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REVISITING THE RECTANGULAR CONSTANT IN BANACH SPACES

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Abstract

Let X be a real Banach space. The rectangular constant $\mu(X)$ and some generalisations of it, $\mu_p(X)$ for $p \geq 1$, were introduced by Gastinel and Joly around half a century ago. In this paper we make precise some characterisations of inner product spaces by using $\mu_p(X)$, correcting some statements appearing in the literature, and extend to $\mu_p(X)$ some characterisations of uniformly nonsquare spaces, known only for $\mu(X)$. We also give a characterisation of two-dimensional spaces with hexagonal norms. Finally, we indicate some new upper estimates concerning $\mu(l_p)$ and $\mu_p(l_p)$.

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1. Introduction

LetX be a real Banach space. Let us denote by $B(X)$ and $S(X)$ the unit ball and the unit sphere, respectively.The vector x is (Birkhoff–James) orthogonalto y (which we denote by $x \perp y$) if $\|x\| \leq \|x + \lambda y\|$ for every real λ. In [8] (see also [9]), the rectangular constant was introduced:

$$\mu(X) = \sup\left\{\frac{1+\lambda}{\|x+\lambda y\|} : x, y \in S(X),\ x \perp y,\ \lambda \geq 0\right\}.$$

In [8], Joly proved that $\sqrt{2} \leq \mu(X) \leq 3$ and, for $\dim(X) \geq 3$, that $\mu(X) = \sqrt{2}$ if and only if X is a Hilbert space. In [4], the equivalence was extended to two-dimensional spaces. Moreover, in [1], the following result was proved: $\mu(X) = 3$ if and only if the space X is nonuniformly nonsquare. We recall that a space X is nonuniformly nonsquare (non-UNS for short) if for every $\epsilon > 0$ there exist $x, y \in S(X)$ such that $\|x \pm y\| > 2 - \epsilon$.

In [6], Gastinel and Joly extended the definition of the rectangular constant: for $p \geq 1$, $x, y \in S(X)$ and $x \perp y$,

$$\mu_p(x,y) = \sup_{\lambda \geq 0}\left\{\frac{(1+\lambda^p)^{1/p}}{\|x+\lambda y\|}\right\}$$

and

$$\mu_p(X) = \sup\{\mu_p(x, y) : x, y \in S(X),\ x \perp y\}.$$

We note that $\mu_1(X) = \mu(X)$. The following properties are proved in [6].

(A) We have $2^{(2-p)/2p} \le \mu_p(X) \le 3$. We remark that $\mu_p(X)$ is never smaller than 1 and so the left-hand inequality is meaningful only for $1 \le p < 2$. In Theorem 2.1 we will prove better estimates.

(B) If $\dim(X) \ge 3$, then X is a Hilbert space if and only if $\mu_p(X) = 2^{(2-p)/2p}$. By the preceding remark this is true only for $1 \le p \le 2$. In Theorem 2.1 we will revise this result by proving that, for $p \ge 2$, X is a Hilbert space if and only if $\mu_p(X) = 1$.

In Section 3 we will extend the characterisation of nonuniformly nonsquare spaces in terms of the parameter $\mu_p(X)$. More precisely we will prove that a space X is non-UNS if and only if $\mu_p(X) = (1 + 2^p)^{1/p}$ for every $p \ge 1$.

In Section 4 we will give a characterisation of two-dimensional spaces with symmetric orthogonality by using the parameter $\mu_p(X)$ and, finally, in the last section we will improve some upper bounds obtained in [5] for the parameter $\mu(l_p)$.

2. Revisiting the Hilbert space characterisation

As we have already remarked, Proposition 7.2.4 in [6] is correct only for $1 \le p \le 2$. In the following theorem we give the correct result for $p > 2$.

THEOREM 2.1. *Let X be a real Banach space and $p \ge 1$.*

(i) *We have* $\max\{1,\ 2^{1/p-1}\mu(X)\} \le \mu_p(X) \le \min\{\mu(X),\ (1 + 2^p)^{1/p}\}$.

(ii) *If $p \ge 2$ and $\dim(X) \ge 3$, then X is a Hilbert space if and only if $\mu_p(X) = 1$.*

PROOF. From the inequality $a^p + b^p \le (a + b)^p \le 2^{p-1}(a^p + b^p)$, where a and b are nonnegative scalars, it follows immediately that $2^{1/p-1}\mu(X) \le \mu_p(X) \le \mu(X)$. The inequality $\mu_p(X) \ge 1$ is trivial. Finally, since $\|x + \lambda y\| \ge 1$ and $\|x + \lambda y\| \ge |\lambda - 1|$,

$$\frac{1 + \lambda^p}{\|x + \lambda y\|^p} \le \min\left(1 + \lambda^p,\ \frac{1 + \lambda^p}{|\lambda - 1\,|^p}\right) \le 1 + 2^p.$$

This concludes the proof of the first statement.

Suppose now that $\mu_p(X) = 1$. If $x, y \in S(X)$ and $x \perp y$, then $(1 + \lambda^p)/\|x + \lambda y\|^p \le 1$. This implies that $\lambda \le \|x + \lambda y\|$ or equivalently $1 \le \|x/\lambda + y\|$ for every $\lambda > 0$. Replacing x by $-x$, for every λ we have $\|\lambda x + y\| \ge 1 = \|y\|$, which means that $y \perp x$. So, orthogonality is symmetric and so by [7] the space X is a Hilbert space. It is easy to prove that if X is a Hilbert space, then $\mu_p(X) = 1$. □

Therefore, the correct characterisation of Hilbert spaces in terms of $\mu_p(X)$ is the following: if $\dim(X) \ge 3$, then X is a Hilbert space if and only if $\mu_p(X) = \max\{1, 2^{(2-p)/2p}\}$.

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3. Uniformly nonsquare spaces

In this section we extend Theorem 4 in [1]. We recall that the property that X is non-UNS can equivalently be defined in the following way: for every $\epsilon > 0$, there exist $x, y \in S(X)$ such that $\|x \pm y\| < 1 + \epsilon$.

THEOREM 3.1. *The following conditions are equivalent:*

(a) *X is non-UNS;*
(b) *for every $p \geq 1$, $\mu_p(X) = (1 + 2^p)^{1/p}$;*
(c) *there exists $p \geq 1$ such that $\mu_p(X) = (1 + 2^p)^{1/p}$.*

PROOF. (c $\Rightarrow$ a) If $\mu_p(X) = (1 + 2^p)^{1/p}$, then, for every $\epsilon > 0$, there exist $\lambda_\epsilon > 0$ and $x_\epsilon, y_\epsilon \in S(X)$ with $x_\epsilon \perp y_\epsilon$ such that

$$1 + 2^p - \epsilon \leq \frac{1 + \lambda_\epsilon^p}{\|x_\epsilon + \lambda_\epsilon y_\epsilon\|^p} \leq 1 + 2^p.$$

It is easy to show that this implies that $2^p - \epsilon < \lambda_\epsilon^p < 2^p + \delta(\epsilon)$ with $\delta(\epsilon) \to 0$ when $\epsilon \to 0$. From this,

$$\|x_\epsilon + \lambda_\epsilon y_\epsilon\| \leq \left(\frac{1 + \lambda_\epsilon^p}{1 + 2^p - \epsilon}\right)^{1/p} \leq \left(\frac{1 + 2^p + \delta(\epsilon)}{1 + 2^p - \epsilon}\right)^{1/p}$$

and so $1 \leq \|x_\epsilon + \lambda_\epsilon y_\epsilon\| \leq 1 + \eta(\epsilon)$ with $\eta(\epsilon) \to 0$ when $\epsilon \to 0$. Next, $f(t) = \|x_\epsilon + t y_\epsilon\|$ is a convex function such that $1 \leq f(t)$, $f(0) = 1$ and $f(\lambda_\epsilon) \leq 1 + \eta(\epsilon)$, so it follows that $1 \leq \|x_\epsilon + y_\epsilon\| < 1 + \eta(\epsilon)$. Let $z = (x_\epsilon + y_\epsilon)/\|x_\epsilon + y_\epsilon\|$. Then

$$\begin{aligned}\|z + y_\epsilon\| = \frac{\|x_\epsilon + y_\epsilon + \|x_\epsilon + y_\epsilon\|\, y_\epsilon\,\|}{\|x_\epsilon + y_\epsilon\|} &\leq \|x_\epsilon + \lambda_\epsilon y_\epsilon + (1 + \|x_\epsilon + y_\epsilon\| - \lambda_\epsilon) y_\epsilon\,\| \\ &\leq \|x_\epsilon + \lambda_\epsilon y_\epsilon\| + |(1 + \|x_\epsilon + y_\epsilon\| - \lambda_\epsilon)| \\ &\leq 1 + \eta(\epsilon) + |1 - \|x_\epsilon + y_\epsilon\|| + |\lambda_\epsilon - 2| = 1 + \delta_1(\epsilon)\end{aligned}$$

with $\delta_1(\epsilon) \to 0$ when $\epsilon \to 0$. Similarly,

$$\begin{aligned}\|z - y_\epsilon\| = \frac{\|x_\epsilon + y_\epsilon - \|x_\epsilon + y_\epsilon\| y_\epsilon\|}{\|x_\epsilon + y_\epsilon\|} &\leq \|x_\epsilon + (1 - \|x_\epsilon + y_\epsilon\|) y_\epsilon\| \\ &\leq \|x_\epsilon\| + |\,1 - \|x_\epsilon + y_\epsilon\|\,| \leq 1 + \eta(\epsilon).\end{aligned}$$

So, X is non-UNS.

(a $\Rightarrow$ b) Let X be non-UNS. Fix ϵ with $0 < \epsilon < 1/2$. There exist $x, y \in S(X)$ such that $\|x \pm y\| > 2 - \epsilon^2$. By convexity, $\|\lambda x \pm (1 - \lambda) y\| \geq 1 - 2\epsilon^2$ for every $\lambda \in [0, 1]$. Moreover, $\|x + \epsilon y\| > 1$. Indeed, $2 - \epsilon^2 < \|x + y\| \leq \|x + \epsilon y\| + 1 - \epsilon$ and so $\|x + \epsilon y\| > 1 + \epsilon - \epsilon^2$. Let $F(\lambda) = \|\epsilon y + \lambda(x - (x + \epsilon y)/\|x + \epsilon y\|)\|$. It is easy to show that F is a convex function with $F(1) = F(\|x + \epsilon y\|)$ and so there exists $\lambda_0 > 1$ such that F attains its minimum. It follows that the two vectors $a = \epsilon y + \lambda_0(x - (x + \epsilon y)/\|x + \epsilon y\|)$ and

$b = x - (x + \epsilon y)/\|x + \epsilon y\|$ are orthogonal. Moreover,

$$\|b\| = \left(\frac{\|x + \epsilon y\| - 1 + \epsilon}{\|x + \epsilon y\|}\right)\left\|\frac{\|x + \epsilon y\| - 1}{\|x + \epsilon y\| - 1 + \epsilon}x + \frac{\epsilon}{\|x + \epsilon y\| - 1 + \epsilon}(-y)\right\|$$

$$\geq \left(\frac{\|x + \epsilon y\| - 1 + \epsilon}{\|x + \epsilon y\|}\right)(1 - 2\epsilon^2) \geq \left(\frac{(1 + \epsilon)\left\|\frac{x}{1+\epsilon} + \frac{\epsilon y}{1+\epsilon}\right\| - 1 + \epsilon}{1 + \epsilon}\right)(1 - 2\epsilon^2)$$

$$\geq \frac{1 - 2\epsilon^2}{1 + \epsilon}((1 + \epsilon)(1 - 2\epsilon^2) - 1 + \epsilon) = \frac{1 - 2\epsilon^2}{1 + \epsilon}\epsilon(2 - 2\epsilon - 2\epsilon^2) = \epsilon(2 - \eta(\epsilon))$$

with $\eta(\epsilon) \to 0$ as $\epsilon \to 0$. Finally, recalling that $\lambda_0 > 1$,

$$\mu_p^p(X) \geq \frac{\|a\|^p + \lambda_0^p\|b\|^p}{\|a - \lambda_0 b\|} \geq \frac{1}{\epsilon^p}\left(\left\|\epsilon y + \lambda_0\left(x - \frac{x + \epsilon y}{\|x + \epsilon y\|}\right)\right\|^p + \epsilon^p(2 - \eta(\epsilon))^p\right)$$

$$\geq \frac{1}{\epsilon^p}\left(\lambda_0\left\|x - \frac{x + \epsilon y}{\|x + \epsilon y\|}\right\| - \epsilon\right)^p + \epsilon^p(2 - \eta(\epsilon))^p\Big)$$

$$\geq \frac{1}{\epsilon^p}((\epsilon(2 - \eta(\epsilon)) - \epsilon)^p + \epsilon^p(2 - \eta(\epsilon))^p) = (1 - \eta(\epsilon))^p + (2 - \eta(\epsilon))^p. \qquad \square$$

We remark that it is easy to show (see [6]) that if $\mu(X) = 3$ is attained, that is, if there exist x and y such that $x \perp y$ and $\mu_1(x, y) = 3$, then there is a segment of length 2 on the unit sphere. (See also [10] for an extension of this result.) The space $X = (\prod_{n=2}^{\infty} l_n^2)_2$ is a non-UNS space but it is strictly convex (see [2, page 185]), so in this space $\mu(X) = 3$ but it is not attained. This gives an affirmative answer to Remark 2.2 in [10].

4. Symmetric orthogonality

We have already remarked that if $\dim(X) \geq 3$, the symmetry of Birkhoff–James orthogonality implies that X is a Hilbert space. However, there are two-dimensional spaces which are not Hilbert spaces but orthogonality is still symmetric. A simple example is the space X with the 'hexagonal' norm, that is, the norm generated by a regular hexagon. An easy evaluation shows that $\mu_p(X) = 2^{1/p}$ for any $p \geq 1$. In this section we give a necessary and sufficient condition for a two-dimensional space X to be isometric to a space with 'hexagonal' norm. We denote by $J_x^+(y)$ the right derivative of the norm at x, that is, $J_x^+(y) = \lim_{\lambda \to 0^+} (\|x + \lambda y\| - \|x\|)/\lambda$ and similarly by $J_x^-(y)$ for the left derivative. The following properties of $J_x^{\pm}(y)$ are easy to prove.

LEMMA 4.1. *For $x, y \in X$:*

(1) $J_x^+(y) = \sup\{f(y) : f \in S(X^*),\ f(x) = \|x\|\}$;
(2) $J_x^-(y) = \inf\{f(y) : f \in S(X^*),\ f(x) = \|x\|\}$;
(3) $J_x^+(x + y) = J_x^+(x) + J_x^+(y)$ *and* $J_x^-(x + y) = J_x^-(x) + J_x^-(y)$;
(4) $J_x^+(-y) = -J_x^+(y)$ *and* $J_x^-(-y) = -J_x^-(y)$;
(5) $J_x^-(y) \leq 0 \leq J_x^+(y)$ *if and only if* $x \perp y$.

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LEMMA 4.2. *Let $x \perp y$ with $x, y \in S(X)$ and $p > 1$. Let λ_0 be such that*

$$\mu_p^p(x,y) = \frac{1+\lambda_0^p}{\|x+\lambda_0 y\|^p}.$$

Then $x + \lambda_0 y \perp \lambda_0^{p-1}x - y$.

PROOF. Let $F(\lambda) = (1+\lambda^p)/\|x+\lambda y\|^p$ and suppose that $F(\lambda_0) \geq F(\lambda)$ for every $\lambda \geq 0$. Then

$$F'_+(\lambda) = \frac{p\lambda^{p-1}\|x+\lambda y\|^p - p(1+\lambda^p)\|x+\lambda y\|^{p-1}J^+_{x+\lambda y}(y)}{\|x+\lambda y\|^{2p}}.$$

So, $F'_+(\lambda_0) \leq 0$ and, by Lemma 4.1,

$$[\lambda_0^{p-1}\|x+\lambda_0 y\| - (1+\lambda_0^p)J^+_{x+\lambda_0 y}(y)] = \lambda_0^{p-1}J^-_{x+\lambda_0 y}(x+\lambda_0 y) + J^-_{x+\lambda_0 y}((-1-\lambda_o^p)y) \leq 0.$$

Moreover,

$$J^-_{x+\lambda_0 y}(\lambda_0^{p-1}x + \lambda_0^p y - y - \lambda_0^p y) = J^-_{x+\lambda_0 y}(\lambda_0^{p-1}x - y) \leq 0.$$

Finally, in the same way, $J^+_{x+\lambda_0 y}(\lambda_0^{p-1}x - y) \geq 0$ and this implies that $x + \lambda_0 y \perp \lambda_0^{p-1}x - y$. □

THEOREM 4.3. *Let $\dim(X) = 2$ and suppose that Birkhoff–James orthogonality is symmetric. Then the following statements are equivalent:*

(a) *X has a 'hexagonal' norm;*
(b) *$\mu_p(X) = 2^{1/p}$ for every $p \geq 1$;*
(c) *there exists $p \geq 1$ such that $\mu_p(X) = 2^{1/p}$.*

PROOF. Suppose that $\mu_p(X) = 2^{1/p} = (1+\lambda_0^p)^{1/p}/\|x+\lambda_0 y\|$. Since the orthogonality is symmetric,

$$2 = \frac{1+\lambda_0^p}{\|x+\lambda_0 y\|^p} \leq 1 + \lambda_0^p$$

and

$$2 = \frac{1+\lambda_0^p}{\|x+\lambda_0 y\|^p} \leq \frac{1+\lambda_0^p}{\lambda_0^p}$$

and this implies that $\lambda_0 = 1$. So, $\|x+\lambda y\| = 1$ for $\lambda \in [0,1]$ and, by Lemma 4.2, we obtain $x + y \perp x - y$. Consider the linear map $T : X \to \mathbb{R}^2$ with $T(x) = (1,0)$ and $T(y) = (-1,1)$ and define $|||T(z)||| = \|z\|$. Then $\|x+\lambda y\| = 1$ implies that $|||(1-\lambda,\lambda)||| = 1$ for $\lambda \in [0,1]$. Again, $|||\lambda(-1,1) + (1-\lambda)(0,1)||| = \|y + 2(1-\lambda)x\| \geq \|y\| = 1$ for $\lambda \in [0,1]$ and by convexity $|||\lambda(-1,1)+(1-\lambda)(0,1)||| = 1$. Since $\|x+y\| = 1 = \|y\| \leq \|x+y+\lambda x\|$, it follows that $x \perp x+y$. Finally, we observe that $|||\lambda(-1,1)+(1-\lambda)(-1,0)||| = \|x - \lambda(x+y)\| \geq 1$ and again, by convexity, we have $|||\lambda(-1,1)+(1-\lambda)(-1,0)||| = 1$ for every $\lambda \in [0,1]$. □

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We conclude this section by showing that in the class of two-dimensional spaces with symmetric orthogonality we always have $\mu_p(H) \leq \mu_p(X) \leq 2^{1/p}$, where H denotes the Euclidean plane. As shown by Theorem 4.3, the upper bound is attained by the hexagonal norm.

THEOREM 4.4. *Let X be a two-dimensional space with symmetric orthogonality. Then $\mu_p(X) \leq 2^{1/p}$.*

PROOF. Let $x, y \in S(X)$ and $x \perp y$ (and so $y \perp x$). Then

$$\sup_{\lambda \geq 0} \frac{(1+\lambda^p)^{1/p}}{\|x+\lambda y\|} = \sup_{\lambda > 0} \frac{(1+(1/\lambda^p))^{1/p}}{\|x+(1/\lambda)y\|} = \sup_{\lambda \geq 0} \frac{(1+\lambda^p)^{1/p}}{\|y+\lambda x\|},$$

so that $\mu_p(x, y) = \mu_p(y, x)$. Let λ_0 be such that $\mu_p(x, y) = (1+\lambda_0^p)^{1/p}/\|x+\lambda_0 y\|$. Then

$$\mu_p(x, y) = \frac{(1+\lambda_0^p)^{1/p}}{\|x+\lambda_0 y\|} = \frac{(1+(1/\lambda_0)^p)^{1/p}}{\|y+(1/\lambda_0)x\|} \leq \mu_p(y, x) = \mu_p(x, y).$$

Since λ_0 or $1/\lambda_0$ is less than or equal to 1, this proves the theorem. □

5. Estimates in l_p spaces

The exact value of the parameter $\mu_p(X)$ is in general unknown. However, as we have already claimed, if X is a Hilbert space, then $\mu(X) = \sqrt{2}$. It is also easy to obtain $\mu(l_1) = \mu(l_\infty) = 3$. These results also follow from Theorem 3.1. Some bounds for l_p spaces are given in [5]: $\mu(l_p) \leq (5+\sqrt{p})/(1+\sqrt{p})$ for $1 \leq p \leq 2$ and $\mu(l_p) \leq 3 - 2/3p$ for $p \geq 2$. In the next theorems, we will improve these estimates.

LEMMA 5.1. *Let $p \geq 2$, $x, y \in S(l_p)$ and $x \perp y$. Then, for every $\lambda \geq 0$,*

$$\|x+\lambda y\|^p \geq 1 + \frac{\lambda^p}{2^{p-1}-1}.$$

PROOF. The proof follows easily if we prove that for every N,

$$\|x+\lambda y\|^p \geq 1 + \Big(\sum_{n=1}^{N} 2^{n(1-p)}\Big)\lambda^p. \tag{5.1}$$

From the well-known Clarkson inequality,

$$2(\|u\|^p + \|v\|^p) \leq \|u+v\|^p + \|u-v\|^p,$$

choosing $u = x + \lambda y$ and $v = \lambda y$,

$$2(\|x+\lambda y\|^p + \lambda^p) \leq \|x+2\lambda y\|^p + 1. \tag{5.2}$$

Since $\|x+\lambda y\| \geq 1$, it follows that $\|x+2\lambda y\|^p \geq 1 + 2\lambda^p$. From this,

$$\|x+\lambda y\|^p \geq 1 + 2^{1-p}\lambda^p. \tag{5.3}$$

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This shows that (5.1) is true for $N = 1$. Let us suppose that (5.1) is true. Then, by (5.2),

$$\begin{aligned}\|x + 2\lambda y\|^p &\geq 2\|x + \lambda y\|^p + 2\lambda^p - 1 \\ &\geq 2\Big(1 + \Big(\sum_{n=1}^{N} 2^{n(1-p)}\Big)\lambda^p\Big) + 2\lambda^p - 1 = 1 + \Big(\sum_{n=0}^{N} 2^{n(1-p)+1}\Big)\lambda^p.\end{aligned}$$

This implies that

$$\|x + \lambda y\|^p \geq 1 + \Big(\sum_{n=0}^{N} 2^{n(1-p)+1}\Big)\frac{\lambda^p}{2^p} = 1 + \Big(\sum_{n=1}^{N+1} 2^{n(1-p)}\Big)\lambda^p. \qquad \square$$

THEOREM 5.2. *For $p \geq 2$,*

$$\mu(l_p) \leq (1 + (2^{p-1} - 1)^{1/(p-1)})^{(p-1)/p}.$$

PROOF. Suppose that $x, y \in S(l_p)$ with $x \perp y$. Then, by Lemma 5.1,

$$\mu(x, y) = \sup_{\lambda \geq 0} \frac{1 + \lambda}{\|x + \lambda y\|} \leq \sup_{\lambda \geq 0} \frac{1 + \lambda}{(1 + \frac{\lambda^p}{2^{p-1}-1})^{1/p}}.$$

It is easy to show that the function

$$\phi(\lambda) = \frac{1 + \lambda}{(1 + \frac{\lambda^p}{2^{p-1}-1})^{1/p}}$$

attains its maximum value for $\lambda = (2^{p-1} - 1)^{1/(p-1)}$, so

$$\mu(x, y) \leq \phi((2^{p-1} - 1)^{1/(p-1)}) = (1 + (2^{p-1} - 1)^{1/(p-1)})^{(p-1)/p}. \qquad \square$$

LEMMA 5.3. *Let $1 < p \leq 2$, $x, y \in S(l_p)$ and $x \perp y$. Then, for every $\lambda \geq 0$,*

$$\|x + \lambda y\|^q \geq 1 + \frac{\lambda^q}{2^{q-1} - 1},$$

where q and p are conjugate indices.

PROOF. Starting from the inequality

$$2^{q-1}(\|u\|^q + \|v\|^q) \geq \|u + v\|^q + \|u - v\|^q,$$

the proof follows with similar arguments to those in Lemma 5.1. $\square$

THEOREM 5.4. *For $1 < p \leq 2$,*

$$\mu(l_p) \leq (1 + (2^{1/(p-1)} - 1)^{p-1})^{1/p}.$$

PROOF. The proof of the theorem is similar to that of Theorem 5.2 with the help of Lemma 5.3. $\square$

THEOREM 5.5. *For* $1 < p \leq 2$,

$$\mu(l_p) \leq \sqrt{\frac{p}{p-1}}.$$

PROOF. For $f, g \in l_p$, we use the inequality (see [3])

$$2\|f\|^2 + 2\|g\|^2 \geq \|f+g\|^2 + (p-1)\|f-g\|^2.$$

Let $x, y \in S(l_p)$ with $x \perp y$. If $f = \frac{1}{2}x + \lambda y$ and $g = \frac{1}{2}x$,

$$2\left\|\frac{x}{2} + \lambda y\right\|^2 + 2\left\|\frac{x}{2}\right\|^2 \geq \|x + \lambda y\|^2 + (p-1)\|\lambda y\|^2$$

or equivalently

$$\|x + 2\lambda y\|^2 \geq 2\|x + \lambda y\|^2 + 2(p-1)\lambda^2 - 1. \tag{5.4}$$

Since $\|x + \lambda y\| \geq 1$, we obtain $\|x + 2\lambda y\|^2 \geq 1 + 2(p-1)\lambda^2$ and, from this,

$$\|x + \lambda y\|^2 \geq 1 + \frac{p-1}{2}\lambda^2.$$

It is now easy to prove by induction that

$$\|x + \lambda y\|^2 \geq 1 + \frac{2^n - 1}{2^n}(p-1)\lambda^2$$

and so $\|x + \lambda y\|^2 \geq 1 + (p-1)\lambda^2$. This last inequality yields

$$\mu(x, y) = \sup_{\lambda \geq 0} \frac{1+\lambda}{\|x + \lambda y\|} \leq \sup_{\lambda \geq 0} \frac{1+\lambda}{(1 + (p-1)\lambda^2)^{1/2}}.$$

Finally, simple calculations show that, for $\lambda > 0$,

$$\frac{1+\lambda}{(1 + (p-1)\lambda^2)^{1/2}} \leq \sqrt{\frac{p}{p-1}}.$$ □

From the last two theorems,

$$\mu(l_p) \leq \min\left((1 + (2^{1/(p-1)} - 1)^{p-1})^{1/p}, \sqrt{\frac{p}{p-1}}\right).$$

A numerical evaluation shows that

$$(1 + (2^{1/(p-1)} - 1)^{p-1})^{1/p} \leq \sqrt{\frac{p}{p-1}}$$

for $1 < p \leq p_0$ with $p_0 \simeq 1.188$.

With the aid of Lemmas 5.1 and 5.3, we also obtain some estimates for $\mu_p(l_p)$.

THEOREM 5.6. *For* $p \geq 2$,

$$\mu_p(l_p) \leq (2^{p-1} - 1)^{1/p}.$$

PROOF. Suppose that $x, y \in S(l_p)$ with $x \perp y$. Then, by Lemma 5.1,

$$\mu_p^p(x, y) = \sup_{\lambda \geq 0} \frac{1 + \lambda^p}{\|x + \lambda y\|^p} \leq \sup_{\lambda \geq 0} \frac{1 + \lambda^p}{1 + \frac{\lambda^p}{2^{p-1}-1}} = \sup_{\lambda \geq 0} \phi(\lambda) \quad \text{(say)}.$$

It is easy to show that the function $\phi(\lambda)$ is increasing, so

$$\mu_p(x, y) \leq \lim_{\lambda \to \infty} \left(\frac{1 + \lambda^p}{1 + \frac{\lambda^p}{2^{p-1}-1}} \right)^{1/p} = (2^{p-1} - 1)^{1/p}. \qquad \square$$

THEOREM 5.7. *For* $1 < p < 2$,

$$\mu_p(l_p) \leq (1 + (2^{1/(p-1)} - 1)^{(p-1)/(2-p)})^{(2-p)/p}.$$

PROOF. The proof is similar to that of Theorem 5.6. Suppose that $x, y \in S(l_p)$ with $x \perp y$. Then, by Lemma 5.3,

$$\|x + \lambda y\|^q \geq 1 + \frac{\lambda^q}{2^{q-1} - 1},$$

where q and p are conjugate indices. Consequently,

$$\mu_p^p(x, y) = \sup_{\lambda \geq 0} \frac{1 + \lambda^p}{\|x + \lambda y\|^p} \leq \sup_{\lambda \geq 0} \frac{1 + \lambda^p}{(1 + \frac{\lambda^q}{2^{q-1}-1})^{p/q}} = \sup_{\lambda \geq 0} \phi(\lambda) \quad \text{(say)}.$$

Finally, the maximum of the function $\phi(\lambda)$ is the upper bound in the theorem. $\square$

In [6], Gastinel and Joly proved that $\mu(l_p) = \mu(l_p^2)$, where l_p^2 is the two-dimensional l_p space and they gave a table with some numerical estimates of the rectangular constant for l_p^2 spaces. In their subsequent Remark 11.4.1, they suggested that probably $\mu(l_p) = \mu(l_q)$, where p and q are conjugate. We conclude with a similar table (see Table 1) obtained with more accurate calculations, which shows instead that, in general, they are different.

TABLE 1. Values of $\mu(l_p)$ and $\mu(l_q)$.

p	$\mu(l_p)$	q	$\mu(l_q)$
2	1.4142	2	1.4142
3	1.7285	3/2	1.6554
4	1.9337	4/3	1.8264
5	2.0772	5/4	1.9554
10	2.4328	10/9	2.3099
15	2.5826	15/14	2.4742
30	2.7598	30/29	2.6819
50	2.8429	50/49	2.7854
100	2.9131	100/99	2.8771

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