Notes on Number Theory V

# INSOLVABILITY OF $\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$

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A number of interesting Diophantine equations are of the form $f(n) = f(a) f(b)$. Thus the case $f(a) = a^a$ has been studied by C. Ko [2] and W.H. Mills [3] and a class of non-trivial solutions has been found, though whether these give all the solutions is still unsettled. The case $f(n) = n!$ has been mentioned by W. Sierpinski. Here the situation is that besides the trivial solutions $m! = m!\ 1!$ and $(m!-1)!\ m! = (m!)!$ and the special solution $10! = 7!\ 6!$ no other solutions are known, nor are they known not to exist. In the present note we show that the equation in the title has no solutions. A sketch of a somewhat different proof that this equation has at most a finite number of solutions was recently communicated to the author by P. Erdös. The details and completion of that proof have been supplied by Miss M. Faulkner in part of a Master's thesis at the University of Alberta. The present proof is designed to avoid lemmas based on non-elementary methods and also reduces the amount of numerical work involved.

Our main tool will be the following

LEMMA 1. The product of $r$ consecutive numbers, each greater than $r$, contains a prime factor $\geq \frac{11}{10} r$. This is a slight refinement of a theorem of Sylvester and Schur in which the $\frac{11}{10} r$ is replaced by $r$. The simplest proof of the Sylvester-Schur theorem is due to P. Erdös [1] and rather obvious minor variations in his proof yield our lemma. We

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hope, in a subsequent note in this series, to prove lemma 1 with the $\frac{11}{10} r$ replaced by $\frac{7}{5} r$. By virtue of the example 6.7.8.9.10 the constant $\frac{7}{5}$ would be best possible, if we are not to exclude small $r$. However, one must modify Erdős' proof considerably to obtain the $\frac{7}{5}$ result.

We now suppose that $\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$ with $a \geq b$. We first note that $2a > n$, for otherwise the primes between $n$ and $2n$ would divide the left hand side but not the right hand side of our equation. Now let $n = a+r$ and let

$$(1) \qquad T_r = \binom{2n}{n} / \binom{2a}{a} = \frac{(2a+1)(2a+2)\ldots(2a+2r)}{((a+1)(a+2)\ldots(a+r))^2} .$$

Observe that by our lemma, $T_r$ must contain, in its numerator, a prime factor $p > \frac{22}{10} r$. If $T_r$ is to be an integer, this prime $p$ cannot divide the denominator of $T_r$, for if $p/(a+i)$ then $p/(2a+2i)$ and (since $p > 2r$) no other element of the numerator. Hence $p$ would divide the denominator more often than the numerator. Clearly $T_r < 4^r$, but now we know that $2b > \frac{22}{10} r$, for otherwise $p$ could not divide $\binom{2b}{b}$. We will show that for $r > 15$ this implies $T_r > 4^r$.

By induction over $b$ it is easy to show that $\binom{2b}{b} > \frac{4^b}{2\sqrt{b}}$ and since

$$b > \frac{11}{10} r \text{ we have } T_r = \binom{2b}{b} > \frac{4^b}{2\sqrt{b}} > \frac{4^{\frac{11}{10} r}}{2\sqrt{\frac{11}{10} r}}$$

For $r > 15$ this last inequality contradicts $T_r < 4^r$.

For $r \leq 15$ the result can be established by a variety of special considerations. We might note that if we had the lemma with $\frac{7}{5}$ replacing $\frac{11}{10}$ the above argument would give the theorem directly for all $r > 1$.

It might be assumed from the above that $\binom{2n}{n} / \binom{2a}{a}$ is never an integer. This is false. In fact

$$\binom{2\binom{2n}{n}-2}{\binom{2n}{n}-1}$$ is divisible by $\binom{2n}{n}$ for every $n$.

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