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ON THE DUAL OF PROJECTIVE VARIETIES

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ABSTRACT. Here we give examples and classifications of varieties with strange behaviour for the enumeration of contacts (answering a question raised by Fulton, Kleiman, MacPherson). Then we give upper and lower bounds (in terms of the degree) for the non-zero ranks of a projective variety.

Fulton, Kleiman and MacPherson [2] prove a very nice theorem about the number of varieties in a p-parameter family touching p varieties $g_1^*(V_1), \dots, g_p^*(V_p)$ with $V_i \subset P^N$ and $g_i \in \mathrm{Aut}(P^N)$. (There are no restrictions on the V_i and g_i). We will refer to this result as *the main theorem of [2].* The authors in [2] discuss the enumerative significance of their formula and the type of contact for general g_i's. In Section 4 they present a number of open questions. The aim of the first section of this note is to give an answer (not *the* answer) to the first question raised there concerning (c, iv) of the main theorem of [2]. At the bottom of page 180 of [2], this question is recast in the following form:

Find integral varieties A, A' in $\mathbf{P}^N$ (possibly $A = A'$) with the same dimension, say $\dim(A) = m$ (with $m \geq 2$), such that there is an irreducible $E \subset A \times A'$ with $\dim(E) = 2m-1$, and such that for all $(x, y) \in E$ $T_xA \neq T_yA'$, $T_xA \cap T_yA'$ contains the line $[x; y]$ thru x and y and $A \cup A'$ does not contain $[x; y]$.

We will say that a variety A (resp. a pair (A, A') with $A \neq A'$) has property (&) (resp. (&&)) if (A, A) (resp. (A, A')) satisfies the condition just given. The bitangency problem in [2] arises when one of the schemes involved, say V_1, contains integral components A, A' with A satisfying (&) or (A, A') satisfying (&&).

For the notions used (dual variety, reflexivity, ranks, ...) and their properties, see the nice papers [2],[3],[5] and [6].

In § 1 (see Remark 1.1) we will show that when the algebraically closed base field $\mathbf{F}$ has characteristic 2, for every even m there are explicit examples (first found in [1] and used there for other purposes) of m-dimensional varieties with property (&). Then we will show (see Theorem 1.2) that the only ordinary varieties with property (&) are the ones described in 1.1. At the beginning of the proof of 1.2 we will discuss also where the restrictions on $\mathrm{char}(\mathbf{F})$ and the dimension m come from. In 1.3 we will describe (when $\mathrm{char}(\mathbf{F}) = 2$) a class of pairs (A, A') satisfying (&&) with A and A' ordinary varieties. Theorem 1.3 will show that there is no other pair (A, A') satisfying (&&) with A and A' ordinary varieties.

In the first part of § 2 we prove a result (Proposition 2.1) about the dual variety of the Veronese embedding of a projective variety. Then we prove a result (Theorem 2.2)

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which gives information about the dual variety of the Segre embedding obtained from two embeddings of a variety in projective spaces.

In § 3 we assume char($\mathbf{F}$) $= 0$ and give a quantitative version (see Theorem 3.1) of a non-vanishing theorem of Hefez and Kleiman ([3], 4.13, or see [6], th. (7) on p. 190) about the ranks of a projective variety; Theorem 3.1 gives a lower bound and an upper bound (both in terms of the degree) for the non-zero ranks of a projective variety. Theorem 3.1 improves very much [7], Prop. 5.3.

1. In this paper every scheme will be algebraic over an algebraically closed field $\mathbf{F}$; essentially the only interesting cases for this section (and for the first part of Proposition 2.1) arise when char($\mathbf{F}$) $= 2$.

Now we describe a nice class of hypersurfaces (defined if char($\mathbf{F}$) $= 2$) introduced in [1] for other purposes (there it was proved that they are exactly the only varieties satisfying a certain property ($)). Fix an even integer m. The examples will be hypersurfaces of $\mathbf{P}^{m+1}$. Set $2r-2 := m$ and fix homogeneous coordinates $z_1, \ldots, z_r, y_1, \ldots, y_r$ of $\mathbf{P}^{m+1}$. Fix an even integer d (the degree of the hypersurfaces) and two homogeneous polynomials h and b in the variables z_i, y_i with $\deg(b) = \deg(h) + 1 = d/2$. Set

$$f = \left(\sum_{i=1}^{r} y_i z_i\right) h^2 + b^2. \tag{1}$$

Set $X := \{f = 0\}$. There are many examples of polynomials f given by (1) for which X is reduced and irreducible. Any such X will be said to be described by (1) (i.e. char($\mathbf{F}$) $=$ $2, m = 2r-2$, X is a hypersurface in $\mathbf{P}^{m+1}$ and its equation is given by (1) for suitable d, b, and h).

REMARK 1.1. *The integral varieties described by (1) have property (&).*

PROOF. It was proved in [1], § 1, that every integral variety X described by (1) has the following property (%):

(%) For every $x \in X_{\text{reg}}$ and every $y \in (X_{\text{reg}}) \cap T_xX$, we have $x \in T_yX$.

In particular by (%) $T_xX \cap T_yX$ contains the line $[x; y]$. Since $\dim(X \cap T_xX) = m-1$ and $X \cap (T_xX)$ is not a cone with vertex x (for general x), we see that X has property (&). ∎

This remark settles the existence asked in [2], beginning of § 4. But of course we want more: under (very strong) assumptions (i.e., that the variety is ordinary) we will show that these are the only examples with property (&) (see Theorem 1.2). Then in 1.3 and 1.4 we will do essentially the same for the property (&&).

THEOREM 1.2. *Let V be an ordinary nondegenerate variety with property (&). Then* char($\mathbf{F}$) $= 2$ *and V is described by (1).*

PROOF. Fix an ordinary (hence reflexive) nondegenerate variety $V \subset \mathbf{P}^N$, V satisfying (&), with $m := \dim(V) < N$. By [2], end of part (c) of the statement of the

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theorem in § 2, we have char($\mathbf{F}$) $= 2$. By a result of N. Katz ([4], note on p. 3, or see [6], Corollary (18) on p. 189) m is even.

(a) First assume $N = m+1$. Let H be a general hyperplane of $\mathbf{P}^{m+1}$. Set $Y := V \cap H$. Fix a general $z \in Y$. By the assumption (&) $(T_zV) \cap V$ contains a variety B with $\dim(B) = m-1$, and such that for every $y \in B$, $z \in T_yV$. We distinguish two subcases: $m = 2$ or $m > 2$.

(a1) First assume $m = 2$, hence $\dim(B) = 1$. By the last part of condition (&) B is not a line. Thus B spans T_zV, i.e. $\{(T_zV) \cap (T_uV) : u \in B\} = \{[z;u] : u \in B\}$ is an infinite set of lines thru z and contained in T_zV. By the second half of step 1 in § 2 of [1] this implies that V is described by (1).

(a2) Assume $m > 2$. By the case $m = 2$, we see that the intersection of V with a general 3-dimensional linear space is described by (1). In particular this gives the irreducibility of $(T_zV) \cap V$ for general V, hence that B is an open subset of $(T_zV) \cap V$ and that the set $\{(T_zV) \cap (T_uV) : u \in B\}$ is open in the set of hyperplanes of T_zV containing z. Again, by the last part of the proof of step 1 in § 2 of [1], this implies that V is described by (1).

(b) Now assume $N > m+1$. A general projection of V in $\mathbf{P}^{m+1}$ is reflexive and ordinary ([6], th. (5) on p. 189) and of course satisfies (&). Fix a general linear subspace $L \subset \mathbf{P}^N$, $\dim(L) = N-m-2$, and a general $z \in V_{\mathrm{reg}}$; denote by $t_L \colon \mathbf{P}^N \setminus L \longrightarrow \mathbf{P}^{m+1}$ the projection from L. Denote by $[U;U']$ the spanning in $\mathbf{P}^N$ of the subsets U and U'. Fix a general $y \in V \cap [L;T_zV]$. By part (a), $t_L(V)$ is given (in suitable coordinates) by (1) and in particular it satisfies the condition (%) ([1],§ 1). Thus $z \in [L;T_yV]$. Take a general linear subspace L' of $[L;T_zV]$ with $\dim(L') = \dim(L)$. Since $[L';T_zV] = [L;T_zV]$ and we know that $t_{L'}(V)$ satisfies (1), we get easily that either $T_yV \subset [L;T_zV]$ or $z \in T_yV$. First assume $z \in T_yV$ (for all such general z and y). Thus V satisfies (1) (use that V is reflexive, hence V and $[L;T_zV]$ have order of contact 2 at z, and that V is not a quadric hypersurface), contradicting [1]. Now assume $T_yY \subset [l;T_zZ]$. Thus the tangent space to $t_L(V)$ at $t_L(z)$ is tangent to $t_L(V)$ along its $(m-1)$-dimensional intersection with $t_L(V)$, contradicting for instance the fact that $t_L(V)$ is ordinary. ■

Now we may show that, if char($\mathbf{F}$) $= 2$ and m is even, there are pairs (A, A') of ordinary m-dimensional hypersurfaces satisfying (&&). In 1.4 we will show that these are the only such examples (up to projective transformations).

REMARK 1.3. *Assume* char($\mathbf{F}$) $= 2$; *fix an even integer m; set $2r-2 := m$. Fix a system of homogeneous coordinates y_i, z_i, $1 \le i \le r$, of $\mathbf{P}^{m+1}$. Fix homogeneous polynomials h, b, h', b' with* $\deg(b) = \deg(h)+1$, $\deg(b') = \deg(h')+1$ *and call A (resp. A') the hypersurface with equation (1) (resp. with equation (1) with (h',b') instead of (h,b)). Assume $A \neq A'$ and that A and A' are integral. Then (A,A') has property (&&).*

PROOF. We describe here one (equivalent and not depending on any choice of coordinates) description of every variety X described by (1). There is a linear isomorphism (a null-correlation) $t \colon \mathbf{P}^{m+1} \longrightarrow \mathbf{P}^{m+1^*}$ such that for every $z \in \mathbf{P}^{m+1}$ all the tangent spaces to X at the points of $(X_{\mathrm{reg}}) \cap t(z)$ pass thru z; if $X \cap t(z)$ is reduced, this means that $X \cap t(z)$ is a

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strange variety with z as strange point. The isomorphism t does not depend on X satisfying (1) if we have fixed the coordinates (i.e. depends only on the part $\sum y_i z_i$ of (1)). Thus the isomorphisms t, t' induced by A and A' are the same. Thus (A, A') has the property (&&), taking as E the set $\{(x,y) \in A \times A' : y \in t(x)\} = \{(x,y) \in A \times A' : x \in t(y)\}$ (the last equality being a consequence of the fact that t is a null-correlation). ∎

THEOREM 1.4. *Fix a pair (A, A') satisfying (&&) with A and A' ordinary varieties. Then* $\mathrm{char}(\mathbf{F}) = 2$, $m := \dim(A)$ *is even and* (A, A') *is, up to a projective transformation, one of the pairs described in 1.3.*

PROOF. By [2], part c(iv) of the main theorem, we have $\mathrm{char}(\mathbf{F}) = 2$; thus, as quoted at the beginning of the proof 1.2 ([4]) m is even. For a general $x \in A$ (resp. $y \in A'$), call $E(x,)$ (resp. $E(,y)$) the $(m-1)$-dimensional subvariety $E \cap (\{x\} \times A')$ (resp. $E \cap (A \times \{y\})$), with $E \subset A \times A'$ as in the definition of the property (&&).

(i) First assume $m = 2$ and $N = 3$. Since A and A' are reflexive and $A \neq A'$ $\{T_xA : x \in A_{\mathrm{reg}}\}$ and $\{T_yA' : y \in A'_{\mathrm{reg}}\}$ are different varieties. Thus for general $x \in A$, T_xA is transversal to A'. We get that we may assume that for general $x \in A$, $(T_xA) \cap A'$ is irreducible; hence we may assume $E(x,)$ dense in $(T_xA) \cap A'$. Similarly for general y $E(,y)$ is dense in $(T_yA') \cap A$. For general $a, b \in A$, and every $y \in E(a,) \cap E(b,)$, T_yA' is the linear span $[\{a,b,y\}]$ of the set $\{a,b,u\}$; note that by the density just asserted and the generality of a and b, $E(a,) \cap E(b,) = A' \cap (T_aA \cap T_bA)$. Similarly for general u, v in A'. Thus we see that the Gauss map $\mathbf{g}'\colon A'_{\mathrm{reg}} \longrightarrow \mathbf{P}^{3^*}$ maps 3 general collinear points to 3 collinear points (i.e. to 3 planes thru the same line). As in the proof of [1], last part of step 1 in §2, we get that $\mathbf{g}'$ is induced by a linear isomorphism $t'\colon \mathbf{P}^3 \longrightarrow \mathbf{P}^{3^*}$ (a null-correlation) and A' is induced by (1) for a suitable choice of homogeneous coordinates (and of functions h', b'). By symmetry the Gauss map of A is induced by a linear isomorphism $t\colon \mathbf{P}^3 \longrightarrow \mathbf{P}^{3^*}$ and A is described by (1) for a (possibly different) choice of homogeneous coordinates. The discussion of the meaning of the collineations t, t' given in the proof of 1.3 and property (&&) show that $t = t'$, i.e. that A and A' are described by (1) (for suitable (h,b) and (h',b')) with respect to the same system of coordinates.

(ii) Now assume $m > 2$ and $N = m + 1$. By step (i) we get the irreducibility of $(T_xA) \cap A'$ for general $x \in A$, and the same proof as in step 1 works.

(iii) Assume $N > m + 1$. We may assume that $A \cup A'$ spans $\mathbf{P}^N$; by the definition of (&&) we get easily that either A spans $\mathbf{P}^N$ or A spans a hyperplane H. In the second case, since $\dim(E) = 2m - 1$, by the definition of (&&) all T_xA, $x \in A_{\mathrm{reg}}$, contain $A' \cap H$; this is obviously false for non-linear A'. Thus we may assume that A spans $\mathbf{P}^N$. Since we know that a general projection of A into $\mathbf{P}^{m+1}$ satisfies (&) and is described by (1), we find a contradiciton as in the last step of the proof of 1.2. ∎

2. First we prove the following result.

PROPOSITION 2.1. *Let V be an integral nondegenerate subvariety of* $\mathbf{P}^N$, *d an integer, $d \geq 2$, and v_d the d-ple Veronese embedding of V, say in* $\mathbf{P}^t$. *Let $v_d(V)^*$ be the dual variety*

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of $v_d(V)$. Then $v_d(V)^*$ is a hypersurface. More precisely, for every $x \in v_d(V)_{\mathrm{reg}}$ a general hyperplane tangent to $v_d(V)$ at $v_d(x)$ is tangent to $v_d(V)_{\mathrm{reg}}$ only at that point.

PROOF. It is sufficient to prove the second part of 2.1.

Fix a smooth point x of V and let $L := T_xV$. Let $W(d)$ be the linear system of degree d hypersurfaces tangent to V at x. $W(2)$ has no base point on $\mathbf{P}^N \setminus \{x\}$. Thus we see easily that if $d \geq 3$ the linear system $W(d)$ gives an embedding j of $\mathbf{P}^N \setminus \{x\}$ into a projective space $\mathbf{P}$. Let U be the closure of $j(V \setminus \{x\})$ in $\mathbf{P}$. Applying Bertini's theorem to U, we see that if $d \geq 3$ a general degree d hypersurface tangent to V at x is not tangent to V at any point of $V_{\mathrm{reg}} \setminus \{x\}$.

Now assume $d = 2$. Set $G := \{g \in \mathrm{Aut}(\mathbf{P}^N) : g(x) = x$ and $g(L) = L)$ and $Y := \{(y, M) : y \in \mathbf{P}^N \setminus \{x\}$ and M is a linear space with $\dim(M) = \dim(L)$ and $y \in M\}$. G acts on Y and its orbits are distinguished by the dimensions of $[L; y]$ and $[L; M]$. Fix $(y, M) \in Y$. First assume $y \notin L$. Using reducible quadrics we see that the codimension in $W(2)$ of the set of quadrics thru y and tangent to M at y is $m + 1$. Now assume $y \in L$. Set $M' := M \cap L$ and $k := \dim(M')$. $\mathrm{Aut}(L)$ acts on the possible pairs (y, M') (with k fixed) with exactly 2 orbits (if $k < m$), distinguished by the condition that $[x; y] \subset M'$ or not. Since every element of $\mathrm{Aut}(L)$ is the restriction of some element of $\mathrm{Aut}(\mathbf{P}^n)$, we see that (even when $k = m$) the codimension in $W(2)$ of the set of quadrics tangent to M' at y is k if $[x; y] \subset M'$, $k + 1$ otherwise. Then use reducible quadrics to pass from M' to M and show that (even if $k = m$) the set of quadrics in $W(2)$ containing y and tangent to M has codimension m in $W(2)$ if $[x; y] \subseteq M$ (i.e. $x \in M$) and codimension $m + 1$ otherwise. Since V is not a linear space, $\dim(V \cap L) < \dim(V)$. Thus the thesis follows from a dimensional count. ∎

The first part of 2.1 was known ([4], th. 2.5, or see [6], th. (20) on p. 180) except when $\mathrm{char}(\mathbf{F}) = 2$ and $\dim(V)$ is odd.

Now we prove the following theorem related to the Segre embedding.

THEOREM 2.2. *Let V be an integral complete variety and $i: V \longrightarrow \mathbf{P}^k$, $j: V \longrightarrow \mathbf{P}^r$ two embeddings; set $m := \dim(V)$. Let u be the embedding of V in a projective space $\mathbf{P}$ corresponding to the composition of (i, j) with the Segre embedding. Then:*

(a) if $\mathrm{char}(\mathbf{F}) \neq 2$ or m is even, then $u(V)$ is ordinary;

(b) if $\mathrm{char}(\mathbf{F}) = 2$ and m is odd, then $u(V)$ is semiordinary.

PROOF. The proof is an easy modification of the proof of [6], th. (20) on p. 180. Fix $P \in V_{\mathrm{reg}}$.

(a) Choose systems of inhomogeneous coordinates $T_1, \ldots, T_k$ at $i(P)$ (resp. $L_1, \ldots, L_r$ at $j(P)$) such that $T_1, \ldots, T_m$ (resp. $L_1, \ldots, L_m$) form a regular system of parameters for i(V) (resp. $j(V)$) at $i(P)$ (resp. $j(P)$) and such that $i^*(L_i) \equiv j^*(T_i)$ modulo the square of the maximal ideal of P in V. In $\mathbf{P}$ the form corresponding to $T_1L_1 + \cdots + T_mL_m$ (resp. $T_1L_{s+1} + \cdots + T_sL_m$ if $\mathrm{char}(\mathbf{F}) = 2$ and $m = 2s$) satisfies the Hessian criterion of [3], 3.2, (or see [6], th. (12) on p. 176) at $u(P)$. Thus $u(V)$ is reflexive. Hence the general tangent hyperplane to $u(V)$ is tangent along a linear space; since $u(V)$ contains no positive dimensional linear space, $u(V)^*$ is a hypersurface. Thus $u(V)$ is ordinary.

(b) Assume m odd, say $m = 2s + 1$. Then use the same form on $\mathbf{P}$ as in the case m even, $\mathrm{char}(\mathbf{F}) = 2$. ∎

3. Now we want to give (when $\mathrm{char}(\mathbf{F}) = 0$) a quantitative bound for a result of Hefez and Kleiman ([3], 4.13, or see [6], th. (7) on p. 190) about the non-vanishing of the ranks of any variety $W \subset \mathbf{P}^N$.

THEOREM 3.1. *Assume* $\mathrm{char}(\mathbf{F}) = 0$. *Fix an integral variety* $V \subset \mathbf{P}^N$; *set* $n := \dim(V)$, $d := r_n(V)$ *(the degree of V) and let c be the maximal integer with* $r_{n-c}(V) \neq 0$ *(the codefect of V). Then for every i with* $n - c \leq i < n$, *we have*

$$r_i(V) \leq d(d-1)^{n-i} \text{ and } d \leq r_i(V)\big(r_i(V)-1\big)^{n-i} \tag{2}$$

PROOF. Taking general hyperplane sections, we may assume $c = 0$ ([6], th. (5) on p. 189) and reduce both inequalities to the case $i = 0$ ([6], th. (5) on p. 189). Taking a general projection, we may assume that V is a hypersurface in $\mathbf{P}^{n+1}$ ([6], th. (5) on p. 189). Thus V is in the closure of the family of smooth hypersurfaces. For every smooth hypersurface Y of degree d, we have $r_0(Y) = d(d-1)^n$. Fix a general linear space L with $\dim(L) = n - 1$; we assume that there are exactly $r_0(V)$ hyperplanes containing L and tangent to V (at points of V_{reg}). There is an open subset U of the family $S(d)$ of smooth degree d hypersurfaces for which this is true. For $u \in U$, let V_u be the corresponding hypersurface. Set $\Gamma := \{(H,u) \in \mathbf{P}^{n+1^*} \times U : L \subset H \text{ and } H \text{ is tangent to } V_u\}$. Every hyperplane H containing L and tangent to V is such that (H,u) is in the closure in $\mathbf{P}^{n+1^*} \times S(d)$ of Γ (and even there is a subvariety J of U with $[V]$ in its closure in $S(d)$ and $(H,u) \in \Gamma$ for every $u \in J$). Thus $r_0(V) \leq d(d-1)^n$. Note that $d = r_n(V)$ ([6], prop. (2)(i) on p. 156). Thus we have the first inequality in 3.1. Since $\mathrm{char}(\mathbf{F}) = 0$, every variety X is reflexive and, if X has dimension n, $r_i(X) = r_{n-i}(X^*)$ ([6], th. (4) on p. 189). Thus applying the first part to V^*, we get the second inequality. ∎

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