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UNCOUNTABLY MANY NON-BINARY SHIFTS ON THE HYPERFINITE II_1-FACTOR

BY

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ABSTRACT. We shall construct uncountably many nonconjugate non-binary shifts with index two on the hyperfinite II_1-factor R using rational functions over a finite field.

1. **Introduction.** R. T. Powers [5] defined a shift on the hyperfinite II_1-factor R to be an identity preserving $*$-endomorphism σ of R such that $\bigcap_{k=1}^{\infty} \sigma^k(R) = \mathbb{C}I$ and defined the index of σ as the Jones index $[R:\sigma(R)]$ (cf. [3]). Powers called a shift σ of R a binary shift if there is a unitary $u \in R$ with $u^2 = I$ such that $R = \{u, \sigma(u), \sigma^2(u), \cdots\}''$ and $u\sigma^k(u) = \pm\sigma^k(u)u$ for $k \in \mathbb{N}$. The unitary u is called a σ-generator. The index of a binary shift is two. Two shifts σ and τ of R are conjugate (resp. outer conjugate) if $\theta\sigma\theta^{-1} = \tau$ for some automorphism θ of R (resp. $\theta\sigma\theta^{-1} = \tau \cdot Adw$ for some θ and unitary $w \in R$). Powers constructed in [5] a countable infinity of non outer conjugate binary shifts on R, and an uncountable infinity of non conjugate binary shifts on R. M. Choda [2] generalized this to the case of a shift with a generating unitary u such that $u^m = I\,(m \in \mathbb{N})$, and constructed a countable infinity of outer conjugacy classes of shifts on R with any given index ($\in \{4\cos^2(\frac{\pi}{n}); n = 3, 4, \ldots\} \cup [4, \infty)$). On the other hand, in [6], G. L. Price constructed a shift σ on R of index two which is not a binary shift. Inspired by the construction of Price's non-binary shift with index two, we shall show the existence of uncountable many non-conjugate non-binary shifts on R with index two. To construct such shifts, we shall consider the shift on the group von Neumann algebra $R_m(G)$ of a group G twisted by a multiplier m, induced from a shift on the group G. In our construction, G will be a vector space over the field $\mathbb{Z}/2\mathbb{Z}$. This method of construction (which includes the examples of Powers and Price) was used by D. Bures and H-S. Yin in [1]. D. Bures and H-S. Yin obtained an intrinsic characterization of such shifts (which they call group shifts), and for those σ satisfying $\sigma(R)' \cap R = \mathbb{C}$, a classification up to conjugacy.

2. **Shifts on von Neumann algebras induced from shifts on groups.** Let G be a countable discrete abelian group and m a multiplier on G. For $x \in G$, define a unitary operator $\lambda_m(x)$ on $\ell^2(G)$ by

$$(\lambda_m(x)\xi)(y) = m(x, x^{-1}y)\xi(x^{-1}y) \quad \text{for } \xi \in \ell^2(G).$$

Then λ_m is a projective representation of G with respect to m. Let $R_m(G)$ denote the von Neumann algebra generated by $\{\lambda_m(x); x \in G\}$. We shall call $R_m(G)$ the (twisted) group von Neumann algebra. We can construct shifts on $R_m(G)$ as follows.

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Let σ be a shift on a group G, that is, an injective homomorphism σ on G such that $\bigcap_{k=1}^{\infty}\sigma^k(G)=\{1\}$. Suppose that σ preserves the multiplier m, that is, that $m(\sigma(x),\sigma(y)) = m(x,y)$ for $x,y\in G$. Then σ induces a shift σ_m on the (twisted) group von Neumann algebra $R_m(G)$ such that $\sigma_m(\lambda_m(x))=\lambda_m(\sigma(x))$ for $x\in G$. Furthermore, if $R_m(G)$ is a factor, then $[R_m(G):\sigma_m(R_m(G))]=[G:\sigma(G)]$. Define $\omega_m\colon G\times G\longrightarrow \mathbb{T}$ by $\omega_m(x,y) = m(x,y)\overline{m(y,x)}$. Then ω_m is an anti-symmetric bicharacter on G (cf. [4]). It is known that if ω_m is non-degenerate, that is, $\omega_m(x,G)=\{1\}$ implies that $x=1$, then $R_m(G)$ becomes a hyperfinite II_1-factor (cf. Slawny [8]). We put

$$X=\coprod_{i=0}^{\infty}G_i,\quad\text{where}\quad G_i\cong\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}=\{0,1\}.$$

A sequence $a\colon\mathbb{Z}\longrightarrow\{0,1\}$ with $a(0)=0$ and $a(n)=a(-n)$ is called a *signature sequence* ([6], [7]). A signature sequence $a\colon\mathbb{Z}\longrightarrow\{0,1\}$ is periodic if there exists an $n\in\mathbb{Z}$ such that $a(j+n)=a(j)$ for any $j\in\mathbb{Z}$. For $x=(x(i))$ and $y=(y(j))$ in X, consider the multiplier m_a (in fact, a bicharacter) defined by

$$m_a(x,y)=(-1)^{\sum_{i>j}a(i-j)x(i)y(j)}. \tag{2.1}$$

Price [6] showed that the group von Neumann algebra $R_{m_a}(X)$ is a factor if and only if the signature sequence a is non-periodic. This was generalized by Price in [7] and by Bures and Yin in [1] to the case of arbitrary integral index.

PROPOSITION 2.1. ([6], Theorem 2.3) *Let $X=\coprod_{i=0}^{\infty}G_i$, $G_i\cong\mathbb{Z}_2$. Let a be a signature sequence on $\mathbb{Z}$ and denote by m_a the corresponding multiplier (2.1). Then the following statements are all equivalent:*

(1) the group von Neumann algebra $R_{m_a}(X)$ is a factor;

(2) the anti-symmetric bicharacter ω_{m_a} is non-degenerate;

(3) the signature sequence a is non-periodic.

EXAMPLE 2.2 (binary shifts of Powers [5]). Let α be a binary shift on R with a unitary generator u. Put $S=\{k\in\mathbb{N};u\alpha^k(u)=-\alpha^k(u)u\}$.

Define the sequence $a\colon\mathbb{Z}\longrightarrow\{0,1\}$ by $a(n)=1$ if $|n|\in S$ and $a(n)=0$ if $|n|\notin S$. Suppose that a is not periodic. Let m_a be as in (2.1).

For $x=(x(0),\dots,x(n),0,0,\dots)\in X=\coprod_{i=0}^{\infty}G_i$, $G_i\cong\mathbb{Z}_2$, we put $u(x) = u^{x(0)}\alpha(u)^{x(1)}\alpha^2(u)^{x(2)}\cdots\alpha^n(u)^{x(n)}$. Then there exists an isomorphism $\theta\colon R\longrightarrow R_{m_a}(X)$ such that $\theta(u(x))=\lambda_{m_a}(x)$.

Define the canonical shift σ on the group X by $(\sigma(x))(j)=x(j-1)$ for $j\geq 1$ and $(\sigma(x))(0)=0$. Since $m_a(\sigma(x),\sigma(y))=m_a(x,y)$, σ induces a shift σ_{m_a} on the von Neumann algebra $R_{m_a}(X)$. Then, with θ as above, $\theta\alpha\theta^{-1}=\sigma_{m_a}$. Thus the binary shift α is exactly σ_{m_a} under the isomorphism θ.

3. **Uncountably many non-binary shifts of index two.** Powers [5] completely classified binary shifts up to conjugacy on a hyperfinite II_1-factor R. Subsequently, Price [6] ingeniously found a non-binary shift with index two on R. We shall now construct uncountably many non-binary shifts on R of index two.

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Let $a: \mathbb{Z} \longrightarrow \{0, 1\}$ be a signature sequence. Let $X = \coprod_{i=0}^{\infty} G_i$, $G_i \cong \mathbb{Z}_2$. Let m_a be the corresponding multiplier on X. Let σ be the canonical shift. Then clearly σ preserves this multiplier m_a. Similarly, let $Y = \coprod_{i=0}^{\infty} H_i$, $H_i \cong \mathbb{Z}_2$. Let $F[t]$ denote the polynomial ring over the finite field $F = \{0, 1\}$. Fix a monic polynomial $p(t) = c_0 + c_1 t + \cdots + c_k t^k \in F[t]$ with $c_0 = 1$. Set $F[t]/p(t) = \{f(t)/p(t); f(t) \in F[t]\}$. Consider the embedding $\Psi: F[t] \longrightarrow F[t]/p(t)$ defined by $\Psi(f(t)) = p(t)f(t)/p(t) = f(t)$. First recall the following elementary fact. Let G be a countable discrete group such that $g^2 = 1$ for any $g \in G$. Then G is isomorphic to $\coprod_{i=0}^{\infty} G_i$, $G_i \cong \mathbb{Z}_2$. We shall denote the group operation by addition. Clearly, G is a vector space over F. In fact the sum is given by the addition of G and the scalar multiplication is given by $0 \cdot x = 0$ and $1 \cdot x = x$.

Define a group isomorphism $\theta: X \longrightarrow F[t]$ by, for $x = (x(i)) \in X$, $\theta(x) = \sum_{i \geq 0} x(i)t^i \in F[t]$ and also define a group isomorphism

$$\gamma: Y \longrightarrow F[t]/p(t) \text{ by, for } y = (y(i)) \in Y,\ \gamma(y) = (\sum_{i \geq 0} y(i)t^i)/p(t).$$

DEFINITION 3.1. For $X = \coprod_{i=0}^{\infty} G_i$, $Y = \coprod_{i=0}^{\infty} H_i$, where $G_i \cong H_i \cong \mathbb{Z}_2$, and a given polynomial $p(t) \in F[t]$, consider the group injection $\Phi_p: X \longrightarrow Y$, $\Phi_p = \gamma^{-1}\Psi\theta$, where γ, Ψ, and θ are as above. Then, for $x = (x(i))$, $(\Phi_p(x))(n) = \sum_{i+j=n} c_i x(j)$. The group injection $\Phi_p: X \longrightarrow Y$ will be called the one defined by (multiplication by) the polynomial p.

Consider the multiplication operator σ_t by t on $F[t]$ (or $F[t]/p(t)$): $\sigma_t(f(t)) = tf(t)$ (or $\sigma_t(f(t)/p(t)) = tf(t)/p(t)$) for $f(t) \in F[t]$. Then $\sigma_t = \theta\sigma\theta^{-1}$ on $F[t]$ and $\sigma_t = \gamma\sigma\gamma^{-1}$ on $F[t]/p(t)$. Thus the canonical shift is realized as the multiplication by t. Therefore $\Phi_p \cdot \sigma = \sigma \cdot \Phi_p$ on X.

The following lemma is a refinement of a result of Price's (Theorem 5.1 of [6]).

LEMMA 3.2. *Let $a: \mathbb{Z} \longrightarrow \{0, 1\}$ be a non-periodic signature sequence and $p \in F[t]$ a monic polynomial with a nonzero constant term. Then there exists a non-periodic signature sequence $b: \mathbb{Z} \longrightarrow \{0, 1\}$ such that*

$$m_b(\Phi_p(x), \Phi_p(y)) = m_a(x, y) \text{ for any } x, y \in X.$$

PROOF. For brevity (and clarity), consider Φ_p to be just multiplication by $p = \sum_{i=0}^{k} c_i t^i$ on $F[t]$. The conditions on b are exactly the following:

(1) $m_b(p, pt^n) = m_a(1, t^n)$, $n = 0, 1, 2, \ldots$;

(2) $m_b(t^n p, p) = m_a(t^n, 1)$, $n = 1, 2, \ldots$.

On expanding, these conditions become

(1) $\sum_{j=0}^{k-n} q(n+j)b(j) = 0$, $n = 0, 1, \ldots, k$,

(2) $\sum_{i=0}^{n} q(i)b(n-i) + \sum_{i=1}^{k} q(i)b(n+i) = a(n)$, $n = 1, 2, \ldots$,

where $q(0) = c_0c_0 + \cdots + c_kc_k$, $q(1) = c_1c_0 + \cdots + c_kc_{k-1}$, $q(k) = c_kc_0$.

Condition (1) is certainly satisfied if $b(0) = \cdots = b(k) = 0$. From $q(k) \neq 0$ it follows immediately that there is a unique solution of (2) such that $b(0) = \cdots = b(k) = 0$. Clearly, by (2), a is periodic if b is. Q. E. D.

Given a signature sequence a and a sequence of monic polynomials $p_\ell(t) = c_{\ell,0} + c_{\ell,1}t + \cdots + c_{\ell,k(\ell)}t^{k(\ell)}$ with $c_{\ell,0} = 1$, defining $X_\ell = \coprod_{i=0}^{\infty} G_i^{(\ell)}$ with $G_i^{(\ell)} \cong \mathbb{Z}_2$ for

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$l = 1, 2, \ldots$, we may apply Lemma 3.2 repreatedly to get a sequence of group injections $\Phi_{p_\ell}: X_\ell \longrightarrow X_{\ell+1}$, defined by the polynomials p_ℓ, and multipliers m_{a_ℓ} on X_ℓ induced by non-periodic signature sequences a_ℓ on $\mathbb{Z}$, which satisfy $m_{a_{\ell+1}}(\Phi_{p_\ell}(x), \Phi_{p_\ell}(y)) = m_{a_\ell}(x, y)$ for $x, y \in X_\ell$ and $a_1 = a$. (Of course, the sequence $\{a_\ell ; \ell = 1, 2, \ldots\}$ is not unique, unless we require the initial conditions specified in the proof of Lemma 3.2)

Now, set $X_{[p]} = \varinjlim(X_l, \Phi_{p_l})$. Define a multiplier $m_{[a,p]}$ on $X_{[p]}$ by $m_{[a,p]}(x, y) = m_{a_\ell}(x, y)$ if $x, y \in X_\ell$. Then $R_{m_{[a,p]}}(X_{[p]})$ is the hyperfinite II_1-factor, since the anti-symmetric bicharacter $\omega_{m_{[a,p]}}$ is non-degenerate by Proposition 2.1. The canonical group endomorphism $\sigma_{[p]}$ is a shift on $X_{[p]}$. Hence $\sigma_{[p]}$ induces a shift $\sigma_{[a,p]}$ on $R_{m_{[a,p]}}(X_{[p]})$.

DEFINITION 3.3. With the above notation, for sequences $p = (p_1, p_2, \ldots)$ of monic polynomials p_ℓ with nonzero constant terms and a non-periodic signature sequence a, the shifts $\sigma_{[a,p]}$ on $R_{m_{[a,p]}}(X_{[p]})$ are called *shifts of Price type*.

The normalizer of a shift σ on a hyperfinite II_1-factor R, denoted by $N(\sigma)$ (cf. [5]), consists of those unitary elements $u \in R$ such that $u\sigma^k(R)u^* = \sigma^k(R)$ for all $k = 1, 2, \ldots$. The normalizer of a shift of Price type is the set of elements of the underlying group up to scalar multiples. This fact is proved by Price in [6], [7].

PROPOSITION 3.4. *Let there be given two sequences of monic polynomials with nonzero constant terms, $p = (p_i)$ and $q = (q_i)$ for $i = 1, 2, \ldots$, and two non-periodic signature sequences a and b. If the two shifts of Price type $\sigma_{[a,p]}$ and $\sigma_{[b,q]}$ are conjugate on the hyperfinite II_1-factor, then $(\sigma_{[p]}, X_{[p]})$ and $(\sigma_{[q]}, X_{[q]})$ are conjugate, where $\sigma_{[p]}$ denotes the shift induced by $\sigma_{[a,p]}$ on $X_{[p]}$.*

PROOF. The shifts $\sigma_{[a,p]}$ on $R_{m_{[a,p]}}(X_{[p]})$ induce shifts $\tilde{\sigma}_{[a,p]}: N(\sigma_{[a,p]})/\mathbb{T} \longrightarrow N(\sigma_{[a,p]})/\mathbb{T}$. By the above fact, $N(\sigma_{[a,p]})/\mathbb{T} \cong X_{[p]}$ and $\tilde{\sigma}_{[a,p]} = \sigma_{[p]}$. Therefore if $\sigma_{[a,p]}$ and $\sigma_{[b,q]}$ are conjugate, then $(\sigma_{[p]}, X_{[p]})$ and $(\sigma_{[q]}, X_{[q]})$ are conjugate. Q. E. D.

In the following we shall construct uncountably many non-binary shifts. First, choose a sequence of distinct irreducible monic polynomials $p_k(t)(\neq t)$, $k = 1, 2, \ldots$. Let $c = (c(1), c(2), c(3), \ldots) \in \prod_{i=1}^{\infty} \mathbb{Z}_2$. Put

$$X^c = \Big\{ g(t)/f(t); g(t), f(t) \in F[t], \\ \text{and if } f(t) = p_1(t)^{k_1} \cdots p_n(t)^{k_n}, k_i \neq 0, \text{ then } c(i) \neq 0. \Big\}$$

That is, X^c is the set of rational functions whose denominator may have $p_i(t)$ as a factor only if $c(i) \neq 0$. X^c is, of course, isomorphic to $\coprod_{i=0}^{\infty} G_i$, where $G_i \cong \mathbb{Z}_2$. Let us denote the shift σ_t on X^c by σ^c.

LEMMA 3.5. *Let c and d be elements in $\prod_{i=1}^{\infty} \mathbb{Z}_2$. Then $c = d$ if and only if (σ^c, X^c) and (σ^d, X^d) are conjugate.*

PROOF. If $c \neq d$, then there exists an n_0 such that either $(c(n_0) = 1$ and $d(n_0) = 0)$ or $(c(n_0) = 0$ and $d(n_0) = 1)$. Hence we may suppose that $c(n_0) = 1$ and $d(n_0) = 0$. If σ^c and σ^d are conjugate, then $p_{n_0}(\sigma^c)$ and $p_{n_0}(\sigma^d)$ are conjugate. But $\text{Im}(p_{n_0}(\sigma^c)) = X^c$ and $\text{Im}(p_{n_0}(\sigma^d)) \neq X^d$. In fact, take an element $g(t)/f(t) \in X^c$. Then $g(t)/(p_{n_0}(t)f(t)) \in X^c$ and $g(t)/f(t) = p_{n_0}(t)g(t)/p_{n_0}(t)f(t) \in \text{Im}(p_{n_0}(\sigma^c))$. Hence $\text{Im}(p_{n_0}(\sigma^c)) = X^c$. On the other hand, $1 \in X^d$, but $1 \notin \text{Im}(p_{n_0}(\sigma^d))$. If $p_{n_0}(t)g(t)/f(t) = 1$, then $p_{n_0}(t)g(t) = f(t)$.

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But $p_{n_0}(t)$ does not divide $f(t)$. This is a contradiction; therefore $1 \notin \mathrm{Im}(p_{n_0}(\sigma^d))$. Thus $\mathrm{Im}(p_{n_0}(\sigma^d)) \neq X^d$. Q. E. D.

Put $X_0^c = F[t]$, $X_1^c = F[t]/p_1(t)^{c(1)}, \ldots, X_\ell^c = F[t]/\left(p_1(t)^{c(1)}p_2(t)^{c(2)}\cdots p_\ell(t)^{c(\ell)}\right)^\ell$. Then we have $\bigcup_{\ell=0}^{\infty} X_\ell^c = X^c$. Furthermore, the embedding from X_ℓ^c to $X_{\ell+1}^c$ is defined by multiplication by the polynomial $p_1(t)^{c(1)}p_2(t)^{c(2)}\cdots p_\ell(t)^{c(\ell)}p_{\ell+1}(t)^{(\ell+1)c(\ell+1)}$. In particular, the Powers binary shifts are associated to the sequence $c = (c(1), c(2), \ldots) = (0, 0, 0, \ldots)$, by Example 2.2. Thus we get the following theorem.

THEOREM 3.6. *There exist uncountable many non-conjugate non-binary shifts of index two on the hyperfinite II_1-factor.*

REMARK. A similar result to this theorem holds in the case of general index. We shall publish it elsewhere.

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