FINITE SOLVABLE c-GROUPS

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(Received 24 June 1966)

Introduction

A group G is called a c-group if each of its subnormal subgroups is characteristic in G. It is the object of this note to give a characterization of finite solvable c-groups.

All groups considered are assumed finite. Let $L(G)$ denote the first term of the lower nilpotent series of G, G' the commutator subgroup of G, and $Z(G)$ the center of G. Then we wish to prove the following theorems:

THEOREM 1. *Let G be a solvable group whose 2-Sylow subgroups are abelian. Then G is a c-group if and only if the following hold:*

1. $G = G'K$, *where* $G' \cap K = (1)$ *and G' is a cyclic Hall subgroup of G;*
2. $G'Z(G)$ *is cyclic;*
3. G/G' *is cyclic.*

THEOREM 2. *Let G be a solvable group whose 2-Sylow subgroups are non-abelian and which possesses no non-trivial abelian direct factor. Then G is a c-group if and only if the following hold:*

1. *The 2-Sylow subgroups of G are generalized quaternion;*
2. *G has exactly one element u of order 2;*
3. $G = L(G) \cdot K$ *where* $L(G) \cap K = (1)$ *and $L(G)$ is a cyclic Hall-subgroup of G;*
4. $G/\langle u \rangle$ *is a c-group.*

PRELIMINARIES. It should be remarked that if G is a solvable c-group then G is supersolvable. This follows from the fact that the chief factors of G are abelian.

A group G is called an A-group if each of its Sylow subgroups is abelian. If G is a solvable A-group then [5] $G' \cap Z(G) = (1)$ and $G'Z(G)$ is the Fitting subgroup of G.

Finally, a group G is called a t-group if each of its subnormal subgroups is normal in G. A theorem of Gaschutz [3] states that if G is a solvable t-group, then $L(G)$ is a Hall-subgroup of G of odd order.

Before proceeding to the proofs of Theorems 1 and 2, we prove the following

LEMMA 1. *Let G be a nilpotent group. Then G is a c-group if and only if G is cyclic.*

PROOF. If G is cyclic then it is clear that G is a c-group. Suppose G is a c-group. Since G is nilpotent, every subgroup of G is subnormal in G. Hence every subgroup of G is characteristic in G. It follows that G is abelian or Hamiltonian. If G is Hamiltonian then $G = A \times B \times Q_n$, where A is abelian of odd order, B is an elementary abelian 2-group, and $Q_n = \langle a, b\rangle$ with $a^{2^{n-1}} = b^4 = 1, ab = ba^{-1}, a^{2^{n-2}} = b^2$, and $n \geqq 3$. The map $\theta : Q_n \to Q_n$ given by

$$a^\theta = a$$
$$b^\theta = ab$$

defines an automorphism of Q_n which can be lifted to G. Thus $\langle b\rangle \neq \langle ab\rangle$ and $\langle b\rangle^\theta = \langle ab\rangle$ and we have a contradiction. It follows that G is abelian. Let G_p be a p-Sylow subgroup of G, say

$$G_p = \langle x_1\rangle \times \langle x_2\rangle \times \cdots \times \langle x_s\rangle,$$

with $s > 1$. We can assume without loss that x_1 is of maximal order in G_p. Then the map $\phi : G_p \to G_p$ given by

$$x_1^\theta = x_1 x_2$$
$$x_i^\theta = x_i, \qquad i \geqq 2,$$

is an automorphism of G_p which can be lifted to G. But $\langle x_1\rangle \neq \langle x_1 x_2\rangle$ and $\langle x_1\rangle^\theta = \langle x_1 x_2\rangle$ and we have a contradiction. So G_p is cyclic, and hence, so is G.

PROOF OF THEOREM 1. First suppose G is a c-group, say $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the canonical factorization of $|G|$ such that $p_1 < p_2 < \cdots < p_r$. Since G is supersolvable, it has a normal series

$$G = H_0 > H_1 > \cdots > H_{r-1} > H_r = (1), \tag{1}$$

where $|H_{i-1} : H_i| = p_i^{\alpha_i}$ for $i = 1, \cdots, r$. Thus if $2 \mid |G|$, then G/H_1 is isomorphic to a 2-Sylow subgroup of G. Since G is a c-group and H_{r-1} is a nilpotent normal subgroup of G of odd order, H_{r-1} is abelian. Similarly, H_{i-1}/H_i is abelian for $i = 2, \cdots, r-2$. So G is an A-group. Thus $G' \cap Z(G) = (1)$ and $G'Z(G)$ is the Fitting subgroup of G. Hereafter, let $Z = Z(G)$ and $L = L(G)$.

Since G is supersolvable, G' is nilpotent and hence, abelian. Since G/L is nilpotent, it must be abelian, thus $G' = L$. So [3] G' is a Hall-subgroup of G and $G = G'K$ with $G' \cap K = (1)$. Since G' is abelian, the automorphisms of G must induce power automorphisms on G'. We claim that G' is cyclic. Let $\theta \in \mathrm{Aut}\,(G')$ and let $g \in G$, say $g = a \cdot b$ where $a \in G'$

https://doi.org/10.1017/S1446788700005553 Published online by Cambridge University Press

and $b \in K$. Define $\bar{\theta} : G \to G$ by $g^{\bar{\theta}} = a^{\theta} b$. Then $\bar{\theta}$ is an automorphism of G. Hence G' is cyclic.

We now show that Z is cyclic. Since $Z \cap G' = (1)$, $Z \leqq K$. Let p be any prime divisor of $|Z|$ and let K_p be the p-Sylow subgroup of K, say

$$K_p = \langle u_1 \rangle \times \langle u_2 \rangle \times \cdots \times \langle u_t \rangle$$

with $t > 1$. Let b be an element of Z of order p. Assume $b \notin \langle u_1 \rangle$ and define $\alpha : K_p \to K_p$ by

$$\begin{aligned} u_1^{\alpha} &= b u_1 \\ u_i^{\alpha} &= u_i, \qquad i > 1, \end{aligned}$$

then α can be extended to an automorphism $\bar{\alpha}$ of G which induces the identity on G'. Say $G' = \langle c \rangle$. Then $H_1 = \langle c, u_1 \rangle$ and $H_2 = \langle c, bu_1 \rangle$ are normal in G and $H_1 \neq H_2$. But $H_1^{\bar{\alpha}} = H_2$ so we have a contradiction. Thus K_p is cyclic. In particular, Z is cyclic. Since G' is a Hall-subgroup of G and $G' \cap Z = (1)$, $G'Z$ is cyclic.

Since $G'Z$ is the Fitting subgroup of G, G/Z has a trivial center. Hence K/Z acts faithfully on $G'Z/Z$, a cyclic group of odd order. Thus K/Z is cyclic. Hence, it follows that G/G' is cyclic.

Conversely, suppose conditions 1—3 hold and let H be a subnormal subgroup of G. We can assume $H \nleqq G'Z$ and $G' \nleqq H$. Consider $H/H \cap G'$; $H/H \cap G'$ is subnormal in $G/H \cap G'$ and hence in $HG'/H \cap G'$. Thus there is a chain of subgroups

$$H_s = HG' > H_{s-1} > \cdots > H_2 > H_1 = H,$$

such that

$$H_i/H \cap G' \trianglelefteq H_{i+1}/H \cap G' \text{ for } i = 1, \cdots, s-1.$$

Since

$$(|H_2 : H|, \ |H : H \cap G'|) = 1, \ H/H \cap G'$$

is characteristic in $H_2/H \cap G'$. So $H/H \cap G' \trianglelefteq H_3/H \cap G'$. Proceeding in this fashion we get that $H/H \cap G'$ is characteristic in $HG'/H \cap G'$. Since HG' is characteristic in G, it follows that H is characteristic in G. So G is a c-group.

Proof of theorem 2. Assume G is a c-group and let $L = L(G)$, $Z = Z(G)$. Then L is a Hall-subgroup of G of odd order. Hence, L is abelian and $G = L \cdot K$ with $L \cap K = (1)$. As in the proof of Theorem 1, we have that L is cyclic. G/L is non-abelian since the 2-Sylow subgroups of G are non-abelian. Hence, L is a proper subgroup of G'. Let S be a 2-Sylow subgroup of G, then $S = B \times Q_n$ where B is an elementary abelian 2-group and Q_n is generalized quaternion of order 2^n. So $S/S \cap G' \cong SG'/G'$ and therefore $S \cap G' \neq (1)$. So if $Q_n = \langle a, b \rangle$ with $b^4 = 1$, then $b^2 \in G'$. Thus

$b^2 \in G' \cap Z$ since $\langle b^2 \rangle \trianglelefteq G$. Now $G/L \cdot \langle b^2 \rangle$ is nilpotent and all of its Sylow subgroups are abelian, so $G' = L \cdot \langle b^2 \rangle$.

$G/\langle b^2 \rangle$ is an A-group so $Z(G/\langle b^2 \rangle) \cap G'/\langle b^2 \rangle = (1)$. Hence, $G' \cap Z = \langle b^2 \rangle$. Let $Z_1/\langle b^2 \rangle = Z(G/\langle b^2 \rangle)$. Then $G'Z_1/\langle b^2 \rangle$ is the Fitting subgroup of $G/\langle b^2 \rangle$. Since $b^2 \in Z$, $G'Z_1$ is the Fitting subgroup of G. Since $G'Z_1/Z_1 \cong L$, we have that $G/G'Z_1$ is cyclic. Since Z_1 is a nilpotent normal subgroup of G, $Z_1 = A_1 \times B_1 \times Q_m$ where A_1 is an abelian group of odd order, B_1 is an elementary abelian 2-group and Q_m is generalized quaternion of order 2^m. So $B_1 \leqq Z$ and $A_1 \leqq Z$. Hence, it follows that B_1 is an abelian direct factor of G. So $B_1 = (1)$ and $Z_1 = A_1 \times Q_m$. Moreover, since K/Z_1 is cyclic, its 2-Sylow subgroups are cyclic. Now Q_m is a proper subgroup of Q_n, for otherwise Q_n would be a direct factor of G and G would not be a c-group. Thus the 2-Sylow subgroups of G have form Q_n. So G has exactly one element of order 2.

Using methods similar to those used in the proof of Theorem 1, one gets that A_1 is cyclic and hence that $Z_1/\langle b^2 \rangle$ is cyclic. In the process we also find that if p is a prime divisor of $|A_1|$, then the p-Sylow subgroup of K is cyclic. Thus the Sylow subgroups of G/G' are cyclic and hence G/G' is cyclic. Thus, $G/\langle b^2 \rangle$ is a c-group.

Conversely, suppose conditions 1—4 hold and let H be a subnormal subgroup of G. Let $\theta \in \mathrm{Aut}\,(G)$. Then since $G/\langle u \rangle$ is a c-group, $H\langle u \rangle/\langle u \rangle$ is characteristic in $G/\langle u \rangle$. Now either $2 \mid |H|$ and hence, $u \in H$, or $2 \nmid |H|$. If $u \in H$ then $H^\theta = H$. If $2 \nmid |H|$, let $y \in H$ and suppose $y^\theta = y_1 u^s$. Here $s = 0$ or 1 and $y_1 \in H$. If $y^\theta = y_1 u$, then $|y^\theta| = 2|y_1|$ and we have a contradiction. So $y^\theta \in H$ and $H^\theta = H$. So G is a c-group.

Corollary 1. *If G is a solvable c-group and G has an abelian direct factor, then this factor is a Hall-subgroup of G.*

Proof. Let A be an abelian direct factor of G, say $G = A \times B$. Thus A must be cyclic by Lemma 1. We can assume that B has the form given in Theorem 1 or 2 and that B has no abelian direct factor. Then we still have $G = L \cdot K$ with $K \cap L = (1)$ and L a cyclic Hall-subgroup of G. Suppose p is a prime divisor of $(|A|, |B|)$. Let K_p be the p-Sylow subgroup of K. If p is odd then K_p is abelian, say

$$K_p = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_t \rangle.$$

Let a be the element of A of order p. Since $p \mid (|A|, |B|)$, $t > 1$, say $a \in \langle x_t \rangle$. Then the map $\theta : K_p \to K_p$ given by

$$x_1^\theta = a x_1$$
$$x_i^\theta = x_i, \qquad i > 1,$$

is an automorphism of K_p which can be lifted to $\bar\theta \in \mathrm{Aut}\,(G)$. Then, if

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$L = \langle c \rangle$, $H_1 = \langle c, x_1 \rangle$ and $H_2 = \langle c, x_1 a \rangle$ are normal subgroups of G and $H_1^{\bar{q}} = H_2$. Hence, we have a contradiction. A similar technique can be applied if $p = 2$.

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