Glasgow Math. J. **56** (2014) 9–12. © Glasgow Mathematical Journal Trust 2013.
doi:10.1017/S0017089512000870.

2-LOCAL DERIVATIONS ON SEMI-FINITE VON NEUMANN ALGEBRAS

SHAVKAT AYUPOV

Institute of Mathematics, National University of Uzbekistan, Tashkent, Uzbekistan, and the Abdus Salam International Centre for Theoretical Physics (ICTP) Trieste, Italy
e-mail: sh_ayupov@mail.ru

and FARKHAD ARZIKULOV

Institute of Mathematics, National University of Uzbekistan, Tashkent, and Andizhan State University, Andizhan, Uzbekistan
e-mail: arzikulovfn@rambler.ru

(Received 29 September 2012; accepted 1 November 2012; first published online 25 February 2013)

Abstract. In the present paper we prove that every 2-local derivation on a semi-finite von Neumann algebra is a derivation.

2002 *Mathematics Subject Classification.* Primary 46L57, Secondary 46L40.

1. Introduction. The present paper is devoted to 2-local derivations on von Neumann algebras. Recall that a 2-local derivation is defined as follows: Given an algebra A, a map $\triangle : A \to A$ (not linear in general) is called a 2-local derivation if for every $x, y \in A$, there exists a derivation $D_{x,y} : A \to A$ such that $\triangle(x) = D_{x,y}(x)$ and $\triangle(y) = D_{x,y}(y)$.

In 1997, Šemrl [**7**] introduced the notion of 2-local derivation and described 2-local derivations on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H. A similar description for the finite-dimensional case appeared later in [**5**]. In the paper by Lin and Wong [**6**], 2-local derivations have been described on matrix algebras over finite-dimensional division rings.

In [**2**] the authors suggested a new technique and have generalized the above-mentioned results of [**7**] and [**5**] for arbitrary Hilbert spaces, namely they considered 2-local derivations on the algebra $B(H)$ of all linear-bounded operators on an arbitrary (no separability is assumed) Hilbert space H and proved that every 2-local derivation on $B(H)$ is a derivation.

In [**1**] we also suggested another technique and generalized the above-mentioned results of [**7**], [**5**] and [**2**] for arbitrary von Neumann algebras of type I and proved that every 2-local derivation on these algebras is a derivation. In [**3**] (Theorem 3.4) a similar result was proved for finite von Neumann algebras.

In the present paper we extended the above results and give a short proof of the theorem for arbitrary semi-finite von Neumann algebras.

2. Preliminaries. Let M be a von Neumann algebra.

Definition. A linear map $D : M \to M$ is called a derivation if $D(xy) = D(x)y + xD(y)$ for any two elements $x, y \in M$.

A map $\Delta : M \to M$ is called a 2-local derivation if for any two elements $x, y \in M$ there exists a derivation $D_{x,y} : M \to M$ such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$.

It is known that any derivation D on a von Neumann algebra M is an inner derivation, that is there exists an element $a \in M$ such that

$$D(x) = ax - xa, x \in M.$$

Therefore, for a von Neumann algebra M the above definition is equivalent to the following one: A map $\Delta : M \to M$ is called a 2-local derivation if for any two elements $x, y \in M$ there exists an element $a \in M$ such that $\Delta(x) = ax - xa$ and $\Delta(y) = ay - ya$.

Let $\mathcal{M}$ be a von Neumann algebra, $\triangle : \mathcal{M} \to \mathcal{M}$ be a 2-local derivation. It is easy to see that $\triangle$ is homogeneous. Indeed, for each $x \in \mathcal{M}$, and for $\lambda \in \mathbb{C}$ there exists a derivation $D_{x,\lambda x}$ such that $\triangle(x) = D_{x,\lambda x}(x)$ and $\triangle(\lambda x) = D_{x,\lambda x}(\lambda x)$. Then,

$$\triangle(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \triangle (x).$$

Hence, $\triangle$ is homogenous. Further, for each $x \in \mathcal{M}$, there exists a derivation D_{x,x^2} such that $\triangle(x) = D_{x,x^2}(x)$ and $\triangle(x^2) = D_{x,x^2}(x^2)$. Then,

$$\triangle(x^2) = D_{x,x^2}(x^2) = D_{x,x^2}(x)x + xD_{x,x^2}(x) = \triangle(x)x + x \triangle (x).$$

A linear map satisfying the above identity is called a Jordan derivation. It is proved in **[4]** that any Jordan derivation on a semi-prime algebra is a derivation. Since every von Neumann algebra $\mathcal{M}$ is semi-prime (i.e. $a\mathcal{M}a = \{0\}$ implies that $a = \{0\}$), in order to prove that a 2-local derivation $\triangle : \mathcal{M} \to \mathcal{M}$ is a derivation it is sufficient to show that the map $\triangle : \mathcal{M} \to \mathcal{M}$ is additive.

3. 2-local derivations on semi-finite von Neumann algebras. Let $\mathcal{M}$ be a semi-finite von Neumann algebra and let τ be a faithful normal semi-finite trace on $\mathcal{M}$. Denote by m_τ the definition ideal of τ, i.e the set of all elements $a \in \mathcal{M}$ such that $\tau(|a|) < \infty$. Then m_τ is a $*$-algebra, and moreover m_τ is a two-sided ideal of $\mathcal{M}$ (see **[8]**, Definition 2.17).

It is clear that any derivation D on $\mathcal{M}$ maps the ideal m_τ into itself. Indeed, since D is inner, i.e. $D(x) = ax - xa, x \in \mathcal{M}$ for an appropriate $a \in \mathcal{M}$, we have $D(x) = ax - xa \in m_\tau$ for all $x \in m_\tau$. Therefore, any 2-local derivation on $\mathcal{M}$ also maps m_τ into itself.

THEOREM. *Let $\mathcal{M}$ be a semi-finite von Neumann algebra, and let $\triangle : \mathcal{M} \to \mathcal{M}$ be a 2-local derivation. Then $\triangle$ is a derivation.*

Proof. Let $\triangle : \mathcal{M} \to \mathcal{M}$ be a 2-local derivation and let τ be a faithful normal semi-finite trace on $\mathcal{M}$. For each $x \in \mathcal{M}$ and $y \in m_\tau$ there exists a derivation $D_{x,y}$ on $\mathcal{M}$ such that $\triangle(x) = D_{x,y}(x)$, $\triangle(y) = D_{x,y}(y)$. Since every derivation on $\mathcal{M}$ is inner, there exists an element $a \in \mathcal{M}$ such that

$$[a, xy] = D_{x,y}(xy) = D_{x,y}(x)y + xD_{x,y}(y) = \triangle(x)y + x\triangle(y),$$

i.e.

$$[a, xy] = \triangle(x)y + x\triangle(y).$$

https://doi.org/10.1017/S0017089512000870 Published online by Cambridge University Press

We have

$$|\tau(axy)| < \infty.$$

Since m_τ is an ideal and $y \in m_\tau$, the elements axy, xy, xya and $\triangle(y)$ also belong to m_τ and hence we have

$$\tau(axy) = \tau(a(xy)) = \tau((xy)a) = \tau(xya).$$

Thus,

$$0 = \tau(axy - xya) = \tau([a, xy]) = \tau(\triangle(x)y + x\triangle(y)),$$

i.e.

$$\tau(\triangle(x)y) = -\tau(x\triangle(y)).$$

For arbitrary $u, v \in \mathcal{M}$ and $w \in m_\tau$ set $x = u + v$, $y = w$. Then $\triangle(w) \in m_\tau$ and

$$\begin{aligned}\tau(\triangle(u+v)w) &= -\tau((u+v)\triangle(w))\\ &= -\tau(u\triangle(w)) - \tau(v\triangle(w)) = \tau(\triangle(u)w) + \tau(\triangle(v)w)\\ &= \tau((\triangle(u) + \triangle(v))w),\end{aligned}$$

and so

$$\tau((\triangle(u+v) - \triangle(u) + \triangle(v))w) = 0,$$

for all $u, v \in \mathcal{M}$ and $w \in m_\tau$. Denote $b = \triangle(u+v) - \triangle(u) + \triangle(v)$. Then,

$$\tau(bw) = 0 \quad \forall w \in m_\tau \quad (1).$$

Now take a monotone increasing net $\{e_\alpha\}_\alpha$ of projections in m_τ such that $e_\alpha \uparrow 1$ in $\mathcal{M}$. Then $\{e_\alpha b^*\}_\alpha \subset m_\tau$. Hence, (1) implies

$$\tau(be_\alpha b^*) = 0 \quad \forall \alpha.$$

At the same time $be_\alpha b^* \uparrow bb^*$ in $\mathcal{M}$. Since the trace τ is normal, we have

$$\tau(be_\alpha b^*) \uparrow \tau(bb^*),$$

i.e. $\tau(bb^*) = 0$. The trace τ is faithful, so this implies that $bb^* = 0$, i.e. $b = 0$. Therefore,

$$\triangle(u+v) = \triangle(u) + \triangle(v), u, v \in \mathcal{M},$$

i.e. $\triangle$ is an additive map on $\mathcal{M}$. As it was mentioned in 'Preliminaries' this implies that $\triangle$ is a derivation on $\mathcal{M}$. The proof is complete. □

REFERENCES

1. Sh. A. Ayupov and F. N. Arzikulov, 2-local derivations on von Neumann algebras of type I. Available at http://www.arxiv.org v1 [math.OA], accessed 29 December 2011.

2. Sh. A. Ayupov and K. K. Kudaybergenov, 2-local derivations and automorphisms on $B(H)$, *J. Math. Anal. Appl.* **395** (2012), 15–18.

3. Sh. A. Ayupov, K. K. Kudaybergenov, B. O. Nurjanov and A. K. Alauatdinov, Local and 2-local derivations on noncommutative Arens algebras, Mathematica Slovaca (to appear). Available at http://arxiv.org/abs/1110.1557, accessed 7 October 2011.

4. M. Bresar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.* **104** (1988), 1003–1006.

5. S. O. Kim and J. S. Kim, Local automorphisms and derivations on M_n, *Proc. Amer. Math. Soc.* **132** (2004), 1389–1392.

6. Y. Lin and T. Wong, A note on 2-local maps, *Proc. Edinb. Math. Soc.* **49** (2006), 701–708.

7. P. Šemrl, Local automorphisms and derivations on $B(H)$, *Proc. Amer. Math. Soc.* **125** (1997), 2677–2680.

8. M. Takesaki, *Theory of operator algebras I*. (Springer-Verlag, New York, 1979).