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# AVERAGES INVOLVING FOURIER COEFFICIENTS OF NON-ANALYTIC AUTOMORPHIC FORMS(1)

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1. **Introduction.** Let $f(\tau)$ be a complex valued function, defined and analytic in the upper half of the complex $\tau$ plane ($\tau=x+iy$, $y>0$), such that $f(\tau+\lambda)=f(\tau)$ where $\lambda$ is real and $f(-1/\tau)=\gamma(-i\tau)^k f(\tau)$, $k$ being a complex number. The function $(-i\tau)^k$ is defined as $e^{k\log(-i\tau)}$ where $\log(-i\tau)$ has the real value when $-i\tau$ is positive and $\gamma$ is a complex number with absolute value 1. Such functions have been studied by E. Hecke [4] who calls them functions with signature $(\lambda, k, \gamma)$. We further assume that $f(\tau)=O(|y|^{-c})$ as $y$ tends to zero uniformly for all $x$, $c$ being a positive real number. It then follows that $f(\tau)$ has a Fourier expansion of the type $f(\tau)=a_0+\sum a_n \exp(2\pi in\tau/\lambda)$ $(n=1, 2, \ldots)$, the series being convergent absolutely in the upper half plane. $f(\tau)$ is called an automorphic form belonging to the group generated by the transformations $\tau\to\tau+\lambda$, $\tau\to-1/\tau$ and $a_n$ $(n=1, 2, \ldots)$ are called the Fourier coefficients of $f(\tau)$. Examples of such functions $f(\tau)$ are quite numerous. For $\lambda=1$, we get analytic modular forms of dimension $-k$ and $\Delta(\tau)$, the discriminant in the theory of elliptic functions, is a well known example. The corresponding Fourier coefficients $a_n$ then define the well known Ramanujan $\tau$ function. For $\lambda=2$, we get automorphic forms of "stufe 2". If $\theta(\tau)=\sum e^{\pi in^2\tau}(-\infty<n<+\infty)$ then $\theta^k(\tau)\equiv\{\theta(\tau)\}^k$, $k$ being a positive integer, is an automorphic form of stufe 2 and the Fourier coefficient $a_n$ becomes the number of ways in which $n$ can be represented as a sum of $k$ squares. We then consider for a function $f$ with signature $(\lambda, k, \gamma)$ the classical problem of obtaining an asymptotic formula for $R^\delta(x)\equiv\sum a_n(x-n)^\delta$ $(0<n\le x)$, $(\delta>0)$, as $x\to\infty$. In general it turns out that $R^\delta(x)=c_0x^{\delta+k}+P_\delta(x)$, for $\delta\ge\delta_0$, $\delta_0$ being a real number depending on $f(\tau)$, $c_0$ a constant independent of $x$, $P_\delta(x)$ being the "error term". The function $P_\delta(x)$ can be represented as a series of Bessel functions of the first kind. In the case where $f(\tau)=\theta^k(\tau)$, we have

$$P_\delta(x) = -x^\delta+\pi^{-\delta}\Gamma(\delta+1)\sum_{n=1}^{\infty} a_n\left(\frac{x}{n}\right)^{k/4+\delta/2}J_{k/2+\delta}(2\pi\sqrt{nx}),$$

$J_\mu(x)$ being the Bessel function of the first kind and the series on the right converges absolutely for $\delta>\frac{1}{2}(k-1)$, conditionally for $\delta>\frac{1}{2}(k-3)$, and can be summed by Riesz typical means $(R, n, \delta)$ for $0\le\delta\le\frac{1}{2}(k-3)$. The above problem can be posed for the case where $a_n$ represents the number of integral representations of $n$ by a

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positive define quadratic form $\sum s_{ij}x_ix_j$ $(1 \le i, j \le m)$, where the matrix $S=(s_{ij})$ is an integral, symmetric, positive definite, matrix of order $m$ and the corresponding results are known. Let now $S=(s_{ij})$ be a rational, non-singular, symmetric, indefinite matrix of order $m$. Then the number of integral representations of a rational number $t$ by the indefinite quadratic form $\sum s_{ij}x_ix_j$ is infinite in general and C. L. Siegel [9] has associated with the set of integral solutions of $\sum s_{ij}x_ix_j=t$ a non-negative valued function $\mu(S, t)$ called the "measure of representation". The function $\mu(S, t)$ is a generalization of the number of integral representations of $t$ by the quadratic form with matrix $S$ when $S$ is a rational, positive definite, matrix. $\mu(S, t)$ is finite except for a few special cases. We then consider the problem of expressing $\sum \mu(S, t)(x-t)^\delta$ $(0<t\le x)$ as a series of analytic functions. It is known [12] that when $|S|>0$, $|S|$ being the determinant of $S$, $\sum \mu(S, t)(x-t)^\delta$ $(0<t\le x)$ can be expressed as a series of Bessel functions of the first kind and in the case $|S|<0$, as a series of Bessel functions of the type $Y_\nu(x)$, $K_\nu(x)$ and two other series involving functions associated with the Bessel functions, all the series so obtained being convergent absolutely for $\delta>\frac{1}{2}(m-1)$. It is known [10] that the numbers $\mu(S, t)$ can be realized as Fourier coefficients of an analytic automorphic form of the type considered by Hecke in the case $|S|>0$ for suitable values of $\lambda$, $k$ and $\gamma$ and this is not the case when $|S|<0$. A function $f(\tau)$ which yields $\mu(S, t)$ as "Fourier coefficients" (in a generalized sense) has been introduced by Siegel [11] and it turns out that this function is not an analytic function of $\tau$, but transforms like an analytic automorphic form under the transformations $\tau \to \tau+\lambda$, $\tau \to -1/\tau$ in the upper half of the complex $\tau$ plane. H. Maass [6] has introduced a class of nonanalytic functions which generalize the functions introduced by Siegel in the study of indefinite quadratic forms with rational coefficients. Our aim, in this paper, is to represent $\sum a_t(x-t)^\delta$ $(0<t\le x)$ as a convergent series of analytic functions where $\{a_t\}$ is the sequence of "Fourier coefficients" of a non-analytic automorphic form in the sense of Maass [6].

2. **Non-analytic automorphic forms and some properties of the associated Dirichlet series.** Let $z$ denote a complex variable, $z=x+iy$, $x$ and $y$ real and $w=\bar{z}$. We consider a pair of complex valued functions $f(z, w)$ and $g(z, w)$ defined in the upper half plane $y>0$ which are solutions of the elliptic partial differential equation

$$y^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-(\alpha-\beta)iy\frac{\partial v}{\partial x}+(\alpha+\beta)y\frac{\partial v}{\partial y} = 0, \tag{1}$$

$\alpha$ and $\beta$ being real numbers and having the following properties:

$$\begin{cases} f(z+\lambda, w+\lambda) = e^{2\pi i b_1}f(z, w) \\ g(z+\lambda, w+\lambda) = e^{2\pi i b_2}g(z, w), \end{cases} \tag{2}$$

$\lambda$ being a real number and $0 \le b_i < 1$ $(i=1, 2)$;

$$g\left(-\frac{1}{z}, -\frac{1}{w}\right) = \gamma(-iz)^\alpha(iw)^\beta f(z, w), \tag{3}$$

where $\gamma = \pm 1$ and $(-iz)^\alpha$, $(iw)^\beta$ are defined by the principal value of the logarithms;

$$\text{(4)} \qquad \begin{cases} f(z, w) = O(y^{\lambda_1}) & \text{and} \quad g(z, w) = O(y^{\lambda_2}) \quad \text{as } y \to \infty \\ f(z, w) = O(y^{-\mu_1}) & \text{and} \quad g(z, w) = O(y^{-\mu_2}) \quad \text{as } y \to 0 \end{cases}$$

where $\lambda_i$ and $\mu_i (i=1, 2)$ are positive constants, and the estimates are uniform in $-\infty < x < \infty$.

It then follows from a result of Maass [6, Hilfssatz 8] that

$$\text{(5)} \qquad \begin{aligned} f_1(x, y) \equiv f(z, w) &= a_0 u(y, \alpha+\beta) + b_0 \\ &+ \sum_{t \neq o,\, t \equiv b_1 (\text{mod } 1)} a_t W\left(\frac{2\pi|t|}{\lambda} y; \alpha, \beta, \operatorname{sgn} t\right) e^{2\pi i t x/\lambda}, \end{aligned}$$

and

$$\text{(6)} \qquad \begin{aligned} g_1(x, y) \equiv g(z, w) &= c_0 u(y, \alpha+\beta) + d_0 \\ &+ \sum_{t \neq o,\, t \equiv b_2 (\text{mod } 1)} b_t W\left(\frac{2\pi|t|}{\lambda} y; \alpha, \beta, \operatorname{sgn} t\right) e^{2\pi i t x/\lambda}, \end{aligned}$$

the series on the right of (5) and (6) being absolutely convergent, where

$$u(y, \gamma) = \frac{y^{1-\gamma}-1}{1-\gamma} = \sum_{n=1}^{\infty} \frac{(\log y)^n}{n!} (1-\gamma)^{n-1},$$

$$W(y; \alpha, \beta, \varepsilon) = y^{-\frac{1}{2}(\alpha+\beta)} W_{\frac{1}{2}(\alpha-\beta)\varepsilon, \frac{1}{2}(\alpha-\beta-1)}(2y), \quad (\varepsilon = \pm 1)$$

with $W_{l,m}(y)$ the Whittaker solution of the confluent hypergeometric differential equation in reduced form [8], and sgn $t = \pm 1$ according as $t > 0$ or $t < 0$ respectively.

It is useful to note that

$$W_{l,m}(y) = \frac{y^l e^{-\frac{1}{2}y}}{\Gamma(m+\frac{1}{2}-l)} \int_0^\infty t^{m-l-\frac{1}{2}} e^{-t} \left(1+\frac{t}{y}\right)^{m+l-\frac{1}{2}} dt,$$

for $y > 0$ and Re $(m+\frac{1}{2}-l) > 0$.

From the properties of the function $W_{l,m}(y)$, it follows that

$$W(y; \alpha, \beta, \varepsilon) \sim 2^{\frac{1}{2}(\alpha-\beta)\varepsilon} y^{-\frac{1}{2}[(\alpha+\beta)+(\beta-\alpha)\varepsilon]} e^{-y}, \text{ as } y \to \infty$$

and therefore

$$W(y; \alpha, \beta, 1) = O(y^{-\beta} e^{-y}), \text{ for } y \to \infty \ (y \text{ real}).$$

From the power series representation of the Whittaker function it follows [8, p. 116] that

$$W(y; \alpha, \beta, 1) = O(y^{-K}), \quad \text{for } y \to 0 \ (y \text{ real})$$

with

$$K > \tfrac{1}{2}(\alpha+\beta) + \tfrac{1}{2}|\alpha+\beta| - 1.$$

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We will be referring to (5) and (6) respectively as the nonanalytic Fourier expansions of $f(z, w)$ and $g(z, w)$ and the sequences of complex numbers $\{a_t\}$ and $\{b_t\}$ as their Fourier coefficients respectively.

We then introduce the Dirichlet series

$$\phi_1(s) = \sum_{t>0} \frac{a_t}{t^s}, \quad \psi_1(s) = \sum_{t>0} \frac{a_{-t}}{t^s}, \tag{7}$$
$$\phi_2(s) = \sum_{t>0} \frac{b_t}{t^s}, \quad \psi_2(s) = \sum_{t>0} \frac{b_{-t}}{t^s},$$

where $s$ is a complex variable and $t^s = e^{s \log t}$ with $\log t$ real. On account of the estimates (4), it follows that the four Dirichlet series in (7) have finite abscissae of convergence [6, p. 257]. Further it is known [6] that they can be continued analytically into the entire complex $s$ plane and the resulting functions are meromorphic. The functions defined by the Dirichlet series in (7) satisfy [6] a functional equation of the following type.

Let

$$\Gamma(s; \alpha, \beta) = \int_0^\infty W(y; \alpha, \beta, 1) y^{s-1} dy, \tag{8}$$

$$\xi_i(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \{\Gamma(s; \alpha, \beta)\phi_i(s) + \Gamma(s; \beta, \alpha)\psi_i(s)\}, \tag{9}$$

and

$$\eta_i(s) = \left(\frac{2\pi}{\lambda}\right)^{-(s+1)} \{\Gamma(s+1; \alpha, \beta) - \tfrac{1}{2}(\alpha-\beta)\Gamma(s; \alpha, \beta)\}\phi_i(s) \tag{10}$$
$$- \left(\frac{2\pi}{\lambda}\right)^{-(s+1)} \{\Gamma(s+1; \beta, \alpha) + \tfrac{1}{2}(\alpha-\beta)\Gamma(s; \beta, \alpha)\}\psi_i(s) (i = 1, 2).$$

Then

$$\xi_1(\alpha+\beta-s) = \gamma\xi_2(s) \quad \text{and} \quad \eta_1(\alpha+\beta-s) = -\gamma\eta_2(s). \tag{11}$$

3. **Preliminary lemmas.**

LEMMA 1. $\Gamma(s; \alpha, \beta)\Gamma(s+1; \beta, \alpha) + \Gamma(s+1; \alpha, \beta)\Gamma(s; \beta, \alpha) = 2\Gamma(s)\Gamma(s+1-\alpha-\beta)$, *with* $\Gamma(s; \alpha, \beta)$ *as defined in* (8).

This result has been proved by Maass [7, §4].

LEMMA 2. *The function* $\phi_1(s)$ *is meromorphic in the complex s-plane with at most simple poles at* $s = \alpha+\beta$ *and* $s = 1$; *further* $(s-1)(s-\alpha-\beta)\phi_1(s)$ *is an entire function of finite order.*

Let
$$F_1(y) = \sum_{t\equiv b_1(\text{mod } 1), t\neq 0} a_t W\left(\frac{2\pi|t|}{\lambda} y; \alpha, \beta, \operatorname{sgn} t\right),$$

$$G_2(y) = \sum_{t\equiv b_2(\text{mod } 1), t\neq 0} b_t W\left(\frac{2\pi|t|}{\lambda} y; \alpha, \beta, \operatorname{sgn} t\right),$$

$$F_2(y) = \sum_{t\equiv b_1(\text{mod } 1), t\neq 0} ta_t W\left(\frac{2\pi|t|}{\lambda} y; \alpha, \beta, \operatorname{sgn} t\right),$$

$$G_2(y) = \sum_{t\equiv b_2(\text{mod } 1), t\neq 0} tb_t W\left(\frac{2\pi|t|}{\lambda} y; \alpha, \beta, \operatorname{sgn} t\right),$$

and
$$H_i(y) = G_i(y) - \lambda\cdot\frac{\alpha-\beta}{4\pi} F_i(y), \quad (i = 1, 2).$$

It then follows from the work of Maass [6] that

(12)
$$\xi_1(s) = \int_1^\infty F_1(y)y^{s-1}\,dy + \gamma\int_1^\infty G_1(y)y^{\alpha+\beta-s-1}\,dy + \frac{a_0}{s(s+1-\alpha-\beta)} - \frac{b_0}{s}$$
$$+\gamma\left\{\frac{c_0}{(1-s)(\alpha+\beta-s)} - \frac{d_0}{(\alpha+\beta-s)}\right\},$$

and

(13)
$$\eta_1(s) = \int_1^\infty H_1(y)y^{s-1}\,dy - \gamma\int_1^\infty H_2(y)y^{\alpha+\beta-s-1}\,dy$$
$$+\lambda\cdot\frac{\alpha-\beta}{4\pi}\left\{-\frac{a_0}{s(s+1-\alpha-\beta)} + \frac{b_0}{s} + \gamma\left[\frac{c_0}{(1-s)(\alpha+\beta-s)} - \frac{d_0}{(\alpha+\beta-s)}\right]\right\}.$$

From the definition of $\xi_1(s)$, it then follows that

(14)
$$\Gamma(s;\alpha,\beta)\phi_1(s) + \Gamma(s;\beta,\alpha)\psi_1(s)$$
$$= \left(\frac{2\pi}{\lambda}\right)^s\left\{\int_1^\infty F_1(y)y^{s-1}\,dy + \gamma\int_1^\infty G_1(y)y^{\alpha+\beta-s-1}\,dy\right\}$$
$$+\left(\frac{2\pi}{\lambda}\right)^s\left\{\frac{a_0}{s(s+1-\alpha-\beta)} - \frac{b_0}{s} + \gamma\left[\frac{c_0}{(1-s)(\alpha+\beta-s)} - \frac{d_0}{(\alpha+\beta-s)}\right]\right\}$$
$$\equiv P(s) + \left(\frac{2\pi}{\lambda}\right)^s\left\{\frac{a_0}{s(s+1-\alpha-\beta)} - \frac{b_0}{s} + \gamma\left[\frac{c_0}{(1-s)(\alpha+\beta-s)} - \frac{d_0}{(\alpha+\beta-s)}\right]\right\}.$$

It follows from the work of Maass [6, p. 256] that

$$\textstyle\int_1^\infty F_1(y)y^{s-1}dy + \gamma\int_1^\infty G_1(y)y^{\alpha+\beta-s-1}\,dy$$

is an entire function of finite order; hence $P(s)$ is an entire function of finite order.

Similarly

$$\{\Gamma(s+1;\alpha,\beta)-\tfrac{1}{2}(\alpha-\beta)\Gamma(s;\alpha,\beta)\}\phi_1(s)-\{\Gamma(s+1;\beta,\alpha)+\tfrac{1}{2}(\alpha-\beta)\Gamma(s;\beta,\alpha)\}\psi_1(s)$$

$$= \left(\frac{2\pi}{\lambda}\right)^{s+1}\left\{\int_1^\infty H_1(y)y^{s-1}\,dy-\gamma\int_1^\infty H_2(y)y^{\alpha+\beta-s-1}\,dy\right\}$$

$$(15)\qquad +\left(\frac{2\pi}{\lambda}\right)^{s+1}\cdot\frac{\alpha-\beta}{4\pi}\cdot\lambda\left\{-\frac{a_0}{s(s+1-\alpha-\beta)}+\frac{b_0}{s}+\gamma\left[\frac{c_0}{(1-s)(\alpha+\beta-s)}-\frac{d_0}{(\alpha+\beta-s)}\right]\right\}$$

$$= Q(s)+\left(\frac{2\pi}{\lambda}\right)^{s+1}\cdot\frac{\alpha-\beta}{4\pi}\cdot\lambda\left\{-\frac{a_0}{s(s+1-\alpha-\beta)}+\frac{b_0}{s}\right.$$

$$\left.+\gamma\left[\frac{c_0}{(1-s)(\alpha+\beta-s)}-\frac{d_0}{(\alpha+\beta-s)}\right]\right\},$$

where $Q(s)$ is an entire function of $s$ of finite order. We can solve for $\phi_1(s)$ from (14) and (15). Using Lemma 1, it follows that

$$\phi_1(s) = \frac{\Gamma(s+1;\beta,\alpha)+\frac{1}{2}(\alpha-\beta)\Gamma(s;\beta,\alpha)}{2\Gamma(s)\Gamma(s+1-\alpha-\beta)}P(s)+\frac{\Gamma(s;\beta,\alpha)}{2\Gamma(s)\Gamma(s+1-\alpha-\beta)}Q(s)$$

$$+\left(\frac{2\pi}{\lambda}\right)^{s}\left\{\frac{a_0}{s(s+1-\alpha-\beta)}-\frac{b_0}{s}+\gamma\left[\frac{c_0}{(1-s)(\alpha+\beta-s)}-\frac{d_0}{(\alpha+\beta-s)}\right]\right\}$$

$$(16)\qquad \times\frac{\Gamma(s+1;\beta,\alpha)+\frac{1}{2}(\alpha-\beta)\Gamma(s;\beta,\alpha)}{2\Gamma(s)\Gamma(s+1-\alpha-\beta)}+\left(\frac{2\pi}{\lambda}\right)^{s+1}\cdot\frac{(\alpha-\beta)}{4\pi}\cdot$$

$$\times\lambda\left\{-\frac{a_0}{s(s+1-\alpha-\beta)}+\frac{b_0}{s}+\gamma\left[\frac{c_0}{(1-s)(\alpha+\beta-s)}-\frac{d_0}{(\alpha+\beta-s)}\right]\right\}$$

$$\times\frac{\Gamma(s;\beta,\alpha)}{2\Gamma(s)\Gamma(s+1-\alpha-\beta)}.$$

Now

$$(17)\qquad \Gamma(s;\alpha,\beta) = 2^{(\alpha-\beta)/2}\frac{\Gamma(s)\Gamma(s+1-\alpha-\beta)}{\Gamma(s+1-\alpha)}F(\beta,1-\alpha,s+1-\alpha;\tfrac{1}{2}),$$

where $F(\alpha,\beta,\gamma;x)$ denotes the hypergeometric function. It is known that $F(\beta,1-\alpha,s+1-\alpha;\frac{1}{2})/\Gamma(s+1-\alpha)$ is an entire function of $s$. Hence

$$\Gamma(s+1;\beta,\alpha)+\tfrac{1}{2}(\alpha-\beta)\Gamma(s;\beta,\alpha)$$

and $\Gamma(s;\beta,\alpha)$ become entire functions of $s$ after division by $2\Gamma(s)\Gamma(s+1-\alpha-\beta)$ It follows from (16) that $\phi_1(s)$ is meromorphic in the complex $s$ plane and has at most poles at $s=\alpha+\beta$, $\alpha+\beta-1$, 1 and 0. We now prove that $\phi_1(s)$ is regular at $s=\alpha+\beta-1$ and $s=0$ by proving that $\lim_{s\to 0} s\phi_1(s)=0$ and $\lim_{s\to\alpha+\beta-1}(s-\alpha-\beta+1)\phi_1(s)=0$.

It follows from (16) that

$$\lim_{s\to 0} s\phi_1(s) = \left(\frac{a_0}{1-\alpha-\beta}-b_0\right)\left\{\frac{\Gamma(s+1;\beta,\alpha)+\frac{1}{2}(\alpha-\beta)\Gamma(s;\beta,\alpha)}{2\Gamma(s)\Gamma(s+1-\alpha-\beta)}\right\}_{s=0}$$

$$+\frac{\alpha-\beta}{2}\left(-\frac{a_0}{1-\alpha-\beta}+b_0\right)\left\{\frac{\Gamma(s;\beta,\alpha)}{2\Gamma(s)\Gamma(s+1-\alpha-\beta)}\right\}_{s=0} = 0,$$

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by using (17) and elementary properties of the gamma function. Similarly

$$\lim_{s\to\alpha+\beta-1} (s+1-\alpha-\beta)\phi_1(s)$$

$$= \left(\frac{2\pi}{\lambda}\right)^{\alpha+\beta-1} \frac{a_0}{(\alpha+\beta-1)} \left\{\frac{\Gamma(s+1;\beta,\alpha)+\frac{1}{2}(\alpha-\beta)\Gamma(s;\beta,\alpha)}{2\Gamma(s)\Gamma(s+1-\alpha-\beta)}\right\}_{s=\alpha+\beta-1}$$

$$-\lambda\left(\frac{2\pi}{\lambda}\right)^{\alpha+\beta}\cdot\frac{\alpha-\beta}{4\pi}\cdot\frac{a_0}{(\alpha+\beta-1)}\left\{\frac{\Gamma(s;\beta,\alpha)}{2\Gamma(s)\Gamma(s+1-\alpha-\beta)}\right\}_{s=\alpha+\beta-1} = 0.$$

LEMMA 3. *The function $\phi_1(s)$ has the functional equation* $2\Gamma(s)\Gamma(s+1-\alpha-\beta)\phi_1(s) = \gamma(2\pi/\lambda)^{2s-\alpha-\beta}\{\lambda(\alpha,\beta,s)\phi_2(\alpha+\beta-s)+\mu(\alpha,\beta,s)\psi_2(\alpha+\beta-s)\}$, *where*

(18) $$\lambda(\alpha,\beta,s) = \Gamma(s+1;\beta,\alpha)\Gamma(\alpha+\beta-s;\alpha,\beta)-\Gamma(s;\beta,\alpha)\Gamma(\alpha+\beta-s+1;\alpha,\beta),$$

and

(19) $$\mu(\alpha,\beta,s) = \Gamma(s+1;\beta,\alpha)\Gamma(\alpha+\beta-s;\beta,\alpha)+\Gamma(s;\beta,\alpha)\Gamma(\alpha+\beta-s+1;\beta,\alpha) \\ +(\alpha-\beta)\Gamma(s;\beta,\alpha)\Gamma(\alpha+\beta-s;\beta,\alpha).$$

This lemma follows by solving for $\phi_1(s)$ from the two equations defined by (11) and using Lemma 1.

LEMMA 4. $\lambda(\alpha,\beta,s)=O(e^{-\pi|t|})$ *and* $\mu(\alpha,\beta,s)=O(e^{-\pi|t|}|t|^{\beta-\alpha})$ *as* $|t|\to\infty$ *uniformly for* $-\infty<a\le\sigma\le b<\infty$, *where as usual* $s=\sigma+it$.

It is known [1, p. 76] that

$$F(a,b,c;z) = 1+\frac{ab}{c}z+\cdots+\frac{(a)_n(b)_n}{(c)_n}z^n+O(|c|^{-n-1}),$$

as $|c|\to\infty$, for fixed $a$, $b$ and $z$, if $|z|<1$ and $|\arg c|\le\pi-\varepsilon<\pi$, where for a complex number $x$, $(x)_n=(x+1)\ldots(x+n-1)$. Hence $F(\beta,1-\alpha,s+1-\alpha;\frac{1}{2})\sim 1$ as $|t|\to\infty$ uniformly in $-\infty<a\le\sigma\le b<\infty$. From Stirling's approximation for $\Gamma(s)$, it follows that $\Gamma(\sigma+it)\sim\sqrt{2\pi}\,e^{-\frac{1}{2}\pi|t|}|t|^{\sigma-\frac{1}{2}}$ for $t\to\infty$, uniformly in $-\infty<a\le\sigma\le b<\infty$. Hence by (17), it follows that

(20) $$|\Gamma(s;\alpha,\beta)| = O\left(\frac{|\Gamma(s)||\Gamma(s+1-\alpha-\beta)|}{|\Gamma(s+1-\alpha)|}\right) \\ = O(e^{-\frac{1}{2}\pi|t|}|t|^{\sigma-\beta-\frac{1}{2}})$$

as $|t|\to\infty$ uniformly in $-\infty<a\le\sigma\le b<\infty$.

Consequently

$$\lambda(\alpha,\beta,s) = \Gamma(s+1;\beta,\alpha)\Gamma(\alpha+\beta-s;\alpha,\beta)-\Gamma(s;\beta,\alpha)\Gamma(\alpha+\beta-s+1;\alpha,\beta) \\ = O(e^{-\pi|t|})+O(e^{-\pi|t|}) = O(e^{-\pi|t|}),$$

as $|t|\to\infty$ uniformly in $-\infty<a\le\sigma\le b<\infty$.

Similarly

$$\mu(\alpha, \beta, s) = \Gamma(s+1; \beta, \alpha)\Gamma(\alpha+\beta-s; \beta, \alpha)+\Gamma(s; \beta, \alpha)\Gamma(\alpha+\beta-s+1; \beta, \alpha)$$
$$+(\alpha-\beta)\Gamma(s; \beta, \alpha)\Gamma(\alpha+\beta-s; \beta, \alpha)$$
$$= O(e^{-\pi|t|}|t|^{\beta-\alpha})+O(e^{-\pi|t|}|t|^{\beta-\alpha})+O(e^{-\pi|t|}|t|^{\beta-\alpha-1})$$
$$= O(e^{-\pi|t|}|t|^{\beta-\alpha}),$$

as $|t| \to \infty$ uniformly in $-\infty < a \le \sigma \le b < \infty$.

LEMMA 5. *Let $c^* > 0$ be such that the four Dirichlet series defined by* (7) *converge absolutely for $s = c^*$* and $c^* > (\alpha+\beta)/2$. *Then $\phi_1(\sigma+it) = O(|t|^{\lambda_1})$, as $|t| \to \infty$ uniformly in $\alpha+\beta-c^* \le \sigma \le c^*$, where $\lambda_1 = \max\{0, \delta_2-2(\alpha+\beta-c^*)\}$ with $\delta_2 = \max(\alpha+\beta, 2\beta)$.*

By Lemma 4, it follows that

$$\frac{\lambda(\alpha, \beta, s)}{\Gamma(s)\Gamma(s+1-\alpha-\beta)} = O(|t|^{\alpha+\beta-2\sigma}), \tag{21}$$

and

$$\frac{\mu(\alpha, \beta, s)}{\Gamma(s)\Gamma(s+1-\alpha-\beta)} = O(|t|^{2\beta-2\sigma}), \tag{22}$$

as $|t| \to \infty$ uniformly in $-\infty < a \le \sigma \le b < \infty$. By the choice of $c^*$, $\phi_2(s) = O(1)$, $\psi_2(s) = O(1)$ for $\sigma \ge c^*$; hence $\phi_2(\alpha+\beta-s) = O(1)$ and $\psi_2(\alpha+\beta-s) = O(1)$ for $\sigma = \alpha+\beta-c^*$. It then follows from Lemma 3, (21) and (22) that

$$\phi_1(s) = O(|t|^{\alpha+\beta-2\sigma})+O(|t|^{2\beta-2\sigma})$$
$$= O(|t|^{\delta_2-2\sigma}),$$

on the line $\sigma = \alpha+\beta-c^*$ as $|t| \to \infty$, where $\delta_2 = \max(\alpha+\beta, 2\beta)$.

By the choice of $c^*$, it follows that $\phi_1(s) = O(1)$ on $\sigma = c^*$. In view of Lemma 2, $(s-1)(s-\alpha-\beta)\phi_1(s)$ is an entire function of finite order; it then follows by the theorem of Phragmen-Lindelöf that $\phi_1(\sigma+it) = O(|t|^{\eta(\sigma)})$ uniformly in $\alpha+\beta-c^* \le \sigma \le c^*$ for $|t| \ge t_0$ ($t_0$ being a suitable positive constant), where $\eta(\sigma)$ is the linear function joining $(c^*, 0)$ and $(\alpha+\beta-c^*, \delta_2-2\sigma)$. Hence

$$\phi_1(\sigma+it) = O(|t|^{\delta_2-2(\alpha+\beta-c^*)}) \text{ if } \delta_2-2(\alpha+\beta-c^*) \ge 0$$
$$= O(1) \qquad \text{if } \delta_2-2(\alpha+\beta-c^*) < 0$$

uniformly in $\alpha+\beta-c^* \le \sigma \le c^*$.

As $\lambda_1 = \max(0, \delta_2-2(\alpha+\beta-c^*))$, the result follows.

4. **Proof of the main theorem.** We invoke Perron's formula in the classical theory of Dirichlet series [3, p. 81] and apply it to

$$\phi_1(s) = \sum_{t>0} \frac{a_t}{t^s},$$

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which converges absolutely for $\sigma \geq c^*$. Then we have for $x>0$ and $\delta \geq 0$,

$$\sum_{0<t\leq x}{}' a_t(x-t)^\delta = \Gamma(\delta+1)x^\delta \cdot \frac{1}{2\pi i}\int_{c^*-i\infty}^{c^*+i\infty} \frac{x^s\Gamma(s)\phi_1(s)}{\Gamma(\delta+1+s)}\,ds, \tag{23}$$

where the dash on the left of (23) indicates that the last term of the sum on the left side of (23) is to be multiplied by $\frac{1}{2}$ if $\delta=0$ and $x=t_i$ with $a_{t_i}\neq 0$; further the integral on the right of (23) is to be understood as a Cauchy limit. We note that the number of $t$ for which $a_t\neq 0$ and $0<t\leq x$ is finite. We now transform the integral on the right of (23) into an integral taken on the line $\operatorname{Re} s=\alpha+\beta-c^*$. *We now impose the additional condition that the strip $\alpha+\beta-c^*\leq\sigma\leq c^*$ includes all the singularities of $\phi_1(s)$ and that $\Gamma(s)$ and $\phi_1(s)$ are both regular on* $\operatorname{Re} s=\alpha+\beta-c^*$. We consider the integral of

$$\frac{x^{\delta+s}\Gamma(s)}{\Gamma(\delta+1+s)}\phi_1(s) \tag{24}$$

over the rectangle with vertices at $c^*\pm it$, $\alpha+\beta-c^*\pm it$ oriented in the positive sense. Then

$$\frac{1}{2\pi i}\int_{c^*-it}^{c^*+it}+\frac{1}{2\pi i}\int_{c^*+it}^{\alpha+\beta-c^*+it}+\frac{1}{2\pi i}\int_{\alpha+\beta-c^*+it}^{\alpha+\beta-c^*-it}+\frac{1}{2\pi i}\int_{\alpha+\beta-c^*-it}^{c^*-it}$$

is the sum of the residues of (24) inside the rectangle; we denote this sum by $Q_\delta(x)$. By Stirling's approximation and Lemma 5

$$\begin{aligned}\int_{c^*+it}^{\alpha+\beta-c^*+it}\frac{x^{\delta+s}\Gamma(s)}{\Gamma(\delta+1+s)}\phi_1(s)ds &= O\left(\int_{\alpha+\beta-c^*}^{c^*} x^{\delta+\sigma}|t|^{\lambda_1-\delta-1}d\sigma\right)\\ &= O(|t|^{\lambda_1-\delta-1}) = o(1) \text{ as } t\to\infty \text{ if } \lambda_1-\delta-1>0\\ &\qquad\text{or } \delta>\lambda_1-1.\end{aligned}$$

Similarly the integral on the line joining $\alpha+\beta-c^*-it$ and $c^*-it$ tends to zero as $|t|\to\infty$ if $\delta>\lambda_1-1$. Hence for $\delta\geq 0$ and $\delta>\lambda_1-1$,

$$\begin{aligned}\frac{1}{\Gamma(\delta+1)}\sum_{0<t\leq x}{}' a_t(x-t)^\delta &= Q_\delta(x)+\frac{1}{2\pi i}\\ &\quad\times\int_{c^*-i\infty}^{c^*+i\infty}\frac{x^{\delta+\alpha+\beta-s}\Gamma(\alpha+\beta-s)\phi_1(\alpha+\beta-s)}{\Gamma(\delta+1+\alpha+\beta-s)}.\end{aligned} \tag{25}$$

Using Lemma 3 and the facts $\phi_2(s)=\sum_{t>0}\frac{b_t}{t^s}$, $\psi_2(s)=\sum_{t>0}\frac{b_{-t}}{t^s}$, for $s=c^*$, the integral on the right of (25) can be rewritten as

$$\begin{aligned}\frac{\gamma}{2}\left(\frac{2\pi}{\lambda}\right)^{\alpha+\beta}x^{\delta+\alpha+\beta}\frac{1}{2\pi i}\int_{c^*-i\infty}^{c^*+i\infty}\frac{\Gamma(\alpha+\beta-s)}{\Gamma(\delta+1+\alpha+\beta-s)}\left\{\frac{\lambda(\alpha,\beta,\alpha+\beta-s)}{\Gamma(\alpha+\beta-s)\Gamma(1-s)}\sum_{t>0}\frac{b_t}{t^s}\right.\\ \left.+\frac{\mu(\alpha,\beta,\alpha+\beta-s)}{\Gamma(\alpha+\beta-s)\Gamma(1-s)}\sum_{>0}\frac{b_{-t}}{t^s}\right\}\left(\frac{4\pi^2x}{\lambda^2}\right)^{-s}ds.\end{aligned} \tag{26}$$

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We want to exchange the order of integration and summation in (26); this can be done if the series

$$\sum_{t>0} b_t \frac{1}{2\pi i}\int_{c^*-i\infty}^{c^*+i\infty} \frac{\lambda(\alpha,\beta,\alpha+\beta-s)}{\Gamma(\delta+1+\alpha+\beta-s)\Gamma(1-s)}\left(\frac{4\pi^2 tx}{\lambda^2}\right)^{-s} ds, \tag{27}$$

and

$$\sum_{t>0} b_{-t} \frac{1}{2\pi i}\int_{c^*-i\infty}^{c^*+i\infty} \frac{\mu(\alpha,\beta,\alpha+\beta-s)}{\Gamma(\delta+1+\alpha+\beta-s)\Gamma(1-s)}\left(\frac{4\pi^2 tx}{\lambda^2}\right)^{-s} ds, \tag{28}$$

converge absolutely.

Let

$$J(x;\alpha,\beta,\delta,c) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\lambda(\alpha,\beta,\alpha+\beta-s)}{\Gamma(1-s)\Gamma(\delta+1+\alpha+\beta-s)} x^{-2s} ds, \tag{29}$$

$$(0 < c < \tfrac{1}{2}(\delta+\alpha+\beta);\ x > 0)$$

and

$$K(x;\alpha,\beta,\delta,c) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\mu(\alpha,\beta,\alpha+\beta-s)}{\Gamma(1-s)\Gamma(\delta+1+\alpha+\beta-s)} x^{-2s} ds, \tag{30}$$

$$(0 < c < \tfrac{1}{2}(\delta+2\alpha);\ x > 0)$$

where $c$ is such that the path of integration does not include any of the singularities of the integrand. As $F(\beta, 1-\alpha, s+1-\alpha; \frac{1}{2})/\Gamma(s+1-\alpha)$ is an entire function of $s$, it follows from (17) that $\Gamma(s;\alpha,\beta)$ is meromorphic in the complex $s$ plane with at most poles at $s=1-n$, $s=\alpha+\beta-n$, $n$ being any positive integer. It then follows from (18) that $\lambda(\alpha,\beta,s)$ is a meromorphic function of $s$ with at most poles at points congruent to 0 or $\alpha+\beta$ modulo 1. The same is true for the function $\mu(\alpha,\beta,s)$. We now study the convergence of the integrals defined by (29) and (30). From Lemma 4, it follows that the integrand in (29) is

$$O(|t|^{-(\delta+1+\alpha+\beta-2c)}x^{-2c}), \quad \text{where } s = c+it.$$

Hence the integrals in (27) and (29) converge absolutely if $\delta > 2c^*-(\alpha+\beta)$ or $\delta > 2c-(\alpha+\beta)$ respectively. Similarly the integrand in (30) is $O(|t|^{-(\delta+1+2\alpha-2c)}x^{-2c})$, where $s=c+it$. Hence the integrals in (28) and (30) converge absolutely if $\delta>2c^*-2\alpha$ or $\delta>2c-2\alpha$ respectively. It then follows that the series (27) and (28) converge absolutely if $\delta>2c^*-2\alpha$ and $\delta>2c^*-(\alpha+\beta)$. We therefore have proved the following

THEOREM. *Let $c^*>0$ be such that all the singularities of $\phi_1(s)$ lie in the strip $\alpha+\beta-c^*\le\sigma\le c^*$, $c^*$ not congruent to* 0 *or* $\alpha+\beta$ modulo 1 *and also $c^*$ satisfy the*

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*conditions of Lemma 5. If* $\lambda_1$ *is as in Lemma 5,* $\delta \geq 0$, $\delta > 2c^* - 2\alpha$, $\delta > 2c^* - \alpha - \beta$ *and* $\delta > \lambda_1 - 1$, *then*

$$\frac{1}{\Gamma(\delta+1)} \sum_{0<t\leq x}{}' a_t(x-t)^\delta = Q_\delta(x) + \frac{\gamma}{2}\left(\frac{2\pi}{\lambda}\right)^{\alpha+\beta} x^{\delta+\alpha+\beta} \tag{31}$$

$$\times \left\{ \sum_{t>0} b_t J\left(\frac{2\pi}{\lambda}\sqrt{tx};\, \alpha, \beta, \delta, c^*\right) + \sum_{t>0} b_{-t} K\left(\frac{2\pi}{\lambda}\sqrt{tx};\, \alpha, \beta, \delta, c^*\right) \right\},$$

where the summation in the series on the right of (31) is over all real $t > 0$ such that $t \equiv b_2 \pmod 1$ and the series on the right of (31) converges absolutely.

5. **Some special cases.** (1) Let $\beta = 0$. Then by a theorem of Maass [6, Satz 6], $f(z, w)$ can be transformed by the application of a suitable differential operator into the function

$$\bar{a}_0 + \bar{b}_0 + \bar{c}_0 \sum_{n+b_1>0} a_{n+b_1} e^{2\pi i \frac{(n+b_1)}{\lambda} z} \tag{32}$$

$\bar{a}_0$, $\bar{b}_0$ and $\bar{c}_0$ being suitable constants and $n$ an integer. (32) and a similar transform of $g(z, w)$ then have a functional equation under the mapping $z \to -z^{-1}$ as in the classical case of the Ramanujan $\tau$ function and $\sum_{0<n+b_1\leq x} a_{n+b_1}(x-n-b_1)^\delta$ can be expressed as a series of Bessel functions of the first kind. A very good account of such results can be found in [2].

(2) $\alpha = \beta$. In this case we get the so called wave functions [5] and the functions $J(x; \alpha, \alpha, \delta, c)$, $K(x; \alpha, \alpha, \delta, c)$ can be evaluated explicitly in terms of Bessel functions and related functions. It turns out that

$$\Gamma(s; \alpha, \alpha) = \frac{1}{\sqrt{\pi}} 2^{s-\alpha-1} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2} - \alpha\right),$$

and

$$\lambda(\alpha, \alpha, s) = \frac{(2\pi) \sin \pi(s-\alpha)}{\sin(\pi s) \sin \pi(2\alpha - s)};$$

then

$$J(x; \alpha, \alpha, \delta, c^*) = \frac{1}{2\pi i}\int_{c^*-i\infty}^{c^*+i\infty} \frac{2\Gamma(s)}{\Gamma(\delta+1+2\alpha-s)} \frac{\sin \pi(\alpha - s)}{\sin \pi(2\alpha - s)} x^{-2s} ds$$

$$= \frac{1}{2\pi i}\int_{c^*-i\infty}^{c^*+i\infty} \frac{2\Gamma(s)}{\Gamma(\delta+1+2\alpha-s)} \cos(\pi\alpha) x^{-2s} ds \tag{33}$$

$$-2\sin(\pi\alpha)\frac{1}{2\pi i}\int_{c^*-i\infty}^{c^*+i\infty} \frac{\Gamma(s)}{\Gamma(\delta+1+2\alpha-s)} \cot \pi(2\alpha - s) x^{-2s} ds.$$

The first integral in (33) is a Bessel function of the first kind multiplied by a factor and the second integral in (33) can be expressed [12, §5] in terms of the Bessel

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function $Y_\nu(x)$ and the Lommel function $S_{\mu,\nu}(x)$. In the same manner it follows that

$$\mu(\alpha, \alpha, s) = \frac{(2\pi)\sin(\pi\alpha)}{\sin(\pi s)\sin\pi(2\alpha - s)},$$

and

$$(34) \qquad K(x; \alpha, \alpha, \delta, c^*) = \frac{1}{2\pi i}\int_{c^*-i\infty}^{c^*+i\infty} \frac{2\sin(\pi\alpha)\Gamma(s)}{\Gamma(\delta+1+2\alpha-s)} \operatorname{cosec}\pi(2\alpha-s)x^{-2s}ds.$$

The integral in (34) is expressible [12, §5] in terms of the Bessel function $K_\nu(x)$ and a function $G_{\mu,\nu}(x)$ similar to the Lommel function $S_{\mu,\nu}(x)$. The relevant properties of $G_{\mu,\nu}(x)$ can be found in [12, §5].

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