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INNER FUNCTIONS AND TOEPLITZ OPERATORS

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ABSTRACT. We give characterizations of Toeplitz operators on generalised H^2 spaces and derive some properties of the corresponding Toeplitz algebras. The proofs depend essentially on having a "sufficient" supply of inner functions.

1. **Introduction.** This paper is a sequel to our paper [13] and continues the analysis, initiated there, of Toeplitz operators on Hardy H^2 spaces defined by function algebras. This analysis is part of a programme being developed by a number of mathematicians which is aimed at extending the classical theory of Toeplitz operators far beyond its original setting. Some of these generalisations of Toeplitz operator theory have become significant and elegant theories. This is the case for Toeplitz operators on the Bergman space of the open unit disc and of more general regions of $\mathbf{C}$ (for a nice expository account of developments in this direction, see [1]). For Toeplitz theories on generalised domains in $\mathbf{C}^n$, see, for instance, [3, 15, 16]. It is well known that Wiener-Hopf operators on $L^2(\mathbf{R}^+)$ are unitarily equivalent to Toeplitz operators on the Hardy space of the circle, and this gives another direction for generalisation, since one can extend the Wiener-Hopf theory to L^2 spaces of cones more general than $\mathbf{R}^+$ [8]. An approach that is close to the one we pursue here takes as its starting point the role played in the classical case by the fact that the unit circle $\mathbf{T}$ is a compact group with totally ordered dual group $\mathbf{Z}$. In this approach, one studies Toeplitz operators with symbols defined on a connected compact abelian group, where the dual group is endowed with an order structure [5, 9, 10, 11, 12, 13].

The generalised Hardy spaces that we consider play a major role in the theory of function algebras. The setting is this: One has a function algebra A defined on a compact Hausdorff space G and a probability measure m on G that is the unique representing measure for a character of A. The Hardy space H^2 is defined to be the closure of A in $L^2(G, m)$ and the other Hardy spaces H^p are defined similarly. A surprisingly large amount of the theory of the Hardy spaces of the circle extends to this setting. As already indicated, there are important applications to function algebras, especially in connection with the problem of embedding analytic structure into the Gleason parts of the spectrum of a function algebra.

As in the classical case, one can define Toeplitz operators on the space H^2 to be compressions of multiplication operators on $L^2(G, m)$. In [13] we showed that much of the theory of Toeplitz operators on the circle carries over to this setting (for an earlier treatment of a special case, where A is assumed to be a Dirichlet algebra, see [6]). However,

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some of the classical Toeplitz operator theory does not extend, just as some of the corresponding H^p space theory fails to carry over. Indeed, there can be very marked differences. To give just one example, a non-scalar hermitian Toeplitz operator on a generalised Hardy space may have an eigenvalue [13] (of course, this cannot occur in the classical case). Moreover, even in the cases where classical theorems extend, the corresponding proofs may not—they may have to be adapted or completely replaced in the more general setting.

In this paper we are concerned with two aspects of Toeplitz theory. The first is considered in Section 2, where we give various characterisations of Toeplitz operators. The second is treated in Section 3 and involves the investigation of certain C^*-algebras generated by Toeplitz operators. The unifying element in our analysis is the crucial role played by inner functions. The approach taken in this paper is therefore somewhat different to that taken in [13], where inner functions played only a minor part. The Toeplitz algebras were introduced in [13] and some of their fundamental properties and applications were obtained. The derivation of these properties relied on the use of a theorem of Tomiyama and Yabuta [14] whose proof does not involve the theory of generalised Hardy spaces. Here, we obtain these properties using inner functions, and thereby simplify and unify the development. Moreover, we get new results and a more detailed analysis.

2. **Characterisations of Toeplitz operators.** We begin by recalling some concepts from the theory of function algebras. Let A be a function algebra on a compact Hausdorff space G and suppose that τ is a character of A. The Hahn-Banach and Riesz-Kakutani theorems guarantee the existence of a regular probability measure m such that $\tau(f) = \int f\,dm$ for all functions f belonging to A. Any such measure m is called a *representing measure for* τ. The case where τ admits only one representing measure is particularly interesting, since a large amount of the theory of Hardy spaces on the circle extends to this setting (see [4,7] for details).

The motivating example for the theory is provided by the disc algebra A on the circle (this is the algebra of all functions on $\mathbf{T}$ which admit extensions to continuous functions on the closed unit disc that are analytic in its interior). It is easily checked that normalised Lebesgue measure on $\mathbf{T}$ is the unique representing measure for the character of A given by evaluation at the origin.

For large classes of other examples, see [13].

We now make some notational conventions that will apply throughout the rest of the paper: We shall let G denote a compact Hausdorff space and A a function algebra on G. *We shall suppose that m is the unique representing measure for a character of A, and, to avoid trivialities, that m is not a point mass*. For $p \geq 1$, we set $L^p = L^p(G,m)$ and for p finite we let H^p denote the closure of A in L^p. We denote by H^∞ the weak* closure of A in $L^\infty = L^{1*}$. Then H^∞ is a norm-closed subalgebra of L^∞ and H^p is a Banach space for all p, where the norm is, of course, the L^p norm. The elements of H^p are called the *analytic* functions of L^p and the analytic unimodular functions of L^∞ are called *inner* functions.

If $\tilde{G}$ denotes the character space of the Banach algebra L^∞ and the map

$$L^\infty \longrightarrow C(\tilde{G}), \quad \psi \longmapsto \tilde{\psi},$$

is the Gelfand representation, then it is shown in [2] that the Gelfand transforms of the inner functions separate the points of $\tilde{G}$. (This uses the strong logmodularity of H^∞ in L^∞ [4, p. 212].) Since the closed linear span of the functions $\varphi\bar{\psi}$, where φ, ψ are transforms of inner functions, is a C^*-algebra, it follows from the Stone-Weierstrass theorem that this closed span is equal to $C(\tilde{G})$. Therefore, L^∞ is the closed linear span of the set

$$\Lambda = \{\varphi\bar{\psi} \mid \varphi, \psi \text{ are inner functions}\}.$$

The set Λ and this result will play a fundamental role in the sequel. Observe that Λ can be pre-ordered by a relation $\leq$, where the condition $\varphi \leq \psi$ is defined to mean that $\psi = \varphi\theta$ for some function θ belonging to H^∞. This pre-ordering is directed upwards: If φ, ψ belong to Λ, then it is trivially verified that there exists an element θ in Λ such that $\varphi, \psi \leq \theta$. Set

$$\Lambda^+ = \{\varphi \in \Lambda \mid \varphi \geq 1\},$$

so that Λ^+ is the set of inner functions. Clearly, Λ^+ is also a pre-ordered set that is directed upwards. Thus, we can (and frequently will) use Λ^+ as an index set for a net.

Let P denote the projection of the Hilbert space L^2 onto the linear subspace H^2. If $\psi \in L^\infty$, the *Toeplitz* operator $T_\psi \in B(H^2)$ with *symbol* ψ is defined by $T_\psi(f) = P(\psi f)$. Fundamental properties of these operators are established in [13]. Let us note here an easily-verified property that we will use repeatedly: If $\varphi, \psi \in L^\infty$ and φ is analytic, then $T_\psi T_\varphi = T_{\psi\varphi}$ and $T_\varphi^* T_\psi = T_{\bar{\varphi}\psi}$. We shall also make frequent use of the fact that $\|T_\varphi\| = \|\varphi\|_\infty$.

If T is a bounded operator on the H^2 space of the circle and U is the unilateral shift on the canonical orthonormal basis of this space, then it is well known that T is a Toeplitz operator if and only if $U^*TU = T$. The following theorem is an analogue of this result.

THEOREM 2.1. *An operator $T \in B(H^2)$ is a Toeplitz operator if and only if $T_\varphi^* T T_\varphi = T$ for all inner functions φ in H^∞.*

PROOF. It is clear that any Toeplitz operator must satisfy the indicated condition. We therefore assume that T is such that the condition $T_\varphi^* T T_\varphi = T$ holds for each inner function φ and we show that T is a Toeplitz operator.

For any inner function φ, define $S_\varphi \in B(L^2)$ by setting $S_\varphi(f) = \bar{\varphi} TP(\varphi f)$ for $f \in L^2$. Let f and g be arbitrary elements of Λ and choose an element φ_0 of Λ^+ which majorises $\bar{f}$ and $\bar{g}$ in Λ, so that $f_1 = \varphi_0 f$ and $g_1 = \varphi_0 g$ belong to H^∞. Let φ be any element of Λ^+ majorising φ_0, and write $\varphi = \varphi_0\psi$ for some element $\psi \in \Lambda^+$. Since $\varphi f = \psi f_1$ and $\varphi g = \psi g_1$, we have $\langle S_\varphi(f), g\rangle = \langle TP(\varphi f), \varphi g\rangle = \langle TP(\psi f_1), \psi g_1\rangle = \langle \bar{\psi} TP(\psi f_1), g_1\rangle = \langle T_\psi^* T T_\psi(f_1), g_1\rangle = \langle T(f_1), g_1\rangle$ (the equation $T_\psi^* T T_\psi = T$ holds because ψ is inner). This shows that the net $(\langle S_\varphi(f), g\rangle)_{\varphi\in\Lambda^+}$ is convergent for all functions f, g belonging to Λ and therefore also for all f, g belonging to the linear span of Λ. Since this linear span

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is dense in L^2 (because it is dense in L^∞ in the essential supremum norm) and since $\|S_\varphi\| \leq \|T\|$ for all $\varphi \in \Lambda^+$, therefore the net $(\langle S_\varphi(f), g\rangle)_{\varphi\in\Lambda^+}$ is convergent for all f, g in L^2. Hence, there exists an operator $S \in B(L^2)$ such that $\lim_{\varphi\in\Lambda^+} S_\varphi = S$ in the weak operator topology.

If $f, g \in H^2$, then for all $\varphi \in \Lambda^+$, we have $\langle S_\varphi(f), g\rangle = \langle TP(\varphi f), \varphi g\rangle = \langle T_\varphi^* T T_\varphi(f), g\rangle = \langle T(f), g\rangle$. Therefore, T is the compression of S to H^2.

Let $\psi \in L^\infty$. We claim that $SM_\psi = M_\psi S$, where M_ψ is the multiplication operator on L^2 associated to ψ. To show this we may suppose that $\psi \in \Lambda^+$: For, if we show the result in this case, then Fuglede's theorem, applied to the normal operator M_ψ, implies that $SM_{\bar\psi} = M_{\bar\psi} S$. From this we deduce that S commutes with M_ψ for all $\psi \in \Lambda$ and therefore also for all $\psi \in L^\infty$, since L^∞ is the closed linear span of Λ. Suppose then that $\psi \in \Lambda^+$. To prove our claim it suffices to show that $\langle SM_\psi(f), g\rangle = \langle M_\psi S(f), g\rangle$ for all $f, g \in L^2$. Clearly, we may reduce to the case where f, g belong to Λ. It then suffices to show that for φ sufficiently large in Λ^+, we have $\langle S_\varphi M_\psi(f), g\rangle = \langle M_\psi S_\varphi(f), g\rangle$. Choose an element φ_0 of Λ^+ which majorises $\bar f$ and $\bar g$ in Λ, so that $f_1 = \varphi_0 f$ and $g_1 = \varphi_0 g$ belong to H^∞, and let φ be an element of Λ^+ majorising φ_0 and $\psi\varphi_0$. Then, since $\varphi\bar\varphi_0$ and $\varphi\bar\varphi_0\bar\psi$ both belong to H^∞, we have

$$\begin{aligned}
\langle S_\varphi M_\psi(f), g\rangle &= \langle TP(\varphi\psi\bar\varphi_0 f_1), \varphi\bar\varphi_0 g_1\rangle \\
&= \langle \bar\psi TP(\varphi\psi\bar\varphi_0 f_1), \bar\psi\varphi\bar\varphi_0 g_1\rangle \\
&= \langle T_\psi^* T T_\psi(\varphi\bar\varphi_0 f_1), \bar\psi\varphi\bar\varphi_0 g_1\rangle \\
&= \langle T(\varphi\bar\varphi_0 f_1), \bar\psi\varphi\bar\varphi_0 g_1\rangle \\
&= \langle \psi\bar\varphi TP(\varphi\bar\varphi_0 f_1), \bar\varphi_0 g_1\rangle \\
&= \langle M_\psi S_\varphi(f), g\rangle.
\end{aligned}$$

Thus, our claim is proved and S commutes with all of the multiplication operators on L^2. Hence, S itself is a multiplication, that is, $S = M_\psi$ for some $\psi \in L^\infty$. Since T is the compression of S to H^2, therefore T is equal to T_ψ. ∎

A Toeplitz operator T_ψ is said to be *analytic* if its symbol ψ is analytic, that is, if ψ belongs to H^∞.

It is well known that a bounded operator on the H^2 space of the circle is an analytic Toeplitz operator if and only if it commutes with the unilateral shift on the canonical orthonormal basis of the space. The following theorem is an analogue of this result.

THEOREM 2.2. *Let T be a bounded operator on H^2. Then the following are equivalent conditions:*

(1) T is an analytic Toeplitz operator;

(2) $TT_\varphi = T_\varphi T$ for all inner functions φ;

(3) $TT_\varphi = T_\varphi T$ for all functions φ belonging to A.

PROOF. The implications (1) ⇒ (2) and (1) ⇒ (3) are clear. Suppose that condition (2) holds. Then for every inner function φ we have $T_\varphi^* T T_\varphi = T$, since T_φ is an isometry and $TT_\varphi = T_\varphi T$. By the preceding theorem, $T = T_\psi$ for some function ψ belonging

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to L^∞. Let θ be an arbitrary element of Λ and write $\theta = \theta_1\bar{\theta}_2$, for some inner functions θ_1 and θ_2. Then $\langle\psi,\theta\rangle = \langle\psi\theta_2,\theta_1\rangle = \langle T_\psi T_{\theta_2}(1),\theta_1\rangle = \langle T_{\theta_2}T_\psi(1),\theta_1\rangle = \langle T_\psi(1),\theta_1\bar{\theta}_2\rangle = \langle T_\psi(1),\theta\rangle$. Since L^2 is the closed linear span of Λ, it follows that $\psi = T_\psi(1)$. Thus, ψ belongs to $H^2 \cap L^\infty = H^\infty$ [4, p. 221] and $T = T_\psi$ is an analytic Toeplitz operator. We have therefore shown that (2) $\Rightarrow$ (1).

Suppose now condition (3) holds, so that $TT_\varphi = T_\varphi T$ for all elements φ belonging to A. Now assume that φ is an element of H^∞. Then there exists a sequence (φ_n) in A converging pointwise to φ almost everywhere on G (with respect to m) and having the property that $\|\varphi_n\|_\infty \leq \|\varphi\|_\infty$ for all n [4, p. 220]. An application of the dominated convergence theorem shows that for all $f \in H^2$ we have $\lim_{n\to\infty}\|T_{\varphi_n}(f) - T_\varphi(f)\|_2^2 = \lim_{n\to\infty}\int|\varphi_n f - \varphi f|^2\,dm = 0$, so that T_φ is the strong limit of the sequence (T_{φ_n}). Since T commutes with all of the operators T_{φ_n}, therefore T also commutes with T_φ. We have thus shown that (3) $\Rightarrow$ (2) and this completes the proof. ∎

We give below another characterisation of analytic Toeplitz operators (Proposition 3.5). The proof will require Theorem 3.3.

COROLLARY 2.3. *The set of analytic Toeplitz operators on H^2 forms a maximal abelian subalgebra of $B(H^2)$. Moreover, this algebra acts irreducibly on H^2.*

PROOF. We show only that the analytic Toeplitz operators act irreducibly, as the rest is obvious. If T_ψ is an idempotent analytic Toeplitz operator, then ψ is an idempotent of H^∞ and is therefore real-valued. Since the only real-valued elements of H^∞ are the constants [4, p. 225], ψ is equal to 0 or 1. The irreducibility result follows. ∎

COROLLARY 2.4. *The spectrum of an analytic Toeplitz operator is necessarily connected.*

PROOF. If T_ψ is an analytic Toeplitz operator, then its spectrum is the same whether considered relative to $B(H^2)$ or to the maximal abelian (closed) subalgebra B of all analytic Toeplitz operators. By Shilov's idempotent theorem, the spectrum of each element of B is connected, because, by the proof of the preceding corollary, B contains no non-trivial idempotents. ∎

Corollary 2.4 was proved in [13] by a slightly different argument.

A natural question that arises from Theorem 2.2 is this: If an operator $T \in B(H^2)$ commutes with all the operators T_φ, where φ is an inner function belonging to A, is T necessarily Toeplitz? The answer turns out to be no, because there may be no non-constant inner functions in A. For example, there is a subalgebra A of the disc algebra B on the circle which contains no non-trivial inner functions. This algebra is of the form

$$A = \{\psi \in B \mid \psi\circ\theta \in B\},$$

where θ is a homeomorphism of $\mathbf{T}$ which is singular (that is, θ maps a set of Lebesgue measure zero onto one of full Lebesgue measure). Moreover, A is a Dirichlet algebra on $\mathbf{T}$, and therefore clearly Lebesgue measure is the unique representing measure for a character of A. The reason that A contains only constant inner functions is that an inner function ψ of A is such that both ψ and $\psi \circ \theta$ are finite Blaschke products, from which it follows that ψ is a constant. The algebra A was constructed by Browder and Wermer [4, pp. 232–5].

3. **Toeplitz algebras.** In this section we are concerned with investigating the properties of the C^*-subalgebra $\mathbf{L}$ of $B(H^2)$ generated by all of the Toeplitz operators on H^2, and of the C^*-algebra $\mathbf{A}$ generated by the Toeplitz operators with continuous symbols. We call $\mathbf{A}$ the *Toeplitz* algebra and $\mathbf{L}$ the *full Toeplitz* algebra. These algebras can be used to establish some of the spectral properties of Toeplitz operators [10, 13]. The *commutator* ideal of $\mathbf{L}$—the smallest closed ideal generated by all additive commutators $ST - TS$ $(S, T \in \mathbf{L})$—will be denoted by $\mathbf{K}$.

If φ is an inner function, set $P_\varphi = 1 - T_\varphi T_\varphi^*$. Because T_φ is an isometry, P_φ is a projection. Moreover, P_φ belongs to $\mathbf{K}$, since $P_\varphi = T_\varphi^* T_\varphi - T_\varphi T_\varphi^*$. It is easily checked that $(P_\varphi)_{\varphi \in \Lambda^+}$ is an increasing net. In the reverse direction, if φ, ψ are inner functions such that $P_\varphi \leq P_\psi$, then $\varphi \leq \psi$ in Λ. For, in this case, $P_\psi(\psi) = 0$ and $\|P_\varphi(\psi)\| \leq \|P_\psi(\psi)\|$, so $P_\varphi(\psi) = 0$. Hence, $\psi = \varphi P(\bar{\varphi}\psi)$, so $\psi\bar{\varphi} \in H^\infty$, that is, $\varphi \leq \psi$.

The following lemma is important for our analysis of $\mathbf{L}$.

LEMMA 3.1. *The net $(P_\varphi)_{\varphi \in \Lambda^+}$ is an increasing approximate unit for the C^*-algebra* $\mathbf{K}$.

PROOF. Let $\mathbf{L}_0$ denote the linear span of all products $T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_n}$, where $\varphi_1, \ldots, \varphi_n$ belong to Λ, and let $\mathbf{K}_0$ denote the linear span of the corresponding operators $T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_n} - T_{\varphi_1 \varphi_2 \cdots \varphi_n}$. If $\varphi \in \Lambda$, then the equation $T_\varphi(T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_n} - T_{\varphi_1 \varphi_2 \cdots \varphi_n}) = T_\varphi T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_n} - T_{\varphi \varphi_1 \cdots \varphi_n} + T_{\varphi(\varphi_1 \cdots \varphi_n)} - T_\varphi T_{\varphi_1 \varphi_2 \cdots \varphi_n}$ shows that $\mathbf{K}_0$ is a left ideal of the algebra $\mathbf{L}_0$. Clearly, $\mathbf{L}_0$ and $\mathbf{K}_0$ are self-adjoint, so $\mathbf{K}_0$ is also a right ideal of $\mathbf{L}_0$. It follows that the closure $\bar{\mathbf{K}}_0$ is an ideal in the C^*-algebra $\bar{\mathbf{L}}_0$ and clearly $\bar{\mathbf{L}}_0 = \mathbf{L}$, since Λ has norm-closed linear span L^∞. It is obvious that the quotient algebra $\mathbf{L}/\bar{\mathbf{K}}_0$ is abelian, so $\bar{\mathbf{K}}_0$ contains $\mathbf{K}$. We now show that $(P_\varphi)_{\varphi \in \Lambda^+}$ is an approximate unit for $\bar{\mathbf{K}}_0$ and this will imply that $\bar{\mathbf{K}}_0 = \mathbf{K}$ (since the projections P_φ belong to $\mathbf{K}$) and the lemma will be proved.

To show that (P_φ) is an approximate unit as claimed, it suffices to show that for any $T \in \mathbf{K}_0$, there exists an inner function φ such that $T = TP_\varphi$ (in this case, for any inner function $\psi \geq \varphi$, we clearly also have $T = TP_\psi$). We may suppose that T is of the form $T = T_{\varphi_n} \cdots T_{\varphi_1} - T_{\varphi_n \cdots \varphi_1}$, where $\varphi_1, \ldots, \varphi_n$ belong to Λ. Choose φ in Λ^+ such that φ majorises all of the elements $\bar{\varphi}_1, \bar{\varphi}_1\bar{\varphi}_2, \ldots, \bar{\varphi}_1 \cdots \bar{\varphi}_n$ of Λ. Then $T_{\varphi_n} \cdots T_{\varphi_1} T_\varphi = T_{\varphi_n} \cdots T_{\varphi_1 \varphi} = T_{\varphi_n} \cdots T_{\varphi_2 \varphi_1 \varphi} = \cdots = T_{\varphi_n \cdots \varphi_1 \varphi}$, so $TT_\varphi = 0$ and therefore $T = TP_\varphi$ as required. ∎

PROPOSITION 3.2. *The only Toeplitz operator belonging to* $\mathbf{K}$ *is the zero operator.*

PROOF. Let $\psi \in L^\infty$ and suppose that $T_\psi \in \mathbf{K}$. Since the net $(T_\psi P_\varphi)_\varphi$ converges to T_ψ, the net $(T_\psi T_\varphi T_\varphi^*)_\varphi$ converges to zero. But $\|T_\psi\| = \|T_\varphi^* T_\psi T_\varphi T_\varphi^* T_\varphi\| \leq \|T_\psi T_\varphi T_\varphi^*\|$, so $T_\psi = 0$. ∎

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THEOREM 3.3. *There exists a unique* $*$*-homomorphism* $\pi\colon \mathbf{L} \longrightarrow L^\infty$ *such that* $\pi(T_\psi) = \psi$ *for all* $\psi \in L^\infty$. *Moreover,* $\mathbf{K}$ *is the kernel of* π.

PROOF. It suffices to show that the map

$$L^\infty \longrightarrow \mathbf{L}/\mathbf{K}, \quad \psi \longmapsto T_\psi + \mathbf{K},$$

is a $*$-isomorphism. Clearly, it is linear and preserves adjoints. Moreover, it is multiplicative, since $\mathbf{K}$ contains all operators of the form $T_\varphi T_\psi - T_{\varphi\psi}$, (this follows easily from the proof of Lemma 3.1). Injectivity of the map is immediate from Proposition 3.2 and surjectivity follows from the observation that the range of the map is a C^*-algebra containing the elements $T_\psi + \mathbf{K}$ ($\psi \in L^\infty$) and these generate the quotient algebra $\mathbf{L}/\mathbf{K}$. ■

REMARK 3.4. This theorem can also be obtained as a special case of a result of Tomiyama and Yabuta [14], as observed in [13]. The advantage of the direct derivation given here is that is constitutes a more elementary approach. Also, in proving the theorem, we showed that $(P_\varphi)_\varphi$ is an approximate unit for $\mathbf{K}$, a result of independent interest that does not appear to follow from the results of [14].

An application of Theorem 3.3 to the Fredholm theory of Toeplitz operators on H^2 is given in [13]. Here, we give a different application, related to the considerations of Section 2, namely, another characterisation of analytic Toeplitz operators.

PROPOSITION 3.5. *A bounded operator* T *on* H^2 *is an analytic Toeplitz operator if and only if both* T *and* T^*T *are Toeplitz operators.*

PROOF. The forward implication is obvious. To show the reverse implication holds, assume that T and T^*T are Toeplitz operators, so that $T = T_\varphi$ and $T^*T = T_\psi$, for some functions $\varphi, \psi \in L^\infty$. Applying the canonical map $\pi\colon \mathbf{L} \longrightarrow L^\infty$ to the equation $T_\varphi^* T_\varphi = T_\psi$, we get $\bar\varphi\varphi = \psi$. Thus, we only need to show that for any $\varphi \in L^\infty$, we have $T_{\bar\varphi} T_\varphi = T_{\bar\varphi\varphi}$ if and only if φ is analytic.

Assume then that $T_{\bar\varphi} T_\varphi = T_{\bar\varphi\varphi}$. To show that $\varphi \in H^\infty$ it suffices to show that H^2 is an invariant subspace for M_φ, that is, that $PM_\varphi P = M_\varphi P$. Clearly, the equation $T_{\bar\varphi} T_\varphi = T_{\bar\varphi\varphi}$ is equivalent to the equation $PM_\varphi^* PM_\varphi P = PM_\varphi^* M_\varphi P$, so therefore $PM_\varphi^*(1-P)M_\varphi P = 0$, that is, $\big((1-P)M_\varphi P\big)^*(1-P)M_\varphi P = 0$. Hence, $(1-P)M_\varphi P = 0$, as required. ■

One can prove a result for $\mathbf{A}$ similar to that shown for $\mathbf{L}$ in Theorem 3.3, but not by the same methods. The problem is that the underlying function algebra A may contain very few inner functions—indeed, as we observed above, there is a Dirichlet subalgebra of the disc algebra which contains no non-trivial inner functions.

If φ is a function defined on G, we shall denote its restriction to a subset M of G by φ_M. The following result can be derived directly from Theorem 3.3; see [13] for details.

THEOREM 3.6. *Let* M *denote the support of the measure* m. *There exists a unique* $*$*-homomorphism* $\pi\colon \mathbf{A} \longrightarrow C(M)$ *such that for all* $\varphi \in C(G)$ *we have* $\pi(T_\varphi) = \varphi_M$. *Moreover, the commutator ideal of* $\mathbf{A}$ *is equal to the kernel of* π.

REMARK 3.7. The algebra $\mathbf{A}$ is not, in general, a Type I C^*-algebra, and it can have a very complicated structure [9, 12]. The computation of the K-theory of $\mathbf{A}$ is relevant to the index theory of Toeplitz operators with continuous symbols, and has been partially computed in a special case [11]. See also [12], where, using a different approach, we proved a theorem analogous to the Gohberg-Krein index theorem for Toeplitz operators with continuous symbols (again, in a special case). The index theory for the general case has not yet been worked out, and should be interesting.

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