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NORMAL EMBEDDINGS OF p-GROUPS INTO p-GROUPS

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A well known lemma of Burnside is generalised, to give necessary and sufficient conditions for a finite p-group K to be normally embedded in a nilpotent group V, with $K \subseteq \omega(V)$. (Here, ω denotes a single word and $\omega(V)$ is the corresponding verbal subgroup.) Our main result is related to earlier work of Blackburn, Gaschütz and Hobby.

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The following result of Burnside has been an initial step for similar statements: A non-abelian group whose centre is cyclic cannot be the derived group of a p-group ([**2**, Theorem p. 241]). Burnside himself noted the consequence, that a non-abelian group the index of whose derived group is p^2 cannot be the derived group of a p-group ([**2**, Theorem, p. 242]). Later on Hobby showed that a group satisfying one of the two hypotheses just mentioned cannot be the Frattini subgroup of a p-group (see [**4**, Theorem 1 and **2**, p. 209]). On the other hand Blackburn has brought in a positive note by determining exactly those two-generator p-groups which occur as derived groups of p-groups [**1**].

The results of Burnside and Hobby mentioned before can still be strengthened: Given the same hypotheses the group cannot be invariant in a p-group and at the same time included in its Frattini subgroup. This seems to be well known. A positive statement (like that of Blackburn) cannot be expected considering the little information given in the hypotheses.

The purpose of this note is slightly more general: We ask for a necessary and sufficient condition to decide whether a given p-group N can be a normal subgroup of a p-group G and contained in a (preassigned) characteristic subgroup of G. Such a condition is exhibited for verbal subgroups (Main Theorem). It depends on the automorphism group of N only. We need a construction to prove the positive part of the statement (Theorem 3); later on we shall see that the construction can be improved for many special cases.

We end this note with the proof of a statement that allows the following specialization: If $N = A \times B$ where A is of exponent p and of nilpotency class 2 and B is of order p, then there is a p-group G with normal subgroup N^+ which is contained in G' such that N and N^+ are isomorphic (see Proposition 8).

Notation is mostly standard: The derived group (that is, the commutator subgroup) of G is denoted by G', and by G_n we mean the nth term of the lower central series of G (so that $G_1 = G$).

A group G is residually nilpotent if the intersection of all G_n is trivial. The intersection of all maximal subgroups of G is called the Frattini subgroup and denoted by $\Phi(G)$; also $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ are the groups of automorphisms and of inner automorphisms of G respectively. A word $w = w(x_1, \ldots, x_k)$ is a product of elements (considered as variables) of a group. We call $w(G) = \langle w(x_1, \ldots, x_k), x_i \in G \rangle$ the verbal subgroup of G corresponding to the word w. For background information on this subject see H. Neumann [**5**, Chapter 1] or D. J. S. Robinson [**6**, p. 55].

For our construction later on we need a basic statement.

Lemma 1. *For a given word $w(x_1, \ldots, x_n)$ and a natural number k there is a finite p-group S such that the verbal subgroup $w(S)$ is of exponent $m = p^k$ and contained in the centre of S.*

Proof. Consider the free group F of rank n and the verbal subgroup $w(F)$. Since w is nontrivial, also $w(F)$ is nontrivial. Since F is residually nilpotent, there is a number t such that

$$w(F) \text{ is contained in } F_t \text{ but not in } F_{t+1}.$$

Since F is a free group, F/F_{t+1} is torsion free, and

$$w(F)F_{t+1}/(w(F))^m F_{t+1} \text{ is of exact exponent } m.$$

Denote $(w(F))^m F_{t+1}$ by R. Choose a normal subgroup Y of F which is maximal with respect to satisfying the relation $w(F)R \cap Y = R$. Every maximal abelian normal subgroup of F/Y containing $w(F)Y/Y$ is finite and so F/Y must be finite, and F/Y is a p-group since $w(F)Y/Y$ is a p-group; the lemma is shown for $S = F/Y$.

To show that certain embeddings are impossible we have the following lemma.

Lemma 2. *Assume that K is a normal subgroup of the finite p-group G. Consider a Sylow-p-subgroup L of* $\mathrm{Aut}(K)$. *If* $\mathrm{Inn}(K) \nsubseteq w(L)$ *for some word w, then also $K \nsubseteq w(G)$.*

Proof. $G/C(K)$ is isomorphic to a subgroup of L, and this isomorphism maps $KC(K)/C(K)$ onto $\mathrm{Inn}(K)$. Since forming verbal subgroups is a monotonic operation, we deduce that

$$w(G/C(K)) = w(G)C(K)/C(K) \quad \text{does not contain}$$
$$KC(K)/C(K), \text{ and so } w(G) \quad \text{does not contain } K.$$

Remark. The following result is due to Gaschütz [3, Satz 11]: If N is a normal subgroup of the finite group G, then N cannot be contained in the Frattini subgroup $\Phi(G)$ of G if $\mathrm{Inn}(N) \not\subseteq \Phi(\mathrm{Aut}(N))$. Note that, in general, the Frattini subgroup of a finite group is not a verbal subgroup (if G is the holomorph of the group of order five, there is a subgroup U such that $\Phi(U) \not\subseteq U \cap \Phi(G)$ and a normal subgroup R such that $\Phi(G)R/R \neq \Phi(G/R)$).

We can now proceed to the positive part.

Theorem 3. *Assume that K is a finite p-group such that, for some word w,* $\mathrm{Inn}(K)$ *is contained in the verbal subgroup $w(L)$ of a Sylow p-subgroup L of* $\mathrm{Aut}(K)$. *Then there is a finite p-group G such that G possesses a normal subgroup $K^+ \subseteq w(G)$ isomorphic to K.*

Proof. Let $\exp(K) = p^k = m$. By Lemma 1 there is a finite p-group S such that $w(S) \subseteq Z(S)$ with an element $u \in w(S)$ of order m.

We form an extension of the wreath product $K \mathrm{wr} S$ by L in the following manner: L and S centralize each other; and if a^* denotes the inner automorphism defined by $a \in K$, then $a^{*-1}xa^* = a^{-1}xa$ for all x in K. Consequently $a^{-1}a^*$ centralizes K, and

$$\left(\prod_{s\in S} s^{-1}as\right)^{-1} a^* \text{ centralizes } K^S.$$

Now

$$\left(\prod_{s\in S}(s^{-1}as)^{-1}a^*\right)\left(\prod_{s\in S}(s^{-1}bs)^{-1}b^*\right) = \left(\prod_{s\in S} s^{-1}bs\right)^{-1}\left(\prod_{s\in S} s^{-1}as\right)^{-1} a^*b^*$$

$$= \left(\prod_{s\in S} s^{-1}abs\right)^{-1}(ab)^*,$$

and we see that the set $D = \{(\prod_{s\in S} s^{-1}as)^{-1}a^* \mid a \in K\}$ is in fact a subgroup of $M = \langle K, S, L\rangle$. It is easy to check that D is normal in M and that it is isomorphic to K.

For all a in K we know by hypothesis

$$a^* \in w(L).$$

Let R be any transversal of $\langle u\rangle$ in S. Then

$$\prod_{s\in S} s^{-1}as = \prod_{r\in R} r^{-1}\left(\prod_{i=0}^{m-1} u^{-1}au^i\right)r.$$

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Now $\prod_{i=0}^{m-1} u^{-i}au^i=\prod_{i=1}^{m-1}[a,u^i]$, and since $u\in w(S)$ we have $[a,u^i]\in w(\langle a\rangle \operatorname{wr} S)\subseteq w(KS)$. So D is contained in $w(M)$, and the theorem is shown for $M=G$ and $D=K^+$.

We have collected all the details needed for our central result.

Main Theorem. (Theorem 4) *Assume that K is a finite p-group and L is a Sylow p-subgroup of* Aut(K).
$w(V)\not\supseteq K$ is true for every nilpotent extension V of K if and only if $w(L)\not\supseteq \mathrm{Inn}(K)$.

Proof. If $w(L)\not\supseteq \mathrm{Inn}(K)$, then $w(V)\not\supseteq K$ by Lemma 2. If $w(L)\supseteq \mathrm{Inn}(K)$, there is an extension V with $w(V)\supseteq K$ by Theorem 3.

2. Special cases

In this section we will show that the construction used in Theorem 3 can be improved in special cases to obtain smaller extensions for the same purpose.

Analysis of the construction in Theorem 3 shows that the key statement is the inclusion of $DL\wedge K^S$ in $w(KS)$. We will show for certain cases that a smaller group S does this already. In each case we have the hypothesis

$$K \text{ is a finite } p\text{-group and } L \text{ is a Sylow } p\text{-subgroup of } \mathrm{Aut}(K). \qquad (+)$$

We will denote by $d(G)$ the derived length of G (so $G^{(d(G))}=1$) and by $d^*(K)$ the derived length of a Sylow p-subgroup of Hol(K).

Proposition 5. *Let K and L be as in (+) and $L^{p^i}\supseteq \mathrm{Inn}(K)$. Then there is an extension G of K such that*

(i) $G^{p^i}\supseteq K$,

and

(ii) $d(G)\leq d^*(K)+1$.

Proof. We choose $\langle t\rangle$ for S, where $\langle t\rangle$ is of order $\mathrm{Max}(p^i,\exp(K))=m$. The proposition now follows from

$$\prod_{i=0}^{m-1} t^{-i}xt^i=(xt^{-1})^m$$

and $d(\langle K^S,L\rangle)=d^*(K)$.

Proposition 6. *Let K and L be as in (+) and $L_n\supseteq \mathrm{Inn}(K)$. Then there is an extension G of K such that*

(i) $G_n\supseteq K$,

and

(ii) $d(G) \leqq d^*(K)+1$.

Proof. Assume $n \leqq v(p-1)+1$, and $\exp(K)=p^k=m$. We choose for S the direct product of v cyclic groups t_j of order m. Let t be one of them. Since $\prod_{i=0} t^{-i}at^i = a^m[a,t]^{\binom{m}{2}}[[a,t],t]^{\binom{m}{3}}\ldots$ and the binomial coefficient $\binom{m}{d}$ is divisible by m for $d<p$, we find

$$\prod_{i=0}^{m-1} t^{-i}at^i \in \langle t,a\rangle_p,$$

and

$$\prod_{s\in S} s^{-1}as \in \langle K,S\rangle_{v(p-1)+1} \subseteq \langle K,S\rangle_n.$$

The inequality (ii) follows as in Proposition 5.

Proposition 7. *Let K and L be as in $(+)$ and $L'' \supseteq \mathrm{Inn}(K)$. Then there is an extension G of K such that*

(i) $G'' \supseteq K$,

and

(ii) $d(G) \geqq d^*(K)+2$.

Proof. Assume $\exp(K)=p^k=m$. We choose

$$S=\langle u,v \mid u^m=v^m=[[u,v]v]=[[u,v],u]=1\rangle.$$

The commutator $w=[u,v]$ has order m. Let $a\in K$. Now

$$\prod_{s\in S} s^{-1}as = \prod_{i=0}^{m-1} v^{-i}\left(\prod_{j=0}^{m-1} w^{-j}\left(\prod_{t=0}^{m-1} u^{-t}au^{-t}\right)w^j\right)v^i$$

is contained in $[\langle w\rangle, [\langle u\rangle,\langle a\rangle]] \subseteq \langle S,K\rangle''$. Since $S''=1$, (ii) follows.

3. A case of embeddability

As promised we prove here the last statement of the introduction; in fact, we show something more general.

Proposition 8. *Assume that N is the direct product $A\times B$ where A is of exponent p and* *of nilpotency class 2 and B is elementary abelian of rank n. Then there is an extension G of N such that N is contained in* G_{n+1}.

Proof. We can make use of the Main Theorem at once if N is abelian. We assume now that N is nonabelian, that A has a basis $\{a_1,\dots,a_k\}$, and B has a basis $b_1,\dots,b_n$. We consider first the group S of all automorphisms of N that stabilize the series

$$N \supset A'B \supset A'\langle b_2,\dots,b_n\rangle \supset A'\langle b_3,\dots,b_n\rangle \supset \cdots \supset A'\langle b_n\rangle \supset A' \supset 1,$$

that is, the group of all automorphisms leaving all terms of the series invariant and inducing the identity on quotient groups of consecutive terms. Clearly S is a p-group. Let z be any element of A'. We consider elements in S which fix all basis elements except one; among these we single out τ_i mapping a_i onto a_ib_1, σ_j mapping b_j onto b_jb_{j+1} and ρ mapping b_n onto b_nz.

Now $[\dots[\tau_i,\sigma_1],\dots\sigma_{n-1}],\rho]$ is an automorphism fixing all basis elements except a_i which is mapped onto a_iz. This shows that the automorphism stabilizing

$$A \supset A' \supset 1$$

are all contained in S_{n+1}, and clearly all inner automorphisms of A belong to this set. There is a Sylow p-subgroup L of Aut(N) that contains S, and we have found

$$L_{n+1} \supseteq S_{n+1} \supseteq \mathrm{Inn}(N).$$

By the Main Theorem, Proposition 8 is true.

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