! \sum_{i=1}^{\nu} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^\nu \\ |\mathbf{k}|=k_1+\cdots+k_\nu=q}} \binom{-\mathbf{s}}{\mathbf{k}+\mathbf{e}_i}(k_i+1)\mathbf{x}^{\mathbf{k}+\mathbf{e}_i} \Big(\prod_{\substack{j=1\\ j\ne i}}^{\nu}(1+tx_j)^{-s_j-k_j}\Big)(1+tx_i)^{-s_i-k_i-1}\\
&= q! \sum_{i=1}^{\nu} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^\nu \\ |\mathbf{k}|=q+1,\ k_i\ge 1}} \binom{-\mathbf{s}}{\mathbf{k}} k_i \mathbf{x}^{\mathbf{k}} \Big(\prod_{j=1}^{\nu}(1+tx_j)^{-s_j-k_j}\Big)\\
&= q! \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^\nu \\ |\mathbf{k}|=q+1}} \binom{-\mathbf{s}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \Big(\prod_{j=1}^{\nu}(1+tx_j)^{-s_j-k_j}\Big) \sum_{i=1}^{\nu} k_i\\
&= (q+1)! \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^\nu \\ |\mathbf{k}|=q+1}} \binom{-\mathbf{s}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \Big(\prod_{j=1}^{\nu}(1+tx_j)^{-s_j-k_j}\Big).
\end{aligned}$$

Hence we find that (3.2) is also true for $q+1$. This ends our induction argument and completes the proof of (3.2). In particular, for any $q \in \mathbb{N}_0$ we have

$$\varphi^{(q)}(0) = q! \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^\nu \\ |\mathbf{k}|=k_1+\cdots+k_\nu=q}} \binom{-\mathbf{s}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}. \tag{3.3}$$

Let $(\mathbf{s};\mathbf{x}) \in \mathbb{C}^\nu \times (-1,\infty)^\nu$ and let $N \in \mathbb{N}_0$. By applying Taylor's formula (with remainder) to the one variable function $\varphi(t)$ we get

$$\varphi(1) = \sum_{q=0}^{N} \frac{\varphi^{(q)}(0)}{q!} + \frac{1}{N!}\int_0^1 (1-t)^N \varphi^{(N+1)}(t)\,dt. \tag{3.4}$$

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Since from (3.1) and (3.3) we see that

$$G_{\nu,N}(\mathbf{s};\mathbf{x}) = \varphi(1) - \sum_{q=0}^{N} \frac{\varphi^{(q)}(0)}{q!},$$

relations (3.2) and (3.4) imply that for all $(\mathbf{s};\mathbf{x}) \in \mathbb{C}^\nu \times (-1,\infty)^\nu$ we have

$$G_{\nu,N}(\mathbf{s};\mathbf{x}) = (N+1) \sum_{\substack{\mathbf{k}\in\mathbb{N}^\nu \\ |\mathbf{k}|=N+1}} \binom{-\mathbf{s}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \int_0^1 (1-t)^N \prod_{i=1}^{\nu} (1+tx_i)^{-s_i-k_i}\, dt.$$

The lemma easily follows from this expression of $G_{\nu,N}(\mathbf{s};\mathbf{x})$. ∎

Lemma 3.2 *Let $T \in \mathcal{M}_{q\times q}(\mathbb{C})$. When the estimate (2.7) holds, the estimate*

$$\|T\mathbf{F}(\mathbf{m})\| \ll_{\mathbf{F},T} (m_1 + \cdots + m_n)^D \tag{3.5}$$

also holds.

Proof Let $\mathbf{F}(\mathbf{m}) = (f_j(\mathbf{m}))_{1\le j\le q}$ and $T = (c_{ij})_{1\le i,j\le q}$. Then

$$T\mathbf{F}(\mathbf{m}) = \left(\sum_{j=1}^{q} c_{ij} f_j(\mathbf{m}) \right)_{1\le i\le q},$$

and hence

$$\begin{aligned}
\|T\mathbf{F}(\mathbf{m})\| &= \left(\sum_{i=1}^{q} \Big| \sum_{j=1}^{q} c_{ij} f_j(\mathbf{m}) \Big|^2 \right)^{1/2} \\
&\le \Big(\max_{1\le i,j\le q} |c_{ij}| \Big) \left(\sum_{i=1}^{q} \Big(\sum_{j=1}^{q} |f_j(\mathbf{m})| \Big)^2 \right)^{1/2} \\
&\ll_{\mathbf{F},T} \left(\sum_{j=1}^{q} |f_j(\mathbf{m})|^2 \right)^{1/2} = \|\mathbf{F}(\mathbf{m})\| \ll (m_1 + \cdots + m_n)^D,
\end{aligned}$$

which is (3.5). ∎

Lemma 3.3 *Let $Q \in \mathbb{R}[X_1,\dots,X_n]$ be an elliptic polynomial of degree d. Then there exist $\alpha > 0, \beta > 0$ and $R > 0$ such that, for any $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}_+^n$ satisfying $|\mathbf{x}| = x_1 + \cdots + x_n \ge R$, we have*

$$\alpha(x_1 + \cdots + x_n)^d \le Q(\mathbf{x}) \le \beta(x_1 + \cdots + x_n)^d.$$

Proof Let Q_d be the homogeneous part of Q of degree d. Set $E = \{\mathbf{x} \in \mathbb{R}_+^n \mid |\mathbf{x}| = 1\}$. Since E is compact and Q is elliptic, it follows that $\alpha = \inf_{\mathbf{y}\in E} Q_d(\mathbf{y}) > 0$ and $\beta = \sup_{\mathbf{y}\in E} Q_d(\mathbf{y}) > 0$. Now let $\mathbf{x} \in \mathbb{R}_+^n \setminus \{0\}$. Since $\mathbf{y} = \big(x_1/|\mathbf{x}|,\dots,x_n/|\mathbf{x}|\big) \in E$, one has $\alpha \le Q_d\big(x_1/|\mathbf{x}|,\dots,x_n/|\mathbf{x}|\big) \le \beta$, and by homogeneity, we deduce that: $\alpha|\mathbf{x}|^d \le Q_d(\mathbf{x}) \le \beta|\mathbf{x}|^d$. We conclude by using in addition the fact that $Q(\mathbf{x}) = Q_d(\mathbf{x}) + O(|\mathbf{x}|^{d-1})$ as $|\mathbf{x}| \to \infty$. ∎

4 Proof of Theorem 2.3

Now we prove by induction on n that $Z_n(\mathbf{F};\mathbb{P};\mathbf{s})$ has a holomorphic continuation to $\mathbb{C}^n$. The basic idea of the argument here is the same as in the "decalage" method of the first author [8, 15].

For technical reasons, we will prove the following slightly more general theorem than Theorem 2.3. But before stating this theorem (*i.e.,* Theorem 4.1), we will introduce some notations.

Let $\mathbf{u} = (u(0),\dots,u(n)) \in \mathbb{N}_0^{n+1}$ such that $u(0) = 0$ and $u(n) \geq 1$. Let

$$\mathbb{P} = \{P_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq u(i)\}$$

be a family of polynomials such that, for all i, j, $P_{i,j} \in \mathbb{R}[X_1,\dots,X_i]$ is an elliptic polynomial in $[0,\infty)^i$ of degree $d_{i,j}$. Set $\mathbf{d} = \{d_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq u(i)\}$ and $\mathbf{d}_i = (d_{i,1},\dots,d_{i,u(i)})$ for all i. We assume that $\mathbf{d}_n \neq (0,\dots,0)$. Set $\nu(i) = \sum_{l=0}^{i-1} u(l)$ $(i = 1,\dots,n+1)$, especially $\nu = \nu(n+1) = \sum_{l=0}^{n} u(l) = |\mathbf{u}|$. For any $T \in \mathbb{R}$ set

$$V_n(\mathbf{u},\mathbf{d};T) := \Big\{ \mathbf{s} = \sigma + \sqrt{-1}\tau \in \mathbb{C}^\nu \mid \sum_{l=i}^{n}\sum_{j=1}^{u(l)} d_{l,j}\sigma_{\nu(l)+j} > T + n + 1 - i\ (1 \leq i \leq n) \Big\},$$

and

$$B_n(\sigma;\mathbf{u};\mathbf{d};T) := \sup_{1\leq i\leq n} \Big(n + 1 + T - i - \sum_{l=i}^{n}\sum_{j=1}^{u(l)} d_{l,j}\sigma_{\nu(l)+j} \Big).$$

Theorem 4.1 *Let $\mathbf{F}\colon \mathbb{N}^n \to \mathbb{C}^q$ be a vectorial function as in Theorem 2.3. Let $\mathbb{P} = (P_{i,j})$ be a family of elliptic polynomials as above and let $H \in \mathbb{R}[X_1,\dots,X_n]$ be a polynomial of degree h. Consider the generalized vectorial multiple zeta-function*

$$(4.1)\quad Z_n(\mathbf{F};\mathbb{P};H;\mathbf{s}) := \sum_{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n} \frac{H(m_1,\dots,m_n)\,\mathbf{F}(m_1,\dots,m_n)}{\prod_{i=1}^{n}\prod_{j=1}^{u(i)} P_{i,j}(m_1,\dots,m_i)^{s_{\nu(i)+j}}} \quad \big(\mathbf{s} = (s_1,\dots,s_\nu)\big).$$

Then

(i) $\mathbf{s} \mapsto Z_n(\mathbf{F};\mathbb{P};H;\mathbf{s})$ *converges absolutely in the set* $V_n(\mathbf{u};\mathbf{d};h+D)$;
(ii) $\mathbf{s} \mapsto Z_n(\mathbf{F};\mathbb{P};H;\mathbf{s})$ *has a holomorphic continuation to the whole complex space* $\mathbb{C}^\nu$;
(iii) *for all* $\sigma = (\sigma_1,\dots,\sigma_\nu) \in \mathbb{R}^\nu$ *we have*

$$(4.2)\quad Z_n(\mathbf{F};\mathbb{P};H;\mathbf{s}) = Z_n(\mathbf{F};\mathbb{P};H;\sigma + \sqrt{-1}\tau) \ll_{\mathbf{F},\mathbb{P},H,\sigma} 1 + \big(1 + |\tau|\big)^{1+B_n(\sigma;\mathbf{u};\mathbf{d};D+h)}$$

uniformly in $\tau = (\tau_1,\dots,\tau_\nu) \in \mathbb{R}^\nu$.

4.1 Proof of Theorem 4.1(i)

By using Lemma 3.3 and relation (2.7), it is easy to see that, for any $\mathbf{s} = \sigma + \sqrt{-1}\tau \in \mathbb{C}^\nu$, we have

$$\frac{H(m_1,\dots,m_n)\,\mathbf{F}(m_1,\dots,m_n)}{\prod_{i=1}^{n}\prod_{j=1}^{u_i} P_{i,j}(m_1,\dots,m_i)^{s_{v_i+j}}} \ll_{\mathbf{F},\mathbb{P},\sigma} \frac{(m_1+\cdots+m_n)^{h+D}}{\prod_{i=1}^{n}(m_1+\cdots+m_i)^{\langle \mathbf{d}_i,\sigma_{\nu(i)}\rangle}} \quad (\mathbf{m}\in\mathbb{N}^n),$$

where $\sigma_{\nu(i)} = (\sigma_{\nu(i)+1},\dots,\sigma_{\nu(i)+u(i)})$. This fact compared with relation (1.2) completes the proof of Theorem 4.1(i). ■

4.2 A Key Lemma

Let $\mathbf{F}$, $\mathbf{u}$, $\mathbb{P} = (P_{i,j})$, $\mathbf{d} = (d_{i,j})$ be as in the statement of Theorem 4.1. Define $\mathbf{u}' = (u(0)',\dots,u(n-1)') \in \mathbb{N}_0^n$ by $u(i)' = u(i)$ for all $i < n-1$ and $u(n-1)' = u(n-1)+u(n)$. For all $i = 1,\dots,n$ set $\nu(i)' = \sum_{l=0}^{i-1} u(l)'$ and

$$\nu' = \nu(n)' = \sum_{l=0}^{n-1} u(l)' = |\mathbf{u}'| (= |\mathbf{u}| = \nu).$$

For $t \in \{1,\dots,r\}$, we define

$$\mathbb{P}^t = \{P^t_{i,j} \mid 1 \le i \le n-1, \quad 1 \le j \le u(i)'\}$$

by

(i) $P^t_{i,j}(X_1,\dots,X_i) = P_{i,j}(X_1,\dots,X_i)$ if $i < n-1$;
(ii) $P^t_{n-1,j}(X_1,\dots,X_{n-1}) = P_{n-1,j}(X_1,\dots,X_{n-1})$ if $i = n-1$ and $j \in \{1,\dots,u(n-1)\}$;
(iii) $P^t_{n-1,j}(X_1,\dots,X_{n-1}) = P_{n,j-u(n-1)}(X_1,\dots,X_{n-1},t)$ if $i = n-1$ and

$$j \in \{u(n-1)+1,\dots,u(n-1)+u(n) = u(n-1)'\}.$$

Set $\mathbf{d}^t = \{d^t_{i,j} \mid 1 \le i \le n-1, 1 \le j \le u(i)'\}$, where $d^t_{i,j} = \deg(P^t_{i,j})$.

We will also use the notations

(i) $\Delta_r U = U(\mathbf{X} + r\mathbf{e}_n) - U(\mathbf{X})$ for $U(\mathbf{X}) = U(X_1,\dots,X_n) \in \mathbb{R}[X_1,\dots,X_n]$;
(ii) $\mathbf{s}(\mathbf{k}) := \mathbf{s} + \sum_{j=1}^{u(n)} k_j \mathbf{e}_{\nu(n)+j}$ for all $\mathbf{s} = (s_1,\dots,s_\nu) \in \mathbb{C}^\nu$ and for all $k = (k_1,\dots,k_{u(n)}) \in \mathbb{N}_0^{u(n)}$;

and the convention

$$Z_0(\mathbf{F};\mathbb{P}^t;H(.,t);\mathbf{s}) := \frac{H(t)\,\mathbf{F}(t)}{\prod_{j=1}^{u(1)} P_{1,j}(t)^{s_{\nu(1)+j}}}. \tag{4.3}$$

Then we have the following key lemma.

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Lemma 4.2 *Let $n \in \mathbb{N}$ and let $H \in \mathbb{R}[X_1, \dots, X_n]$ of degree h. Assume that $\mathbf{F}$ and $\mathbb{P} = (P_{i,j})$ satisfy all the assumptions of Theorem 4.1. Then for all $N \in \mathbb{N}_0$ and for all $\mathbf{s} = (s_1, \dots, s_v) \in V_n(\mathbf{u}, \mathbf{d}, h + D)$, we have*

(4.4)
$$\begin{aligned}
&Z_n(\mathbf{F}; \mathbb{P}; H; \mathbf{s})\\
&= (I_q - T_n)^{-1} \sum_{t=1}^{r} Z_{n-1}\Big(\mathbf{F}(\,\cdot\,, t); \mathbb{P}^t; H(\,\cdot\,, t); \mathbf{s}\Big)\\
&+ (I_q - T_n)^{-1} T_n \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{u(n)}\\ 1 \le |\mathbf{k}| \le N}} \Big(\prod_{j=1}^{u(n)} \binom{-s_{v(n)+j}}{k_j}\Big) Z_n\Big(\mathbf{F}; \mathbb{P}; H \prod_{j=1}^{u(n)} (\Delta_r P_{n,j})^{k_j}; \mathbf{s}(\mathbf{k})\Big)\\
&+ (I_q - T_n)^{-1} T_n \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{u(n)}\\ |\mathbf{k}| \le N}} \Big(\prod_{j=1}^{u(n)} \binom{-s_{v(n)+j}}{k_j}\Big) Z_n\Big(\mathbf{F}; \mathbb{P}; (\Delta_r H) \prod_{j=1}^{u(n)} (\Delta_r P_{n,j})^{k_j}; \mathbf{s}(\mathbf{k})\Big)\\
&+ R_N(\mathbf{F}; \mathbb{P}; H; \mathbf{s}),
\end{aligned}$$

where I_q is the unit matrix of size q, and $\mathbf{s} \mapsto R_N(\mathbf{F}; \mathbb{P}; H; \mathbf{s})$ converges absolutely, defines a holomorphic function in $V_n(\mathbf{u}; \mathbf{d}; h{+}D{-}N{-}1)$, and satisfies in this region the estimate

(4.5) $$R_N(\mathbf{F}; \mathbb{P}; H; \mathbf{s}) = R_N(\mathbf{F}; \mathbb{P}; H; \sigma + i\tau) \ll_{\mathbf{F}, \mathbb{P}, H, N, \sigma} 1 + (1 + |\tau|)^{N+1} \quad (\tau \in \mathbb{R}^v).$$

Proof of Lemma 4.2 For all $\mathbf{s} = (s_1, \dots, s_v) \in V_n(\mathbf{u}, \mathbf{d}, h + D)$ we have

$$\begin{aligned}
&Z_n(\mathbf{F}; \mathbb{P}; H; \mathbf{s})\\
&= \sum_{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n} \frac{H(m_1, \dots, m_n)\, \mathbf{F}(m_1, \dots, m_n)}{\prod_{i=1}^{n} \prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}}\\
&= \sum_{\mathbf{m} = (\mathbf{m}', m_n) \in \mathbb{N}^{n-1} \times \mathbb{N}} \frac{H(\mathbf{m}', m_n)\, \mathbf{F}(\mathbf{m}', m_n)}{\prod_{i=1}^{n} \prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}}\\
&= \sum_{t=1}^{r} \sum_{\mathbf{m}' \in \mathbb{N}^{n-1}} \frac{H(\mathbf{m}', t)\, \mathbf{F}(\mathbf{m}', t)}{\Big(\prod_{i=1}^{n-1} \prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}\Big) \prod_{j=1}^{u(n)} P_{n,j}(\mathbf{m}', t)^{s_{v(n)+j}}}\\
&+ \sum_{\mathbf{m} = (\mathbf{m}', m_n) \in \mathbb{N}^{n-1} \times \mathbb{N}} \frac{H(\mathbf{m}', m_n + r)\, \mathbf{F}(\mathbf{m}', m_n + r)}{\Big(\prod_{i=1}^{n-1} \prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}\Big) \prod_{j=1}^{u(n)} P_{n,j}(\mathbf{m}', m_n + r)^{s_{v(n)+j}}}\\
&= \sum_{t=1}^{r} Z_{n-1}(\mathbf{F}(\,\cdot\,, t); \mathbb{P}^t; H(\,\cdot\,, t); \mathbf{s})\\
&\qquad + \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{\Big(H(\mathbf{m}) + \Delta_r H(\mathbf{m})\Big)\; T_n \mathbf{F}(\mathbf{m})}{\prod_{i=1}^{n} \prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}} \prod_{j=1}^{u(n)} \Big(1 + \frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\Big)^{-s_{v(n)+j}},
\end{aligned}$$

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where we used (2.6) to verify the last equality. Fix $N \in \mathbb{N}_0$. Applying Lemma 3.1 to the above, for any $\mathbf{s} = (s_1, \dots, s_v) \in V_n(\mathbf{u}, \mathbf{d}, h + D)$ we have

$$
\begin{aligned}
&Z_n(\mathbf{F}; \mathbb{P}; H; \mathbf{s}) \\
&= \sum_{t=1}^{r} Z_{n-1}(\mathbf{F}(\,\cdot\,, t); \mathbb{P}^t; H(\,\cdot\,, t); \mathbf{s}) \\
&+ \sum_{\mathbf{m}\in\mathbb{N}^n} \frac{H(\mathbf{m})\; T_n\mathbf{F}(\mathbf{m})}{\prod_{i=1}^{n}\prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^{u(n)}\\ |\mathbf{k}|\le N}} \prod_{j=1}^{u(n)} \binom{-s_{v(n)+j}}{k_j} \left(\frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\right)^{k_j} \\
&+ \sum_{\mathbf{m}\in\mathbb{N}^n} \frac{\Big(\Delta_r H(\mathbf{m})\Big)\; T_n\mathbf{F}(\mathbf{m})}{\prod_{i=1}^{n}\prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^{u(n)}\\ |\mathbf{k}|\le N}} \prod_{j=1}^{u(n)} \binom{-s_{v(n)+j}}{k_j} \left(\frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\right)^{k_j} \\
&+ \sum_{\mathbf{m}\in\mathbb{N}^n} \frac{\Big(H(\mathbf{m}) + \Delta_r H(\mathbf{m})\Big)\; T_n\mathbf{F}(\mathbf{m})}{\prod_{i=1}^{n}\prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}} \\
&\qquad \times G_{u(n),N}\left((s_{v(n)+j})_{1\le j\le u(n)}; \left(\frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\right)_{1\le j\le u(n)} \right) \\
&= \sum_{t=1}^{r} Z_{n-1}(\mathbf{F}(\,\cdot\,, t); \mathbb{P}^t; H(\,\cdot\,, t); \mathbf{s}) \\
&+ T_n \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^{u(n)}\\ |\mathbf{k}|\le N}} \left(\prod_{j=1}^{u(n)} \binom{-s_{v(n)+j}}{k_j} \right) Z_n\left(\mathbf{F}; \mathbb{P}; H \prod_{j=1}^{u(n)} (\Delta_r P_{n,j})^{k_j}; \mathbf{s}(\mathbf{k}) \right) \\
&+ T_n \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^{u(n)}\\ |\mathbf{k}|\le N}} \left(\prod_{j=1}^{u(n)} \binom{-s_{v(n)+j}}{k_j} \right) Z_n\left(\mathbf{F}; \mathbb{P}; (\Delta_r H) \prod_{j=1}^{u(n)} (\Delta_r P_{n,j})^{k_j}; \mathbf{s}(\mathbf{k}) \right) \\
&+ \sum_{\mathbf{m}\in\mathbb{N}^n} \frac{\Big(H(\mathbf{m}) + \Delta_r H(\mathbf{m})\Big)\; T_n\mathbf{F}(\mathbf{m})}{\prod_{i=1}^{n}\prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}} \\
&\qquad \times G_{u(n),N}\left((s_{v(n)+j})_{1\le j\le u(n)}; \left(\frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\right)_{1\le j\le u(n)} \right).
\end{aligned}
$$

The term corresponding to $\mathbf{k} = \mathbf{0}$ of the second sum on the right-hand side is $T_n Z_n(\mathbf{F}; \mathbb{P}; H; \mathbf{s})$. We move this term to the left-hand side. Since 1 is not an eigenvalue

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of T_n, multiplying both sides by $(I_q - T_n)^{-1}$, we obtain (4.4) with

$$(4.6)\quad R_N(\mathbf{F};\mathbb{P};H;\mathbf{s}) := \sum_{\mathbf{m}\in\mathbb{N}^n} \frac{\Big(H(\mathbf{m}) + \Delta_r H(\mathbf{m})\Big)(I_q - T_n)^{-1} T_n \mathbf{F}(\mathbf{m})}{\prod_{i=1}^{n}\prod_{j=1}^{u(i)} P_{i,j}(m_1,\dots,m_i)^{s_{v(i)+j}}} \times G_{u(n),N}\Big((s_{v(n)+j})_{1\le j\le u(n)}; \Big(\frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\Big)_{1\le j\le u(n)}\Big)$$

for any $\mathbf{s} \in V_n(\mathbf{u},\mathbf{d},h+D)$. Let $j = 1,\dots,u_n$. The fact that $\deg\big(\Delta_r P_{n,j}\big) \le \deg(P_{n,j}) - 1$ and Lemma 3.3 implies that $(\Delta_r P_{n,j}(\mathbf{m}))/P_{n,j}(\mathbf{m}) \ll |\mathbf{m}|^{-1}$ uniformly in $\mathbf{m} \in \mathbb{N}^n$. This and Lemma 3.1 imply that

$$\mathbf{s} \mapsto G_{u(n),N}\Big((s_{v(n)+j})_{1\le j\le u(n)}; \Big(\frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\Big)_{1\le j\le u(n)}\Big)$$

is holomorphic in $\mathbb{C}^{u(n)}$ and that

$$G_{u(n),N}\Big((s_{v(n)+j})_{1\le j\le u(n)}; \Big(\frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\Big)_{1\le j\le u(n)}\Big) \ll_{\mathbf{F};\mathbb{P},N,\sigma} \Big(1 + (1+|\tau|)^{N+1}\Big)\frac{1}{|\mathbf{m}|^{N+1}},$$

uniformly in $\tau \in \mathbb{R}^n$ and $\mathbf{m} \in \mathbb{N}^n$. Applying Lemma 3.2 with $T = (I_q - T_n)^{-1}T_n$, we have

$$\begin{aligned}
&\frac{\Big(H(\mathbf{m}) + \Delta_r H(\mathbf{m})\Big)(I_q - T_n)^{-1} T_n \mathbf{F}(\mathbf{m})}{\prod_{i=1}^{n}\prod_{j=1}^{u(i)} P_{i,j}(m_1,\dots,m_i)^{s_{v(i)+j}}} \\
&\qquad \times G_{u(n),N}\Big((s_{v(n)+j})_{1\le j\le u(n)}; \Big(\frac{\Delta_r P_{n,j}(\mathbf{m})}{P_{n,j}(\mathbf{m})}\Big)_{1\le j\le u(n)}\Big) \\
&\ll \frac{(m_1+\cdots+m_n)^{h+D}}{\prod_{i=1}^{n}\prod_{j=1}^{u(i)}(m_1+\cdots+m_i)^{d_{i,j}\sigma_{v(i)+j}}}\Big(1 + (1+|\tau|)^{N+1}\Big)\frac{1}{|\mathbf{m}|^{N+1}} \\
&= \frac{\Big(1 + (1+|\tau|)^{N+1}\Big)}{\Big(\prod_{i=1}^{n-1}(m_1+\cdots+m_i)^{\langle \mathbf{d}_i,\sigma_{v(i)}\rangle}\Big)(m_1+\cdots+m_n)^{\langle \mathbf{d}_n,\sigma_{v(n)}\rangle - h - D + N + 1}}.
\end{aligned}$$

This, (4.6) and (1.2) imply that $\mathbf{s} \mapsto R_N(\mathbf{F};\mathbb{P};H;\mathbf{s})$ converges absolutely, defines a holomorphic function in $V_n(\mathbf{u};\mathbf{d};h+D-N-1)$ and satisfies in this region the estimate

$$R_N(\mathbf{F};\mathbb{P};H;\mathbf{s}) = R_N(\mathbf{F};\mathbb{P};H;\sigma + \sqrt{-1}\tau) \ll_{\mathbf{F},\mathbb{P},H,N,\sigma} 1 + (1+|\tau|)^{N+1} \quad (\tau \in \mathbb{R}^{\nu}).$$

This concludes the proof of Lemma 4.2. ■

4.3 Proof of Theorem 4.1(ii)

We now prove Theorem 4.1(ii). The proof is by induction on $n \in \mathbb{N}$; it will be clear, by using convention (4.3) above, that the proof that "the $n-1$ case implies the n case" also works for the case $n = 1$. Let $n \geq 1$ and we assume that Theorem 4.1(ii) is true for functions of at most $n-1$ indeterminates. We will prove that it remains true for functions of n indeterminates.

In the following we write $f(\mathbf{s}) \in \mathcal{R}(A)$ if $\mathbf{s} \mapsto f(\mathbf{s})$ is holomorphic in (or can be continued holomorphically to) the region A. Let $\mathbf{F}$, $\mathbf{u}$, $\mathbb{P} = (P_{i,j})$, $\mathbf{d} = (d_{i,j})$ be as in the statement of Theorem 4.1. Set

$$\mathcal{H} := \left\{ \frac{H}{\prod_{j=1}^{u(n)} P_{n,j}^{b_j}} \;\middle|\; H \in \mathbb{R}[X_1, \dots, X_n] \text{ and } \mathbf{b} = (b_1, \dots, b_{u(n)}) \in \mathbb{N}_0^{u(n)} \right\}.$$

For $G = H\Big(\prod_{j=1}^{u(n)} P_{n,j}^{b_j}\Big)^{-1} \in \mathcal{H}$ of degree $g := \deg(H) - \langle \mathbf{d}_n, \mathbf{b} \rangle \in \mathbb{Z}$, we define

$$(4.7)\quad Z_n(\mathbf{F}; \mathbb{P}; G; \mathbf{s}) := \sum_{\mathbf{m}=(m_1,\dots,m_n) \in \mathbb{N}^n} \frac{G(m_1, \dots, m_n)\, \mathbf{F}(m_1, \dots, m_n)}{\prod_{i=1}^{n} \prod_{j=1}^{u(i)} P_{i,j}(m_1, \dots, m_i)^{s_{v(i)+j}}} = Z_n(\mathbf{F}; \mathbb{P}; H; \mathbf{s}(\mathbf{b})),$$

where $\mathbf{s}(\mathbf{b}) = \mathbf{s} + \sum_{j=1}^{u(n)} b_j \mathbf{e}_{v(n)+j}$ for $\mathbf{s} = (s_1, \dots, s_\nu)$. Theorem 4.1(i) implies that $\mathbf{s} \mapsto Z_n(\mathbf{F}; \mathbb{P}; G; \mathbf{s})$ converges absolutely in $V_n(\mathbf{u}; \mathbf{d}; g + D)$, and hence

$$(4.8)\qquad Z_n(\mathbf{F}; \mathbb{P}; G; \mathbf{s}) \in \mathcal{R}(V_n(\mathbf{u}; \mathbf{d}; g + D)).$$

Let $M \in \mathbb{N}_0$ be a fixed integer. We will prove by induction on $g = \deg(G)$ that

$$(4.9)\qquad Z_n(\mathbf{F}; \mathbb{P}; G; \mathbf{s}) \in \mathcal{R}(V_n(\mathbf{u}; \mathbf{d}; D - M)).$$

Step 1: The case $g \leq -M$

In this case, it follows from (4.8) that $\mathbf{s} \mapsto Z_n(\mathbf{F}; \mathbb{P}; G; \mathbf{s})$ is a holomorphic function in $V_n(\mathbf{u}; \mathbf{d}; g + D) \subset V_n(\mathbf{u}; \mathbf{d}; D - M)$. Thus, (4.9) holds for $G \in \mathcal{H}$ such that $\deg(G) \leq -M$.

Step 2: The case $g \geq -M + 1$

We will show that if (4.9) holds for all $G \in \mathcal{H}$ with $\deg(G) \leq g - 1$, then it also holds for $G \in \mathcal{H}$ with $\deg(G) = g$.

Let $g \geq -M + 1$ and suppose that (4.9) holds for all $G \in \mathcal{H}$ such that $\deg(G) \leq g - 1$. Let $G = H\big(\prod_{j=1}^{u(n)} P_{n,j}^{b_j}\big)^{-1} \in \mathcal{H}$ such that $\deg(G) = g$. Denote $h = \deg(H)$. Let $N = \max(M + g - 1, 0) \in \mathbb{N}_0$. By using (4.7), (4.8) and Lemma 4.2, for all

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$\mathbf{s} \in V_n(\mathbf{u}; \mathbf{d}; g + D)$ we have

$$(4.10)\quad Z_n(\mathbf{F}; \mathbb{P}; G; \mathbf{s}) = Z_n(\mathbf{F}; \mathbb{P}; H; \mathbf{s}(\mathbf{b}))$$
$$= (I_q - T_n)^{-1} \sum_{t=1}^{r} Z_{n-1}(\mathbf{F}(\,\cdot\,, t); \mathbb{P}^t; H(\,\cdot\,, t); \mathbf{s}(\mathbf{b}))$$
$$+ (I_q - T_n)^{-1} T_n \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{u(n)} \\ 1 \le |\mathbf{k}| \le N}} \left(\prod_{j=1}^{u(n)} \binom{-s_{v(n)+j}}{k_j} \right) Z_n\Big(\mathbf{F}; \mathbb{P}; G_{\mathbf{k}}; \mathbf{s}\Big)$$
$$+ (I_q - T_n)^{-1} T_n \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^{u(n)} \\ |\mathbf{k}| \le N}} \left(\prod_{j=1}^{u(n)} \binom{-s_{v(n)+j}}{k_j} \right) Z_n\Big(\mathbf{F}; \mathbb{P}; G'_{\mathbf{k}}; \mathbf{s}\Big)$$
$$+ R_N(\mathbf{F}; \mathbb{P}; H; \mathbf{s}(\mathbf{b})),$$

where

$$(4.11)\qquad G_{\mathbf{k}} := G \prod_{j=1}^{u(n)} \Big(\frac{\Delta_r P_{n,j}}{P_{n,j}} \Big)^{k_j}; \quad G'_{\mathbf{k}} := \frac{\Delta_r H}{\prod_{j=1}^{u(n)} P_{n,j}^{b_j}} \prod_{j=1}^{u(n)} \Big(\frac{\Delta_r P_{n,j}}{P_{n,j}} \Big)^{k_j}$$

for any $\mathbf{k} \in \mathbb{N}_0^{u(n)}$, and $\mathbf{s} \mapsto R_N(\mathbf{F}; \mathbb{P}; H; \mathbf{s})$ is a holomorphic function in $V_n(\mathbf{u}; \mathbf{d}; h + D - N - 1)$.

It is easy to see that $\mathbf{s}(\mathbf{b}) \in V_n(\mathbf{u}; \mathbf{d}; h + D - N - 1)$ if and only if $\mathbf{s} \in V_n(\mathbf{u}; \mathbf{d}; g + D - N - 1)$. According to our choice of N we have $g + D - N - 1 \le D - M$, and hence $V_n(\mathbf{u}; \mathbf{d}; D - M) \subset V_n(\mathbf{u}; \mathbf{d}; g + D - N - 1)$. Consequently,

$$(4.12)\qquad R_N\big(\mathbf{F}; \mathbb{P}; H; \mathbf{s}(\mathbf{b})\big) \in \mathcal{R}\big(V_n(\mathbf{u}; \mathbf{d}; D - M)\big).$$

Next, the ellipticity of $P_{n,j}$ implies (see Lemma3.3) that $\deg\big(\frac{\Delta_r P_{n,j}}{P_{n,j}}\big) \le -1$. It follows then from (4.11) that

$$\deg(G_{\mathbf{k}}) \le g - |\mathbf{k}| \le g - 1 \quad (\text{for } \mathbf{k} \in \mathbb{N}_0^{u(n)} \setminus \{0\}) \quad \text{and}$$
$$\deg(G'_{\mathbf{k}}) \le g - 1 - |\mathbf{k}| \le g - 1 \quad (\text{for } \mathbf{k} \in \mathbb{N}_0^{u(n)}).$$

Therefore, the induction hypothesis on $\deg(G)$ implies that

$$Z_n(\mathbf{F}; \mathbb{P}; G_{\mathbf{k}}; \mathbf{s}) \in \mathcal{R}(V_n(\mathbf{u}; \mathbf{d}; D - M))$$

(for all $\mathbf{k} \in \mathbb{N}_0^{u(n)} \setminus \{0\}$) and

$$Z_n(\mathbf{F}; \mathbb{P}; G'_{\mathbf{k}}; \mathbf{s}) \in \mathcal{R}(V_n(\mathbf{u}; \mathbf{d}; D - M))$$

(for all $\mathbf{k} \in \mathbb{N}_0^{u(n)}$). Lastly, for any fixed $t \in \{1, \dots, r\}$, the function $\mathbf{F}(., t)\colon \mathbb{N}^{n-1} \to \mathbb{C}^q$ satisfies

$$\mathbf{F}(\mathbf{m}' + r\mathbf{e}'_k, t) = \mathbf{F}(m_1, \dots, m_{k-1}, m_k + r, m_{k+1}, \dots, m_{n-1}, t) = T_k \mathbf{F}(\mathbf{m}', t)$$

$(1 \le k \le n-1)$. It follows by induction hypothesis on n that

$$Z_{n-1}(\mathbf{F}(\,\cdot\,,t);\mathbb{P}^t;H(\,\cdot\,,t);\mathbf{s}(\mathbf{b})) \in \mathcal{R}(\mathbb{C}^\nu). \tag{4.13}$$

Combining (4.12)–(4.13) and (4.10), we conclude that $\mathbf{s} \mapsto Z_{n-1}(\mathbf{F};\mathbb{P};G;\mathbf{s})$ has a holomorphic continuation to the set $V_n(\mathbf{u};\mathbf{d};D-M)$. This ends the induction argument on $g = \deg(G)$.

Since M is arbitrary, by letting $M \to \infty$, we obtain that Theorem 4.1(ii) is also true for n. This also finishes the induction argument on n and completes the proof of Theorem 4.1(ii). ■

4.4 Proof of Theorem 4.1(iii)

We proceed also by induction on $n \in \mathbb{N}$. As in Subsection 4.3, the following argument also works for $n = 1$. Let $n \ge 1$ and assume that Theorem 4.1(iii) is true for functions of at most $n-1$ indeterminates. We will prove that it also remains true for functions of n indeterminates. Actually we prove by induction on $N \in \mathbb{N}_0$ that estimate (4.2) holds uniformly in $\mathbf{s} = \sigma + \sqrt{-1}\tau \in V_n(\mathbf{u};\mathbf{d};D+h-N)$.

When $N = 0$, the result follows from Theorem 4.1(i) and the absolute convergence of the series $Z_n(\mathbf{F};\mathbb{P};H;\mathbf{s})$ in $V_n(\mathbf{u};\mathbf{d};D+h)$.

Now assume that the estimate (4.2) is true for $N(\ge 0)$. We show that it also remains true for $N+1$. Lemma 4.2 and the analytic continuation proved above imply that for any $\mathbf{s} \in \mathbb{C}^\nu$, we have formula (4.4), whose right-hand side we denote as

$$(I_q - T_n)^{-1}\Sigma_1 + (I_q - T_n)^{-1}T_n\Sigma_2 + (I_q - T_n)^{-1}T_n\Sigma_3 + R_N(\mathbf{F};\mathbb{P};H;\mathbf{s}).$$

In the following, we will evaluate each of the above terms.

Step 1

Let $\mathbf{k} \in \mathbb{N}_0^{u_n} \setminus \{0\}$. It is easy to see that

$$h_{\mathbf{k}} := \deg\Big(H\prod_{j=1}^{u_n}(\Delta_r P_{n,j})^{k_j}\Big) \le h + \sum_{j=1}^{u_n} k_j(d_{n,j}-1) = h + \langle \mathbf{k},\mathbf{d}_n\rangle - |\mathbf{k}|. \tag{4.14}$$

In addition, it is also easy to check that $\mathbf{s} \in V_n(\mathbf{u};\mathbf{d};D+h-N-1)$ if and only if

$$\mathbf{s}(\mathbf{k}) = \mathbf{s} + \sum_{j=1}^{u_n} k_j \mathbf{e}_{\nu_n+j} \in V_n(\mathbf{u};\mathbf{d};D+h+\langle \mathbf{k},\mathbf{d}_n\rangle - N - 1).$$

Since

$$D + h + \langle \mathbf{k},\mathbf{d}_n\rangle - N - 1 \ge D - N - 1 + h_{\mathbf{k}} + |\mathbf{k}| \ge h_{\mathbf{k}} + D - N,$$

we see that if $\mathbf{s} \in V_n(\mathbf{u};\mathbf{d};D+h-N-1)$, then $\mathbf{s}(\mathbf{k}) \in V_n(\mathbf{u};\mathbf{d};D+h_{\mathbf{k}}-N)$. It follows then from the above and the induction hypothesis on N that, uniformly in

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$\mathbf{s} \in V_n(\mathbf{u};\mathbf{d};D+h-N-1)$, we have

$$(4.15)\quad \Sigma_1 \ll_{\mathbf{F},\mathbb{P},H,\sigma} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^{u(n)} \\ 1\le|\mathbf{k}|\le N}} \Big(\prod_{j=1}^{u_n} \big(1+|\tau_{v(n)+j}|\big)^{k_j}\Big) \Big(1+\big(1+|\tau|\big)^{1+B_n(\sigma(\mathbf{k});\mathbf{u};\mathbf{d};D+h_{\mathbf{k}})}\Big)$$

$$\ll_{\mathbf{F},\mathbb{P},H,\sigma} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^{u(n)} \\ 1\le|\mathbf{k}|\le N}} \Big\{ \big(1+|\tau|\big)^{|\mathbf{k}|} + \big(1+|\tau|\big)^{|\mathbf{k}|+1+B_n(\sigma(\mathbf{k});\mathbf{u};\mathbf{d};D+h_{\mathbf{k}})}\Big\},$$

where $\sigma(\mathbf{k}) = \sigma + \sum_{j=1}^{u(n)} k_j \mathbf{e}_{v(n)+j}$. By using (4.14) we have

$$(4.16)\quad B_n(\sigma(\mathbf{k});\mathbf{u};\mathbf{d};D+h_{\mathbf{k}})$$

$$= \sup_{1\le i\le n}\Big(D+h_{\mathbf{k}}+n+1-i-\sum_{l=i}^{n-1}\sum_{j=1}^{u(l)} d_{l,j}\sigma_{v(l)+j} - \sum_{j=1}^{u(n)} d_{n,j}(\sigma_{v(n)+j}+k_j)\Big)$$

$$\le \sup_{1\le i\le n}\Big(D+h+\langle\mathbf{k},\mathbf{d}_n\rangle-|\mathbf{k}|+n+1-i-\sum_{l=i}^{n}\sum_{j=1}^{u(l)} d_{l,j}\sigma_{v(l)+j} - \langle\mathbf{k},\mathbf{d}_n\rangle\Big)$$

$$= B_n(\sigma;\mathbf{u};\mathbf{d};D+h)-|\mathbf{k}|.$$

The bounds (4.15) and (4.16) imply that

$$(4.17)\qquad \Sigma_1 \ll_{\mathbf{F},\mathbb{P},H,\sigma} (1+|\tau|)^N + \big(1+|\tau|\big)^{1+B_n(\sigma;\mathbf{u};\mathbf{d};D+h)}$$

uniformly in $\mathbf{s} \in V_n(\mathbf{u};\mathbf{d};D+h-N-1)$.

Step 2

Our argument here is similar to that in Step 1. Let $\mathbf{k} \in \mathbb{N}_0^{u(n)}$. We see that

$$(4.18)\qquad h'_{\mathbf{k}} := \deg\Big((\Delta_r H)\prod_{j=1}^{u(n)}(\Delta_r P_{n,j})^{k_j}\Big) \le h-1+\sum_{j=1}^{u(n)} k_j(d_{n,j}-1)$$

$$= h-1+\langle\mathbf{k},\mathbf{d}_n\rangle-|\mathbf{k}|$$

and

$$D+h+\langle\mathbf{k},\mathbf{d}_n\rangle-N-1 \ge D-N+h'_{\mathbf{k}}+|\mathbf{k}| \ge h'_{\mathbf{k}}+D-N.$$

Hence, as before, we find that if $\mathbf{s} \in V_n(\mathbf{u};\mathbf{d};D+h-N-1)$, then $\mathbf{s}(\mathbf{k}) \in V_n(\mathbf{u};\mathbf{d};D+h'_{\mathbf{k}}-N)$. Therefore, using the induction hypothesis on N, uniformly in $\mathbf{s} \in V_n(\mathbf{u};\mathbf{d};D+h-N-1)$, we have

$$(4.19)\quad \Sigma_2 \ll_{\mathbf{F},\mathbb{P},H,\sigma} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^{u(n)} \\ |\mathbf{k}|\le N}} \Big(\prod_{j=1}^{u(n)}(1+|\tau_{v(n)+j}|)^{k_j}\Big)\Big(1+\big(1+|\tau|\big)^{1+B_n(\sigma(\mathbf{k});\mathbf{u};\mathbf{d};D+h'_{\mathbf{k}})}\Big)$$

$$\ll_{\mathbf{F},\mathbb{P},H,\sigma} \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^{u(n)} \\ |\mathbf{k}|\le N}} \Big\{(1+|\tau|)^{|\mathbf{k}|} + \big(1+|\tau|\big)^{|\mathbf{k}|+1+B_n(\sigma(\mathbf{k});\mathbf{u};\mathbf{d};D+h'_{\mathbf{k}})}\Big\}.$$

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By using (4.18) we have

(4.20)

$$B_n(\sigma(\mathbf{k});\mathbf{u};\mathbf{d};D+h'_{\mathbf{k}})$$

$$= \sup_{1\le i\le n}\Big(D+h'_{\mathbf{k}}+n+1-i-\sum_{l=i}^{n-1}\sum_{j=1}^{u(l)}d_{l,j}\sigma_{\nu(l)+j}-\sum_{j=1}^{u(n)}d_{n,j}(\sigma_{\nu(n)+j}+k_j)\Big)$$

$$\le \sup_{1\le i\le n}\Big(D+h-1+\langle\mathbf{k},\mathbf{d}_n\rangle-|\mathbf{k}|+n+1-i-\sum_{l=i}^{n}\sum_{j=1}^{u(l)}d_{l,j}\sigma_{\nu(l)+j}-\langle\mathbf{k},\mathbf{d}_n\rangle\Big)$$

$$\le B_n(\sigma;\mathbf{u};\mathbf{d};D+h)-|\mathbf{k}|.$$

The bounds (4.19) and (4.20) imply that

$$\Sigma_2 \ll_{\mathbf{F},\mathbb{P},H,\sigma} (1+|\tau|)^N+\big(1+|\tau|\big)^{1+B_n(\sigma;\mathbf{u};\mathbf{d};D+h)} \tag{4.21}$$

uniformly in $\mathbf{s}\in V_n(\mathbf{u};\mathbf{d};D+h-N-1)$.

Step 3

In this step we will use all the notations of Lemma 4.2 introduced at the beginning of Subsection 4.2. The induction hypothesis on n implies that for any $t=1,\dots,r$, we have

$$Z_{n-1}(\mathbf{F}(\,\cdot\,,t);\mathbb{P}^t;H(\,\cdot\,,t);\mathbf{s}) \ll_{\mathbf{F},\mathbb{P},H,\sigma} 1+\big(1+|\tau|\big)^{1+B_{n-1}(\sigma;\mathbf{u}';\mathbf{d}^t;D+h_t)}, \tag{4.22}$$

where $h_t := \deg\big(H(\,\cdot\,,t)\big)$. Also we see that

(4.23) $B_{n-1}(\sigma;\mathbf{u}';\mathbf{d}^t;D+h_t)$

$$= \sup_{1\le i\le n-1}\Big(D+h_t+n-i-\sum_{l=i}^{n-1}\sum_{j=1}^{u(l)'}d^t_{l,j}\sigma_{\nu(l)'+j}\Big)$$

$$= \sup_{1\le i\le n-1}\Big(D+h_t+n-i-\sum_{l=i}^{n-2}\sum_{j=1}^{u(l)}d_{l,j}\sigma_{\nu(l)+j}$$

$$-\sum_{j=1}^{u(n-1)}d_{n-1,j}\sigma_{\nu(n-1)+j}-\sum_{j=u(n-1)+1}^{u(n-1)+u(n)}d_{n,j-u(n-1)}\sigma_{\nu(n)+j-u(n-1)}\Big)$$

$$= \sup_{1\le i\le n-1}\Big(D+h_t+n-i-\sum_{l=i}^{n}\sum_{j=1}^{u(l)}d_{l,j}\sigma_{\nu(l)+j}\Big)$$

$$\le B_n(\sigma;\mathbf{u};\mathbf{d};D+h)-(h-h_t)-1$$

$$\le B_n(\sigma;\mathbf{u};\mathbf{d};D+h)-1.$$

Estimates (4.22) and (4.23) imply that for any $t = 1, \dots, r$, we have

$$Z_{n-1}(\mathbf{F}(\,\cdot\,,t);\mathbb{P}^t;H(\,\cdot\,,t);\mathbf{s}) \ll_{\mathbf{F},\mathbb{P},H,\sigma} 1+\big(1+|\tau|\big)^{B_n(\sigma;\mathbf{u};\mathbf{d};D+h)} \quad (\tau\in\mathbb{R}^\nu). \tag{4.24}$$

Step 4: Conclusion

Combining relation (4.4) and estimates (4.17), (4.21), (4.24) and (4.5), we conclude that

$$\begin{aligned} Z_n(\mathbf{F};\mathbb{P};H;\mathbf{s}) &\ll_{\mathbf{F},\mathbb{P},H,\sigma} \big(1+|\tau|\big)^{N+1} + \big(1+|\tau|\big)^{1+B_n(\sigma;\mathbf{u};\mathbf{d};D+h)} \\ &\ll_{\mathbf{F},\mathbb{P},H,\sigma} \big(1+|\tau|\big)^{1+B_n(\sigma;\mathbf{u};\mathbf{d};D+h)} \end{aligned}$$

uniformly in $\mathbf{s} \in V_n(\mathbf{u};\mathbf{d};D+h-N-1)\setminus V_n(\mathbf{u};\mathbf{d};D+h-N)$. (The last inequality follows from the fact that if $\mathbf{s} \in V_n(\mathbf{u};\mathbf{d};D+h-N-1)\setminus V_n(\mathbf{u};\mathbf{d};D+h-N)$, then $N+1 \le 1+B_n(\sigma;\mathbf{u};\mathbf{d};D+h)$). This concludes the induction argument on N, therefore, also on n and completes the proof of Theorem 4.1 and also of Theorem 2.3. ■

5 Proofs of Theorem 2.1 and Corollary 2.2

There exists some $l \in \{1,\dots,q\}$ such that $\eta_l(i) = 0$ for all $i \in \{1,\dots,n\}$. Then the l-th coordinate of $Z_n(\mathbf{A};\mathbb{P};\mathbf{s})$ coincides with $\zeta_n^{ш}(\mathbf{a};\mathbb{P};\mathbf{s})$. Therefore Theorem 2.1 immediately follows from Theorem 2.3. ■

Next we prove Corollary 2.2. Let $a_i\colon \mathbb{N}\to\mathbb{C}$ $(i=1,\dots,n)$, $\lambda_1,\dots,\lambda_n\in\mathbb{C}$ be as in Corollary 2.2. Let $q = r^n$, $\mathbf{A}\colon \mathbb{N}^n \to \mathbb{C}^q$ the function defined by (2.2) and let $T_1,\dots,T_n \in \mathcal{M}_{q\times q}(\mathbb{C})$ be the matrices defined by (2.3).

For any $i \in \{1,\dots,n\}$ there exists $C_i = C_i(\lambda_i) \ge 0$ such that $|a_i(m)| \le C_i\,|\lambda_i|^{m/r}$ for any $m\in\mathbb{N}$. In fact, writing $m = hr+m_0$ with $h\in\mathbb{N}_0$, $0\le m_0<r$ and using assumption (2.5), we have $a_i(m) = \lambda_i^h a_i(m_0)$. Since $h = (m/r)+O(1)$, the claim follows.

Therefore, for all $\mathbf{m} = (m_1,\dots,m_n)\in\mathbb{N}^n$, we have

$$\Big|\prod_{i=1}^n a_i(m_1+\cdots+m_i)\Big| \le (C_1\cdots C_n)\prod_{k=1}^n\Big|\prod_{i=k}^n\lambda_i\Big|^{m_k/r} \le C_1\cdots C_n$$

by assumption (2.4). Moreover, assumption (2.5) implies that $T_k = (\prod_{i=k}^n\lambda_i)I_q$ for any $k\in\{1,\dots,n\}$. Assumption (2.4) implies then that 1 is not an eigenvalue of any of the matrices $T_1,\dots,T_n$. Therefore Corollary 2.2 follows from Theorem 2.1. ■

6 Examples

In this section, we consider some examples in the cases $n = 1,2$. Indeed, we can explicitly determine several values of them that come down to known results (see Propositions 6.1 and 6.3, and Examples 6.2 and 6.4).

6.1 The Case $n = 1$

First we consider the zeta-function $Z_1(\mathbf{F}; s)$ defined by (2.9), which is equal to $Z_1(\mathbf{F}; X; 1; s)$ in (4.1).

Let $M \in \mathcal{M}_{l\times l}(\mathbb{C})$ and assume that there exists some $k \in \mathbb{N}$ such that each entry of M^m is of $O(m^k)$. Define $\mathbf{F}\colon \mathbb{N} \to \mathbb{C}^{l^2}$ and $Z_1(\mathbf{F}; s) = Z_1(M; s)$ by

$$\mathbf{F}(m) = \mathbf{F}(M; m) = M^m, \quad Z_1(\mathbf{F}; s) = Z_1(M; s) = \sum_{m=1}^{\infty} \frac{M^m}{m^s}.$$

We show that there exists a matrix $T \in \mathcal{M}_{l^2\times l^2}(\mathbb{C})$ such that $\mathbf{F}(m + 1) = T\,\mathbf{F}(m)$ for any $m \in \mathbb{N}$. In fact, writing $M^m = \big(\mu_{ij}^{(m)}\big)$ $(m \in \mathbb{N})$, we may regard $\mathbf{F}(M; m) = M^m$ as

$$\mathbf{F}(M; m) = \begin{pmatrix} \mu_{11}^{(m)} \\ \mu_{12}^{(m)} \\ \vdots \\ \mu_{ll}^{(m)} \end{pmatrix} \in \mathbb{C}^{l^2}.$$

Therefore we see that

$$\begin{aligned}\mathbf{F}(M; m + 1) &= \begin{pmatrix} \sum_{\nu=1}^{l} \mu_{1\nu}^{(m)} \mu_{\nu 1}^{(1)} \\ \sum_{\nu=1}^{l} \mu_{1\nu}^{(m)} \mu_{\nu 2}^{(1)} \\ \vdots \\ \sum_{\nu=1}^{l} \mu_{l\nu}^{(m)} \mu_{\nu l}^{(1)} \end{pmatrix} = T \begin{pmatrix} \mu_{11}^{(m)} \\ \mu_{12}^{(m)} \\ \vdots \\ \mu_{ll}^{(m)} \end{pmatrix} \\ &= T\mathbf{F}(M; m),\end{aligned}$$

where

$$T = {}^tM \oplus {}^tM \oplus \cdots \oplus {}^tM = {}^tM^{\oplus l} \in \mathcal{M}_{l^2\times l^2}(\mathbb{C}).$$

Suppose that 1 is not an eigenvalue of M, namely not an eigenvalue of T. Then it follows from Theorem 2.3 that $Z_1(M; s)$ can be continued holomorphically to $\mathbb{C}$. In particular when $l = 1$, from Theorem 2.3, we can recover the known fact that for $x \in \mathbb{C}$ with $|x| \le 1$ and $x \ne 1$, the polylogarithm

$$\mathrm{Li}(s; x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}$$

can be continued holomorphically to $\mathbb{C}$.

As an example, we consider the zeta-function associated with the Fibonacci numbers $\{F_n\}_{n\ge 0}$, which are defined by the following linear recurrence relation:

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1} \quad (n \in \mathbb{N}). \tag{6.1}$$

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We recall the well-known results (see, for example, [30]). Let $\alpha = (1+\sqrt{5})/2$ be the golden ratio. Then

$$F_n = \frac{1}{\sqrt{5}}\big(\alpha^n - (-\alpha)^{-n}\big) \quad (n \in \mathbb{N}). \tag{6.2}$$

From the recurrence relation (6.1), we see that

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \quad (n \in \mathbb{N}),$$

namely,

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}, \quad \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_2 \\ F_1 \end{pmatrix} \quad (n \in \mathbb{N}).$$

Therefore we have

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n} \quad (n \in \mathbb{N}). \tag{6.3}$$

Set $Q = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which is often called the Fibonacci matrix, and $S_d = \xi_d \alpha^{-1} Q$ for $d \in \mathbb{N}$, where $\xi_d = e^{2\pi i/d}$ is the d-th primitive root of unity. Now we assume $d \geq 2$. Then, we can easily check that the eigenvalues of S_d are ξ_d and $-\xi_d \alpha^{-2}$ that are not equal to 1. Now we consider

$$Z_1(S_d; s) = \sum_{m=1}^{\infty} \frac{S_d^m}{m^s} = \sum_{m=1}^{\infty} \frac{(\xi_d \alpha^{-1} Q)^m}{m^s},$$

which can be continued holomorphically to $\mathbb{C}$ if $d > 1$, by Theorem 2.3.

In order to evaluate $Z_1(S_d; s)$ at nonpositive integers, we recall the Frobenius–Euler numbers $\{\widetilde{E}_n(\lambda)\}$ (see [18], also [6]) defined by

$$\mathcal{G}(t;\lambda) = \frac{1-\lambda}{e^t - \lambda} = \sum_{n=0}^{\infty} \widetilde{E}_n(\lambda) \frac{t^n}{n!} \quad (\lambda \in \mathbb{C};\ \lambda \neq 1),$$

where $|t| < \sqrt{(\log|\lambda|)^2 + (\arg\lambda)^2}$ with $-\pi \leq \arg\lambda < \pi$. We can check that

$$\widetilde{E}_0(\lambda) = 1,\ \widetilde{E}_1(\lambda) = \frac{1}{\lambda - 1},\ \widetilde{E}_2(\lambda) = \frac{\lambda+1}{(\lambda-1)^2}, \ldots.$$

Then we obtain the following.

Proposition 6.1 *For $h, d \in \mathbb{N}$ with $d \geq 2$ and $(h,d) = 1$,*

(6.4)

$$Z_1(S_d^h; s) = \frac{1}{\sqrt{5}} \begin{pmatrix} \alpha \mathrm{Li}(s; \xi_d^h) + \alpha^{-1} \mathrm{Li}(s; (-\xi_d)^h \alpha^{-2h}) & \mathrm{Li}(s; \xi_d^h) - \mathrm{Li}(s; (-\xi_d)^h \alpha^{-2h}) \\ \mathrm{Li}(s; \xi_d^h) - \mathrm{Li}(s; (-\xi_d)^h \alpha^{-2h}) & \alpha^{-1} \mathrm{Li}(s; \xi_d^h) + \alpha \mathrm{Li}(s; (-\xi_d)^h \alpha^{-2h}) \end{pmatrix}.$$

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In particular, for $N \in \mathbb{N}_0$*,*

$$(6.5)\quad Z_1(S_2;-N)=\frac{1}{\sqrt{5}}\times$$
$$\begin{pmatrix} \alpha(1-2^{N+1})\frac{B_{N+1}}{N+1}-\frac{\alpha^{-3}(-1)^{N+1}}{1-\alpha^{-2}}\widetilde{E}_N(\alpha^{-2}) & (1-2^{N+1})\frac{B_{N+1}}{N+1}+\frac{\alpha^{-2}(-1)^{N+1}}{1-\alpha^{-2}}\widetilde{E}_N(\alpha^{-2}) \\ (1-2^{N+1})\frac{B_{N+1}}{N+1}+\frac{\alpha^{-2}(-1)^{N+1}}{1-\alpha^{-2}}\widetilde{E}_N(\alpha^{-2}) & \alpha^{-1}(1-2^{N+1})\frac{B_{N+1}}{N+1}-\frac{\alpha^{-1}(-1)^{N+1}}{1-\alpha^{-2}}\widetilde{E}_N(\alpha^{-2}) \end{pmatrix}.$$

Proof For our purpose, here we give some formulas on special values of polylogarithms. First, since $\mathrm{Li}(s;-1)=(2^{1-s}-1)\zeta(s)$, we have

$$(6.6)\qquad \mathrm{Li}(-N;-1)=-(2^{N+1}-1)\frac{B_{N+1}}{N+1}\quad (N\in\mathbb{N}_0),$$

where the Bernoulli numbers $\{B_n\}$ are defined by $te^t/(e^t-1)=\sum_{n\geq 0}B_nt^n/n!$ (see [31, Chap. 13]). We prove, when $|\lambda|\leq 1$ and $\lambda\neq 1$, that

$$(6.7)\qquad \mathrm{Li}(-N;\lambda)=\frac{\lambda(-1)^N}{1-\lambda}\widetilde{E}_N(\lambda)\quad (N\in\mathbb{N}_0).$$

If $|\lambda|<1$, this follows immediately from

$$\mathcal{G}(t;\lambda)=\frac{1-\lambda}{\lambda}\sum_{m=1}^{\infty}\lambda^m e^{-mt}=\frac{1-\lambda}{\lambda}\sum_{n=0}^{\infty}(-1)^n\Big\{\sum_{m=1}^{\infty}\lambda^m m^n\Big\}\frac{t^n}{n!}.$$

A proof of (6.7) for the general case can be obtained by using the contour integral expression of polylogarithms, but here we show an alternative proof in the frame of our present method. We consider (4.4) and (4.6) in the case that $n=1$, $\mathbf{F}(m)=\lambda^m$ ($\lambda\in\mathbb{C}\setminus\{1\}$; $|\lambda|\leq 1$), $\mathbb{P}=(X)$, $H=1$, $q=1$, $r=1$, and $T_1=\lambda$. Putting $s=-N$ in (4.4) and (4.6), we have

$$\begin{aligned}(6.8)\quad Z_1(\mathbf{F};-N)&=\frac{\lambda}{1-\lambda}\Big(\sum_{k=1}^{N}\binom{N}{k}Z_1(\mathbf{F};k-N)\Big)+\frac{\lambda}{1-\lambda}\\ &=\frac{\lambda}{1-\lambda}\Big(\sum_{k=0}^{N}\binom{N}{k}Z_1(\mathbf{F};k-N)\Big)-\frac{\lambda}{1-\lambda}Z_1(\mathbf{F};-N)+\frac{\lambda}{1-\lambda}.\end{aligned}$$

Let

$$(6.9)\qquad G(t)=\sum_{N=0}^{\infty}Z_1(\mathbf{F};-N)\frac{t^N}{N!}.$$

Multiplying both sides of (6.8) by $t^N/N!$ and summing up with respect to N, we obtain

$$G(t)=\frac{\lambda}{1-\lambda}G(t)e^t-\frac{\lambda}{1-\lambda}G(t)+\frac{\lambda}{1-\lambda}e^t,$$

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from which we have

$$(6.10)\qquad G(t)=\frac{\lambda}{e^{-t}-\lambda}=\frac{\lambda}{1-\lambda}\frac{1-\lambda}{e^{-t}-\lambda}=\frac{\lambda}{1-\lambda}\sum_{N=0}^{\infty}\widetilde{E}_N(\lambda)\frac{(-t)^N}{N!}.$$

Comparing (6.9) and (6.10), we obtain (6.7).

By (6.2) and (6.3), we have

$$(6.11)\qquad \begin{aligned} S_d^n &= \frac{1}{\sqrt{5}}\Big(\xi_d\alpha^{-1}\Big)^n\begin{pmatrix}\alpha^{n+1}-(-\alpha)^{-n-1} & \alpha^n-(-\alpha)^{-n}\\ \alpha^n-(-\alpha)^{-n} & \alpha^{n-1}-(-\alpha)^{-n+1}\end{pmatrix}\\ &= \frac{1}{\sqrt{5}}\begin{pmatrix}\xi_d^n\alpha+(-\xi_d)^n\alpha^{-2n-1} & \xi_d^n-(-\xi_d)^n\alpha^{-2n}\\ \xi_d^n-(-\xi_d)^n\alpha^{-2n} & \xi_d^n\alpha^{-1}+(-\xi_d)^n\alpha^{-2n+1}\end{pmatrix},\end{aligned}$$

which gives (6.4). In particular, we have

$$(6.12)\qquad Z_1(S_d;s)=\frac{1}{\sqrt{5}}\begin{pmatrix}\alpha\mathrm{Li}(s;\xi_d)+\alpha^{-1}\mathrm{Li}(s;-\xi_d\alpha^{-2}) & \mathrm{Li}(s;\xi_d)-\mathrm{Li}(s;-\xi_d\alpha^{-2})\\ \mathrm{Li}(s;\xi_d)-\mathrm{Li}(s;-\xi_d\alpha^{-2}) & \alpha^{-1}\mathrm{Li}(s;\xi_d)+\alpha\mathrm{Li}(s;-\xi_d\alpha^{-2})\end{pmatrix}.$$

Applying (6.6) and (6.7) to the right-hand side of (6.12), we can explicitly evaluate the values $Z_1(S_d;-N)$ $(N\in\mathbb{N}_0)$. For example, since $\xi_2=-1$, we obtain (6.5). ■

Example 6.2 It is known (see [21]) that $\mathrm{Li}(2;-1)=-\frac{1}{2}\zeta(2)=-\frac{1}{12}\pi^2$, and

$$\mathrm{Li}(2;\alpha^{-2})=\mathrm{Li}\Big(2;\frac{3-\sqrt{5}}{2}\Big)=\frac{1}{15}\pi^2-\Big\{\log\Big(\frac{\sqrt{5}-1}{2}\Big)\Big\}^2.$$

Hence, by (6.2), we obtain

$$\sum_{m\geq 1}\frac{(-\alpha)^{-m}F_m}{m^2}=\frac{1}{\sqrt{5}}\Big(-\frac{3}{20}\pi^2+\Big\{\log\Big(\frac{\sqrt{5}-1}{2}\Big)\Big\}^2\Big).$$

Using this, we can evaluate $Z_1(M;2)$.

6.2 The Case $n=2$

Next we consider the case $n=2$ and $\mathbb{P}=(X_1;X_1+X_2)$. Let $M_1,M_2\in\mathcal{M}_{l\times l}(\mathbb{C})$ with the assumption that $M_1M_2=M_2M_1$. We define $\mathbf{F}\colon\mathbb{N}^2\to\mathbb{C}^{l^2}$ and $Z_2(\mathbf{F};s_1,s_2)=Z_2(M_1,M_2;s_1,s_2)$ by

$$\mathbf{F}(m_1,m_2)=\mathbf{F}(M_1,M_2;m_1,m_2)=M_1^{m_1}M_2^{m_2},$$

$$Z_2(M_1,M_2;s_1,s_2)=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{M_1^{m_1}M_2^{m_2}}{m_1^{s_1}(m_1+m_2)^{s_2}}.$$

Then, as well as the above consideration in the case $n = 1$, it follows from Theorem 2.3 that if 1 is not an eigenvalue of M_1, M_2 and M_1M_2, then $Z_2(M_1M_2, M_2; s_1, s_2)$ and $Z_2(M_1, M_1M_2; s_1, s_2)$ can be continued holomorphically to $\mathbb{C}^2$.

Using the well-known $*$-product argument in the study of multiple zeta values, we have

$$(6.13)\quad Z_1(M_1; s_1)Z_1(M_2; s_2) = \Big(\sum_{1\le m<n} + \sum_{1\le n<m} + \sum_{1\le m=n}\Big)\frac{M_1^m M_2^n}{m^{s_1}n^{s_2}} = Z_2(M_1M_2, M_2; s_1, s_2) + Z_2(M_1, M_1M_2; s_2, s_1) + Z_1(M_1M_2; s_1 + s_2).$$

From Theorem 2.3, we see that (6.13) holds for all $(s_1, s_2) \in \mathbb{C}^2$.

Proposition 6.3 *The double series*

$$\phi(s) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(i\alpha^{-1})^{2m+n}F_{2m+n} + (i\alpha^{-1})^{m+2n}F_{m+2n}}{m^s(m+n)^s}$$

can be continued meromorphically to $\mathbb{C}$*. In particular,*

$$\phi(0) = \frac{1}{18}\Big\{6 - \sqrt{5} + (2 - 3\sqrt{5})i\Big\}.$$

Proof Putting $M_1 = M_2 = S_4 (= i\alpha^{-1}Q)$ and $s_1 = s_2 = s$ in (6.13), we see that

$$Z_2(S_4^2, S_4; s, s) + Z_2(S_4, S_4^2; s, s) = Z_1(S_4; s)^2 - Z_1(S_4^2; 2s)$$

holds for all $s \in \mathbb{C}$ because 1 is not an eigenvalue of S_4 and S_4^2. Compare the $(1, 2)$-entries of the both sides of the above formula. Using (6.3), (6.11), (6.12), and (6.4) and the fact $\alpha + \alpha^{-1} = \sqrt{5}$, we have

$$(6.14)\quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(i\alpha^{-1})^{2m+n}F_{2m+n} + (i\alpha^{-1})^{m+2n}F_{m+2n}}{m^s(m+n)^s} = \frac{1}{\sqrt{5}}\Big\{\mathrm{Li}(s; i)^2 - \mathrm{Li}(s; -i\alpha^{-2})^2\Big\} - \frac{1}{\sqrt{5}}\Big\{\mathrm{Li}(s; -1) - \mathrm{Li}(s; -\alpha^{-4})\Big\},$$

which gives meromorphic continuation of $\phi(s)$. In particular, by (6.6), (6.7), and (6.14), we have

$$\begin{aligned}\phi(0) &= \frac{1}{\sqrt{5}}\Big\{\mathrm{Li}(0; i)^2 - \mathrm{Li}(0; -i\alpha^{-2})^2 - \big(\mathrm{Li}(0; -1) - \mathrm{Li}(0; -\alpha^{-4})\big)\Big\}\\ &= \frac{1}{\sqrt{5}}\Big\{\Big(\frac{i}{1-i}\Big)^2 - \Big(\frac{-i\alpha^{-2}}{1+i\alpha^{-2}}\Big)^2 + \frac{1}{2} - \frac{\alpha^{-4}}{1+\alpha^{-4}}\Big\}\\ &= \frac{1}{18}\big\{6 - \sqrt{5} + (2 - 3\sqrt{5})i\big\}.\end{aligned}$$

■

Example 6.4 Consider the case $S_1 = \alpha^{-1}Q$. We set $M = S_1$. Then it follows from (6.3) and (6.11) that the $(1,2)$-entry of M^m is

$$\alpha^{-m}F_m = (1-(-\alpha^{-2})^m)/\sqrt{5} = O(1) = O(m^{1-\varepsilon}).$$

Hence, by (2.13) in the case $k=1$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha^{-m}F_m+\alpha^{-n}F_n-\alpha^{-m-n}F_{m+n}}{m(m+n)^2} = \sum_{m=1}^{\infty}\frac{\alpha^{-m}F_m}{m^3}.$$

This is a sum formula with Fibonacci numbers on the numerator. Note that this also comes from (2.12).

7 Proof of Theorem 2.4

The method given here is essentially the same as the one introduced in the proof of [27, Theorem 2.1]. We begin by recalling the well-known result

$$\sum_{m=1}^{\infty}\frac{(-1)^m\sin(m\theta)}{m} = -\frac{\theta}{2} \quad (-\pi<\theta<\pi), \tag{7.1}$$

where the left-hand side is uniformly convergent in the wider sense with respect to $\theta \in (-\pi,\pi)$ (see [31, § 3.35 and § 9.11]). It is also known that

$$\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m} \quad (-\pi<\theta<\pi) \tag{7.2}$$

is convergent uniformly in the wider sense, whose value we denote by $C(\theta)$. For $k \in \mathbb{N}$ and $x \in \mathbb{R}$ with $0 \le x < 1$, let

(7.3)

$$\begin{aligned}\mathbf{H}(\theta;x;k) &= 2\Big(\sum_{m=1}^{\infty}\frac{(-1)^m x^m\mathbf{F}(m)\cos(m\theta)}{m^k}\Big)\Big(\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\theta)}{n}+\frac{\theta}{2}\Big)\\ &= \sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}x^m\mathbf{F}(m)\sin((m+n)\theta)}{m^k n}\\ &\quad - \sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{(-1)^{m+n}x^m\mathbf{F}(m)\sin((m-n)\theta)}{m^k n} + \theta\sum_{m=1}^{\infty}\frac{(-1)^m x^m\mathbf{F}(m)\cos(m\theta)}{m^k},\end{aligned}$$

which is uniformly convergent in the wider sense with respect to $\theta \in (-\pi,\pi)$. By (7.1), we see that

$$\mathbf{H}(\theta;x;k) = 0 \quad (\theta \in (-\pi,\pi)).$$

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Therefore we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\theta\,\mathbf{H}(\theta;x;k)d\theta=0. \tag{7.4}$$

By partial integration, it follows from (7.3) that

$$-\sum_{m,n=1}^{\infty}\frac{x^m\mathbf{F}(m)}{m^k n(m+n)}+\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{x^m\mathbf{F}(m)}{m^k n(m-n)}+2\sum_{m=1}^{\infty}\frac{x^m\mathbf{F}(m)}{m^{k+2}}=0. \tag{7.5}$$

Setting $l=m-n$ and $j=n-m$ in the second term on the left-hand side of (7.5) according as $m>n$ and $m<n$ respectively, we obtain

$$\sum_{m,n=1}^{\infty}\frac{x^{m+n}\mathbf{F}(m+n)}{mn(m+n)^k}-2\sum_{m,n=1}^{\infty}\frac{x^m\mathbf{F}(m)}{m^k n(m+n)}+2\sum_{m=1}^{\infty}\frac{x^m\mathbf{F}(m)}{m^{k+2}}=0. \tag{7.6}$$

Moreover, using the relation

$$\frac{1}{ab}=\left(\frac{1}{a}+\frac{1}{b}\right)\frac{1}{a+b}, \tag{7.7}$$

we see that the first term of the left-hand side of (7.6) tends to $2Z_2(\mathbf{F}_3;1,k+1)$ as $x\to 1$. As for the second term of (7.6), using (7.7) repeatedly, we have

$$\begin{aligned}\sum_{m,n=1}^{\infty}\frac{x^m\mathbf{F}(m)}{m^k n(m+n)}&=\sum_{m,n=1}^{\infty}\frac{x^m\mathbf{F}(m)}{m^k(m+n)^2}+\sum_{m,n=1}^{\infty}\frac{x^m\mathbf{F}(m)}{m^{k-1}n(m+n)^2}\\&=\cdots\\&=\sum_{h=2}^{k+1}\sum_{m,n=1}^{\infty}\frac{x^m\mathbf{F}(m)}{m^{k+2-h}(m+n)^h}+\sum_{m,n=1}^{\infty}\frac{x^m\mathbf{F}(m)}{n(m+n)^{k+1}}.\end{aligned} \tag{7.8}$$

We see that each side of (7.8) is absolutely and uniformly convergent with respect to $x\in[0,1]$. Hence (7.8) holds for $x=1$. Thus we have (2.10). This completes the proof. ■

Remark On the right-hand side of (7.3) the order of summation can be interchanged. If $\mathbf{F}=1$, this is true even in the case $x=1$, $k=1$. In fact, let

$$a_{MN}=\sum_{m\le M}\sum_{n\le N}\frac{(-1)^{m+n}\sin((m+n)\theta)}{mn}.$$

Then by (7.1) and (7.2) we have

$$\begin{aligned}&\lim_{M\to\infty}a_{MN}\\&=\sum_{n\le N}\frac{(-1)^n}{n}\Big\{\cos(n\theta)\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sin(m\theta)+\sin(n\theta)\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\cos(m\theta)\Big\}\\&=\sum_{n\le N}\frac{(-1)^n}{n}\Big\{-\frac{\theta}{2}\cos(n\theta)+C(\theta)\sin(n\theta)\Big\}=\alpha_N,\end{aligned}$$

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say. Then $\alpha_N \to -\theta C(\theta)$ as $N \to \infty$. The convergence of (7.1) and (7.2) implies the existence of $A > 0$, independent of N, for which $\left|\sum_{n\le N}(-1)^n n^{-1}\mathrm{cs}(n\theta)\right| < A$ holds for any N (where cs stands for sin or cos). On the other hand, for any $\varepsilon > 0$, there exists a sufficiently large $M = M(\varepsilon)$ for which $\left|\sum_{m\ge M}(-1)^m m^{-1}\mathrm{cs}(m\theta)\right| < \varepsilon$ holds. Therefore

$$|\alpha_N - a_{MN}| \le \Big|\sum_{n\le N}\frac{(-1)^n}{n}\cos(n\theta)\Big|\cdot\Big|\sum_{m\ge M}\frac{(-1)^m}{m}\cos(m\theta)\Big| + \Big|\sum_{n\le N}\frac{(-1)^n}{n}\sin(n\theta)\Big|\cdot\Big|\sum_{m\ge M}\frac{(-1)^m}{m}\sin(m\theta)\Big| < 2A\varepsilon,$$

which implies that the convergence $a_{MN} \to \alpha_N$ (as $M \to \infty$) is uniform in N. Then by a well-known property of double series we can conclude that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}\sin((m+n)\theta)}{mn} = -\theta C(\theta) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}\sin((m+n)\theta)}{mn}.$$

The case involving $\sin((m-n)\theta)$ is similar. The situation (7.3) is simpler because of the factor x^m, $0 \le x < 1$.

8 More General Form of Vectorial Sum Formulas

Based on the consideration in the previous section, we give a generalization of the result in Theorem 2.4, namely, a certain sum formula for values of vectorial zeta-functions (1.6).

We start with the following elementary lemma that can be immediately proved by induction. Note that here and from now on, the empty sum (resp. the empty product) implies 0 (resp. 1).

Lemma 8.1 *For $r \in \mathbb{N}$,*

$$\sin\Big(\sum_{j=1}^{r} x_j\Big) = \sum_{j=1}^{r}\Big(\prod_{\nu=1}^{j-1}\cos x_\nu\Big)\cdot\sin x_j\cdot\Big(\cos\Big(\sum_{\rho=j+1}^{r} x_\rho\Big)\Big).$$

Corresponding to this relation, we define

$$\text{(8.1)}\qquad G_r(\theta) = \sum_{j=1}^{r}\sum_{m_1=1}^{\infty}\cdots\sum_{m_{j-1}=1}^{\infty}\prod_{\nu=1}^{j-1}\frac{(-1)^{m_\nu}\cos(m_\nu\theta)}{m_\nu} \times\Big(\sum_{m_j=1}^{\infty}\frac{(-1)^{m_j}\sin(m_j\theta)}{m_j}+\frac{\theta}{2}\Big) \times\sum_{m_{j+1}=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{(-1)^{\sum_{\rho=j+1}^{r} m_\rho}\cos\Big(\Big(\sum_{\rho=j+1}^{r} m_\rho\Big)\theta\Big)}{\prod_{\rho=j+1}^{r} m_\rho}.$$

As noted at the beginning of the previous section, the right-hand side of (8.1) is uniformly convergent in the wider sense with respect to $\theta \in (-\pi, \pi)$, so is continuous. The order of the last multiple sum on the right-hand side can be interchanged freely, which can be seen as in the remark at the end of the last section. By (7.1), we see that

$$G_r(\theta) = 0 \quad (-\pi < \theta < \pi). \tag{8.2}$$

Similarly to Theorem 2.4, let $\mathbf{F} = (f_1, \dots, f_q) \colon \mathbb{N} \to \mathbb{C}^q$ be a function that satisfies that, for a fixed $k \in \mathbb{N}$, $f_j(m) = O(m^{k-\varepsilon})$ $(1 \le j \le q)$. For $r \in \mathbb{N}$ with $r \ge 2$ and $x \in [0, 1)$, we define

$$\mathbf{H}_r(\theta; x; k) = 2\bigg(\sum_{l=1}^{\infty} \frac{(-1)^l x^l \mathbf{F}(l) \cos(l\theta)}{l^k}\bigg) G_{r-1}(\theta). \tag{8.3}$$

Note that $\mathbf{H}_2(\theta; x; k) = \mathbf{H}(\theta; x; k)$ defined by (7.3). As a multiple analogue of (7.4), we obtain, from (8.2), the following integral representation.

Proposition 8.2 *For $r \in \mathbb{N}$ with $r \ge 2$,*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \theta\, \mathbf{H}_r(\theta; x; k) d\theta = 0. \tag{8.4}$$

This may be regarded as a "primitive" form of vectorial sum formulas. Indeed, as we considered in the previous section, the integral representation (8.4) in the case $r = 2$ gives a vectorial sum formula for double zeta-functions. Similarly, we consider the case $r = 3$ and prove the following.

Theorem 8.3 *For $K \in \mathbb{N}$ with $K > 3$,*

$$\begin{aligned} &\sum_{\substack{k_1,k_2 \ge 1,\ k_3 \ge 2 \\ k_1+k_2+k_3=K}} \bigg\{ \sum_{m_1,m_2,m_3 \in \mathbb{N}} \frac{\mathbf{F}(m_1)}{m_1^{k_1}(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}} \bigg\} \\ &\quad + \sum_{\substack{k_2 \ge 1,\ k_3 \ge 2 \\ k_2+k_3=K-1}} \bigg\{ \sum_{m_1,m_2,m_3 \in \mathbb{N}} \frac{\mathbf{F}(m_2) - \mathbf{F}(m_1+m_2)}{m_1(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}} \bigg\} \\ &\quad + \sum_{m_1,m_2,m_3 \in \mathbb{N}} \frac{\mathbf{F}(m_3) - \mathbf{F}(m_1+m_3) - \mathbf{F}(m_2+m_3) + \mathbf{F}(m_1+m_2+m_3)}{m_1(m_1+m_2)(m_1+m_2+m_3)^{K-2}} \\ &= \sum_{m=1}^{\infty} \frac{\mathbf{F}(m)}{m^K} \end{aligned} \tag{8.5}$$

holds.

In particular when $\mathbf{F}(\cdot) = 1$, we can see that (8.5) implies the ordinary sum formula for triple zeta values

$$\sum_{\substack{k_1,k_2 \ge 1,\ k_3 \ge 2 \\ k_1+k_2+k_3=K}} \bigg\{ \sum_{m_1,m_2,m_3 \in \mathbb{N}} \frac{1}{m_1^{k_1}(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}} \bigg\} = \zeta(K)$$

for $K > 3$.

In view of Theorems 2.4 and 8.3, we propose the following conjecture that implies vectorial sum formulas for multiple series.

Conjecture 8.4 For $r \in \mathbb{N}$ and $K \in \mathbb{N}$ with $K > r$,

$$\sum_{\substack{k_1,\dots,k_{r-1}\geq 1,\ k_r\geq 2\\ k_1+\cdots+k_r=K}} \Big\{ \sum_{m_1,m_2,\dots,m_r\in\mathbb{N}} \frac{\mathbf{F}(m_1)}{m_1^{k_1}(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}\cdots(\sum_{j=1}^r m_j)^{k_r}} \Big\}$$
$$+ \sum_{\substack{k_2,\dots,k_{r-1}\geq 1,\ k_r\geq 2\\ k_2+\cdots+k_r=K-1}} \Big\{ \sum_{m_1,m_2,\dots,m_r\in\mathbb{N}} \frac{\mathbf{F}(m_2)-\mathbf{F}(m_1+m_2)}{m_1(m_1+m_2)^{k_2}(m_1+m_2+m_3)^{k_3}\cdots(\sum_{j=1}^r m_j)^{k_r}} \Big\}$$
$$+ \sum_{\substack{k_3,\dots,k_{r-1}\geq 1,\ k_r\geq 2\\ k_3+\cdots+k_r=K-2}} \Big\{ \sum_{m_1,m_2,\dots,m_r\in\mathbb{N}} \frac{\mathbf{F}(m_3)-\mathbf{F}(m_1+m_3)-\mathbf{F}(m_2+m_3)+\mathbf{F}(m_1+m_2+m_3)}{m_1(m_1+m_2)(m_1+m_2+m_3)^{k_3}\cdots(\sum_{j=1}^r m_j)^{k_r}} \Big\}$$
$$+\cdots$$
$$+ \sum_{m_1,m_2,\dots,m_r\in\mathbb{N}} \frac{\mathbf{F}(m_r)-\sum_{j<r}\mathbf{F}(m_j+m_r)+\sum_{j_1<j_2<r}\mathbf{F}(m_{j_1}+m_{j_2}+m_r)-\cdots}{m_1(m_1+m_2)(m_1+m_2+m_3)\cdots(\sum_{j=1}^{r-1} m_j)(\sum_{j=1}^r m_j)^{K-r+1}}$$
$$= \sum_{m=1}^{\infty} \frac{\mathbf{F}(m)}{m^K}$$

holds.

For example, the case $K = r+1$ (so that the only possible choice is $(k_1,k_2,\dots,k_{r-1},k_r) = (1,1,\dots,1,2)$) implies that

$$(8.6)\quad \sum_{m_1,m_2,\dots,m_r\in\mathbb{N}} \frac{\sum_{j=1}^r \mathbf{F}(m_j)-\sum_{j_1<j_2}\mathbf{F}(m_{j_1}+m_{j_2})+\cdots+(-1)^{r-1}\mathbf{F}(\sum_{j=1}^r m_j)}{m_1(m_1+m_2)(m_1+m_2+m_3)\cdots(\sum_{j=1}^{r-1} m_j)(\sum_{j=1}^r m_j)^2}$$
$$= \sum_{m=1}^{\infty} \frac{\mathbf{F}(m)}{m^{r+1}}.$$

In particular when $\mathbf{F}(\cdot) = 1$, (8.6) coincides with the well-known formula

$$\zeta_{EZ,r}(1,1,\dots,1,2) = \zeta(r+1).$$

In fact, we can numerically check formula (8.6) in the case $r = 4$.

In the rest of this section, we give a proof of Theorem 8.3. First we prove the following lemma. For simplicity, we put $\widetilde{\mathbf{F}}(m;x) = x^m\mathbf{F}(m)$ for $x \in [0,1)$.

Lemma 8.5 *With the above notation, and for $k \in \mathbb{N}$ and $x \in [0,1)$,*

$$(8.7)\quad \sum_{l,m,n=1}^{\infty} \frac{\widetilde{\mathbf{F}}(m+n;x)}{ln(l+m)(m+n)^k} = 3\sum_{l,m,n=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+m+n;x)}{l(l+m)(l+m+n)^{k+1}} + \sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+m;x)}{lm^2(l+m)^k},$$

$$(8.8)\quad \sum_{l,m,n=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l;x)}{l^k n(l+m)(m+n)} = \sum_{l,m,n=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l;x)}{l^k m(m+n)(l+m+n)} + \sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l;x)}{l^k m^2(l+m)}.$$

Proof The left-hand side of (8.7) is equal to

$$\sum_{m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(m+n;x)}{n(m+n)^k}\sum_{l=1}^{\infty}\frac{1}{l(l+m)}=\sum_{m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(m+n;x)}{mn(m+n)^k}\sum_{l=1}^{\infty}\Big(\frac{1}{l}-\frac{1}{l+m}\Big)$$
$$=\sum_{m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(m+n;x)}{mn(m+n)^k}\sum_{l=1}^{m}\frac{1}{l}.$$

We divide the inner sum on the right-most side into two parts according as $l=m$ and $l<m$, and we set $m=l+j$ ($j\in\mathbb{N}$) in the latter case. Then the right-most side is equal to

$$\sum_{m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(m+n;x)}{m^2n(m+n)^k}+\sum_{l,j,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+j+n;x)}{ln(l+j)(l+j+n)^k}. \tag{8.9}$$

Using the relation

$$\frac{1}{a(b+c)}=\frac{1}{a+b+c}\Big(\frac{1}{a}+\frac{1}{b+c}\Big) \tag{8.10}$$

and then (7.7), we see that the second member on the right-hand side of (8.9) can be rewritten to

$$3\sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+m+n;x)}{l(l+m)(l+m+n)^{k+1}}.$$

Thus we obtain (8.7).

Next, by rewriting the left-hand side of (8.8) to

$$\sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^k(l+m)}\sum_{n=1}^{\infty}\frac{1}{n(m+n)},$$

and arguing similarly to the proof of (8.9), we see that this is equal to the right-hand side of (8.8). This completes the proof. ■

Proof of Theorem 8.3 From (8.1) in the case $r=2$, we can easily see that

$$G_2(\theta)=\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}\sin((m+n)\theta)}{mn}+\theta\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m},$$

which is uniformly convergent for $\theta\in(-\pi,\pi)$. Substituting this series into $\mathbf{H}_3(\theta;x;k)$ in (8.3), and calculating (8.4) in the case $r=3$, we obtain

$$0=-\sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^kmn(l+m+n)}+\sum_{\substack{l,m,n=1\\ l\neq m+n}}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^kmn(l-m-n)}$$
$$+2\sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^km(l+m)^2}+2\sum_{\substack{l,m=1\\ l\neq m}}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^km(l-m)^2}+\frac{\pi^2}{3}\sum_{l=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^{k+1}}. \tag{8.11}$$

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We divide the second member of (8.11) into three subsums according as (i) $l > m$, (ii) $l < m$, and (iii) $l = m$. On (i) we set $j = l - m$, while on (ii) we set $j = m - l$. We further divide part (i) into two subsums according as $j > n$, $j < n$. We also divide the fourth member of (8.11) into two subsums according as $l > m$, $l < m$. Applying Lemma 8.5 (to the part $j < n$ of (i) and part (ii)), we can rewrite (8.11) to

$$(8.12)\qquad \begin{aligned} 0 &= -3\sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^k m(m+n)(l+m+n)} + \sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+m+n;x)}{lmn(l+m+n)^k}\\ &\quad -3\sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+m+n;x)}{l(l+m)(l+m+n)^{k+1}} + 2\sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^k m(l+m)^2}\\ &\quad +\sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+m;x)}{lm^2(l+m)^k} + \sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^k m^2(l+m)} + \zeta(2)\sum_{l=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^{k+1}}\\ &= -3A_1 + A_2 - 3A_3 + 2A_4 + A_5 + A_6 + A_7, \end{aligned}$$

say. Applying (8.10) repeatedly, we have

$$A_1 = \sum_{j=0}^{k-1}\sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^{k-j}m(l+m+n)^{j+2}} + \sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(n;x)}{l(l+m)(l+m+n)^{k+1}},$$

and then, using (7.7) repeatedly, we obtain

$$(8.13)\quad \begin{aligned} A_1 &= \sum_{j=0}^{k-1}\sum_{p=0}^{k-j-1}\left(\sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^{k-j-p}(l+m)^{p+1}(l+m+n)^{j+2}}\right)\\ &\quad +\sum_{j=0}^{k-1}\sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(m;x)}{l(l+m)^{k-j}(l+m+n)^{j+2}} + \sum_{l,m,n=1}^{\infty}\frac{\widetilde{\mathbf{F}}(n;x)}{l(l+m)(l+m+n)^{k+1}}. \end{aligned}$$

Also, using (7.7) repeatedly, we obtain

$$(8.14)\qquad A_4 = \sum_{j=1}^{k}\sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^j(l+m)^{k+3-j}} + \sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{m(l+m)^{k+2}}.$$

Now we use a result of [27]. It is easy to see that we can replace factors of the form x^l, which are implicitly included in [27, (3.13)], by $\widetilde{\mathbf{F}}(l;x)$. This implies

$$(8.15)\qquad A_5 + A_6 = A_4 - 3\sum_{m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(m;x)}{m^{k+3}} + 2A_7.$$

Moreover, we have

$$(8.16)\qquad A_7 = \sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+m;x)}{l^2(l+m)^{k+1}} + \sum_{l,m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l;x)}{l^{k+1}(l+m)^2} + \sum_{m=1}^{\infty}\frac{\widetilde{\mathbf{F}}(m;x)}{m^{k+3}}$$

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by the harmonic product relation. Combining (8.14), (8.15), and (8.16), we have

$$\begin{aligned}
&2A_4 + A_5 + A_6 + A_7 \\
&= 3A_4 - 3\sum_{m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(m;x)}{m^{k+3}} + 3A_7 \\
&= 3\sum_{j=1}^{k+1}\sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l;x)}{l^j(l+m)^{k+3-j}} + 3\sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l;x)}{m(l+m)^{k+2}} + 3\sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+m;x)}{l^2(l+m)^{k+1}}.
\end{aligned}$$

Applying Theorem 2.4 to the double sum on the right-hand side, we obtain

$$\begin{aligned}
&2A_4 + A_5 + A_6 + A_7 \\
&\quad = 3\sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+m;x)}{l(l+m)^{k+2}} + 3\sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+m;x)}{l^2(l+m)^{k+1}} + 3\sum_{m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(m;x)}{m^{k+3}} \\
&\quad = 3A_8 + 3A_9 + 3A_{10},
\end{aligned} \tag{8.17}$$

say. Next, since

$$\begin{aligned}
\frac{1}{lmn} &= \Big(\frac{1}{lm} + \frac{1}{mn} + \frac{1}{ln}\Big)\frac{1}{l+m+n} \\
&= \Big\{\Big(\frac{1}{l} + \frac{1}{m}\Big)\frac{1}{l+m} + \Big(\frac{1}{m} + \frac{1}{n}\Big)\frac{1}{m+n} + \Big(\frac{1}{l} + \frac{1}{n}\Big)\frac{1}{l+n}\Big\}\frac{1}{l+m+n},
\end{aligned}$$

we find that $A_2 = 6A_3$. Noting this fact and (8.17), we can rewrite (8.12) as

$$-A_1 + (2A_3 + A_8 + A_9) + A_{10} = A_3. \tag{8.18}$$

Putting $m + n = q$, we have

$$A_3 = \sum_{l,q=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+q;x)}{l(l+q)^{k+1}} \sum_{m=1}^{q-1} \frac{1}{l+m},$$

while putting $l + m = r$, we have

$$A_3 = \sum_{n,r=1}^{\infty} \frac{\widetilde{\mathbf{F}}(n+r;x)}{r(n+r)^{k+1}} \sum_{l=1}^{r-1} \frac{1}{l}.$$

Therefore,

$$\begin{aligned}
&2A_3 + A_8 + A_9 \\
&= \sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+m;x)}{l(l+m)^{k+1}} \Big\{\sum_{j=1}^{m-1} \frac{1}{l+j} + \sum_{j=1}^{l-1} \frac{1}{j} + \frac{1}{l+m} + \frac{1}{l}\Big\} \\
&= \sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+m;x)}{l(l+m)^{k+1}} \sum_{j=1}^{l+m} \frac{1}{j} = \sum_{l,m=1}^{\infty} \frac{\widetilde{\mathbf{F}}(l+m;x)}{l(l+m)^{k+1}} \sum_{j=1}^{\infty}\Big(\frac{1}{j} - \frac{1}{l+m+j}\Big).
\end{aligned} \tag{8.19}$$

Now we use the identity

$$\frac{1}{(l+m)^{k+1}}\Big(\frac{1}{j}-\frac{1}{l+m+j}\Big)=\frac{1}{j(l+m+j)^{k+1}}+\sum_{u=1}^{k}\frac{1}{(l+m)^{u}(l+m+j)^{k+2-u}}$$

to the right-hand side of (8.19), and then apply (7.7) to the first double sum of the resulting expression. We find that (8.19) is

$$=\sum_{l,m,j=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+m;x)}{l(l+j)(l+m+j)^{k+1}}+\sum_{l,m,j=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+m;x)}{j(l+j)(l+m+j)^{k+1}}$$
$$+\sum_{u=1}^{k}\sum_{l,m,j=1}^{\infty}\frac{\widetilde{\mathbf{F}}(l+m;x)}{l(l+m)^{u}(l+m+j)^{k+2-u}}.$$

Substituting this result and (8.13) into (8.18), and putting $K=k+3$, we arrive at a formula that is almost the same as (8.5) but where $\mathbf{F}(\,\cdot\,)$ is replaced by $\widetilde{\mathbf{F}}(\,\cdot\,;x)$. Finally, as in the proof of Theorem 2.4, we can let $x\to 1$ because of the uniform convergence. This completes the proof of Theorem 8.3. ■

Remark At present, it seems to be hard to give the proof of Conjecture 8.4 for general r. In fact, if we were to obtain its proof then we would consequently obtain a brand-new method to prove the sum formulas for Euler–Zagier multiple zeta values that does not depend on Drinfel'd integral expressions.

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