REMARK ON A CRITERION FOR COMMON TRANSVERSALS

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All sets considered will be finite, and $|X|$ will denote the cardinal number of the set X.

Let $\mathfrak{A} = (A_i : i \in I)$ be a family of subsets of a set E. A subset $E' \subseteq E$ is called a *transversal* of $\mathfrak{A}$ if there exists a bijection $\sigma: E' \to I$ such that $e \in A_{\sigma(e)}$ $(e \in E')$. According to a well-known theorem of P. Hall [2], *the family* $\mathfrak{A}$ *has a transversal if and only if* $|\{\bigcup A_i : i \in I'\}| \geqq |I'|$ *for every subset* I' *of* I. Ford and Fulkerson [1] obtained (as a special case of a more general theorem) an analogous criterion for the existence of a common transversal (CT) of two families. We may state their result in the following terms.

The families $\mathfrak{A} = (A_i : i \in I)$, $\mathscr{B} = (B_j : j \in J)$ *of subsets of* E, *where* $|I| = |J| = n$, *have a CT if and only if*

$$|(\bigcup_{i \in I'} A_i) \cap (\bigcup_{j \in J'} B_j)| \geqq |I'| + |J'| - n \tag{1}$$

whenever $I' \subseteq I$, $J' \subseteq J$.

The original proof of this depended on the max-flow min-cut theorem from the theory of flows in networks. This theorem is intimately connected with Menger's graph theorem [3], and it is therefore not surprising that a direct deduction of the Ford–Fulkerson criterion can be made from Menger's theorem. Another treatment [4] depends on the theory of transversal independence. Our purpose here is to indicate a simple argument which relies solely on Hall's theorem.

Assume that $E \cap I = E \cap J = \emptyset$; and consider the family $\mathfrak{X} = (X_k : k \in I \cup E)$ of subsets of $E \cup J$ defined by the requirements:

$$X_k = A_k \ (k \in I), \qquad X_k = \{k\} \cup \{j : j \in J, k \in B_j\} \qquad (k \in E).$$

We assert that $\mathfrak{A}$ *and* $\mathfrak{B}$ *have a CT if and only if* $\mathfrak{X}$ *has a transversal.*

Write $E = \{e_1, \ldots, e_m\}$, $I = J = \{1, \ldots, n\}$. Suppose that $\mathfrak{X}$ has a transversal (which must be the whole of $E \cup J$). This implies, after appropriately ordering the e's, A's and B's, that $n \leqq m$ and

$$e_1 \in X_1 = A_1, \ \ldots, \ e_n \in X_n = A_n,$$

$$e_{n+1} \in X_{e_{n+1}}, \ \ldots, \ e_m \in X_{e_m},$$

$$1 \in X_{e_1}, \qquad \ldots, \ n \in X_{e_n}.$$

The last line is equivalent to the statements

$$e_1 \in B_1, \ldots, e_n \in B_n;$$

and therefore $\mathfrak{A}$ and $\mathfrak{B}$ possess the CT $\{e_1, \ldots, e_n\}$.

The converse is also easy to prove.

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Now, by Hall's theorem, $\mathfrak{X}$ has a transversal if and only if

$$\left|\bigcup_{k\in K'} X_k\right| \geqq |K'|,$$

whenever $K' \subseteq I \cup E$. Write $K' = I' \cup E'$, where $I' \subseteq I$, $E' \subseteq E$; then this is equivalent to the condition

$$\left|\left(\bigcup_{i\in I'} A_i\right)\cup\left(\bigcup_{k\in E'} X_k\right)\right| \geqq |I'|+|E'|, \tag{2}$$

whenever $I' \subseteq I$, $E' \subseteq E$. Further,

$$\left(\bigcup_{i\in I'} A_i\right)\cap\left(\bigcup_{k\in E'} X_k\right) = \left(\bigcup_{i\in I'} A_i\right)\cap E'$$

and

$$\bigcup_{k\in K'} X_k = E'\cup\{j : j\in J, B_j\cap E' \neq \emptyset\};$$

and so, after a simple rearrangement of terms, we may write (2) in the form

$$\left|\left(\bigcup_{i\in I'} A_i\right)\cap(E-E')\right| + \left|\{j : j\in J, B_j\cap E' \neq \emptyset\}\right| \geqq |I'|, \tag{3}$$

whenever $I' \subseteq I$, $E' \subseteq E$.

It remains to establish the equivalence of (1) and (3). To prove the implication (3) $\Rightarrow$ (1), it suffices to define, for each $J' \subseteq J$, the set E' by the equation $E-E' = \bigcup\{B_j : j\in J'\}$. The reverse implication (1) $\Rightarrow$ (3) is proved if, for each $E' \subseteq E$, we take $J' = \{j : j\in J, B_j\cap E' = \emptyset\}$.

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