EXACT STATISTICAL MECHANICS OF A ONE-DIMENSIONAL SELF-GRAVITATING SYSTEM

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Abstract. The statistical mechanics of an isolated self-gravitating system consisting of N uniform mass sheets is considered using both canonical and microcanonical ensembles. The one-particle distribution function is found in closed form. The limit for large numbers of sheets with fixed total mass and energy is taken and is shown to yield the isothermal solution of the Vlasov equation. The order of magnitude of the approach to Vlasov theory is found to be $0(1/N)$. Numerical results for spatial density and velocity distributions are given.

1. Introduction

Consider the one-dimensional model of an N-body self-gravitating system described by the Hamiltonian

$$\begin{aligned} H(\mathbf{p},\mathbf{x}) &= T(\mathbf{p}) + V(\mathbf{x}) \\ &= \sum_{n=1}^{N} p_n^2/(2\sigma_n) + 2\pi G \sum_{n>m} \sigma_n\sigma_m |x_n - x_m| \,. \end{aligned} \tag{1.1}$$

Physically this represents a system of N parallel, uniform mass sheets with positions $\mathbf{x}=(x_1, x_2,\ldots, x_N)$ and momenta $\mathbf{p}=(p_1, p_2,\ldots, p_N)$ that move along the x-axis under the influence of their mutual gravitation. Each sheet has a certain surface mass density σ_n, and it may pass freely through any other sheet, since there is no hard-core included in the interaction potential. G is the gravitational constant.

This model has some direct astrophysical relevance to the problem of the mass distribution normal to a highly flattened galaxy. More indirectly it has been used to investigate such questions as: (1) Relaxation by particle effects and the validity of Vlasov theory for large N; (2) Relaxation by mean field effects and the validity of Lynden-Bell's theory of violent relaxation; (3) Mass segregation effects.

A self-gravitating system of this type, if left to develop for a sufficiently long time, will reach the ultimate relaxed state of thermodynamic equilibrium. The precise description of this state is of considerable interest. For example, the degree of relaxation of any system must be defined relative to this state as a standard. Also the degree of validity of Vlasov theory in the thermodynamic equilibrium state is a useful indicator of the general validity of Vlasov theory for any system. Finally, by comparison to numerical experiments that have reached quasi-equilibrium, one may find evidence for approximate integrals of motion.

It should be made clear, however, that the state of thermodynamic equilibrium for this one-dimensional self-gravitating system sheds no light whatsoever on the general problem, since there is no state of thermodynamic equilibrium for three-dimensional systems. This is easily seen from the virial theorem $E=-T$, which predicts a negative heat capacity for the system, in direct violation of general statistical mechanical

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theorems. The nonexistence of thermodynamic equilibrium for three-dimensional systems is also manifest from the divergence of the relevant partition functions.

For the case of a one-dimensional self-gravitating system with equal masses, $\sigma_n = \sigma$, confined to a 'box' of length L, Salzburg (1965) found the thermodynamic properties of the system. This was done using an extension of the methods of Lenard (1961) and Prager (1961), which were developed for treating the analogous one-dimensional plasma. Nonextensive thermodynamic properties were found when the usual thermodynamic limit of $N \to \infty$, keeping N/L fixed, was taken.

The calculations of Salzburg are not directly applicable to the self-gravitating systems of interest in stellar dynamics, which are generally regarded as existing in free space, not enclosed by external walls. Also, the chemical thermodynamics approach is not sufficiently detailed to answer questions about the intrinsic structure of the system, such as the density distribution relative to the center of mass. Furthermore the appropriate large N limit for such a system is not the usual thermodynamic limit, but rather one in which the total mass and energy are kept fixed. While interesting in its own right, the calculation of Salzburg needs to be extended for the present purposes.

In this paper, by suitable modification of the methods of Lenard, Prager, and Salzburg, the single particle distribution function will be found in closed form for an isolated, equal mass model on the basis of both canonical and microcanonical ensembles. Specific account is taken of the integrals of motion due to the separability of the center of mass motion. This is done by performing the integration over the phase space subject to the constraints $\bar{x}=0$ and $\bar{p}=0$, where

$$\bar{x}(\mathbf{x}) \equiv N^{-1} \sum_n x_n \tag{1.2}$$

$$\bar{p}(\mathbf{p}) \equiv \sum_n p_n \tag{1.3}$$

are the coordinate and momentum of the center of mass. This is equivalent to choosing a frame of reference in which the center of mass is at rest at the origin. The proper investigation of the intrinsic structure of the system requires such a procedure; otherwise certain average properties such as density would be uniform in space due to the uniform motion of the center of mass.

For the microcanonical ensemble there is the additional constraint of fixed total energy E:

$$E = H(\mathbf{p}, \mathbf{x}). \tag{1.4}$$

For the canonical ensemble there is no such constraint, but the integrations are done with the weighting function $\exp(-\beta H)$, where $\beta = (kT)^{-1}$; T is the temperature and k is Boltzmann's constant.

The physical nature of the canonical ensemble is admittedly somewhat obscure in the present instance, since the system has been assumed to be in momentum isolation, and it is difficult to see how it then can also be in energy contact with a heat bath. However, it will be assumed that such a physical arrangement can be made and that the canonical ensemble has some meaning. In any case the canonical results are *necessary mathematical preliminaries to finding the more physically realistic micro-*canonical results.

The possibility of doing the necessary configurational integrations in closed form rests first of all on the fact that it is always possible to reduce any such integration to one over a particular ordering of the coordinates, say $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$. Then the potential takes the simple form

$$V(\mathbf{x}) = 2\pi G\sigma^2 \sum_{n>m} (x_n - x_m) \tag{1.5}$$

without absolute value bars. Letting $\lambda = 2\pi G\sigma^2$ this may be written, after some manipulation,

$$V(\mathbf{x}) = \lambda \sum_{l=1}^{N-1} l(N-l)\cdot(x_{l+1} - x_l). \tag{1.6}$$

This equation has a simple physical interpretation. Consider the work done in reducing an interval $x_{l+1} - x_l$ to zero while keeping rigid connections between the members of each group of particles to the left and to the right, so that all other intervals remain constant. Since the force is independent of distance, this is equivalent to the work done in moving just two sheets of mass $l\sigma$ and $(N-l)\,\sigma$ into coincidence over a distance $x_{l+1} - x_l$, that is, $-\lambda l(N-l)\cdot(x_{l+1} - x_l)$. Each interval may in turn be reduced to zero by a similar process, and eventually all particles will be coincident, so that $V=0$. Thus the potential energy of the original configuration is given by the sum (1.6). The importance of Equation (1.6) is that it expresses the potential as a sum of independent contributions, which makes it possible to do the configurational integrations, after a simple change of variables. This is the essential analytical trick employed by Lenard and Prager in the plasma case.

The combinatorial difficulties of the plasma case fortunately do not arise here in performing the necessary phase averages, because all sheets have the same mass. On the other hand the analytical difficulties in the present calculation exceed those of the plasma case, because of the more detailed information sought, and it is somewhat remarkable that the results come out in such simple form. By treating a system with various masses, one would encounter the combinatorial difficulties as well.

In the limit of large N, with fixed total mass and energy, it is shown that the single particle distribution function approaches the isothermal solution of the Vlasov equation. The order of this approach is $1/N$, which implies something about the order of the two-particle distribution function.

Numerical results are presented for the spatial density of an N-particle system for both the canonical and the microcanonical results. Comparison with the numerical experiments of Hohl (1968) for $N=3$ indicates that there is an approximate integral of motion in this case.

2. The Canonical Ensemble

The canonical one-particle distribution function $f_c(p, x)$ may be defined as the phase space average of the quantity

$$N^{-1}\sum_n \delta(p-p_n)\,\delta(x-x_n) \tag{2.1}$$

with weighting function $\exp(-\beta H)$ and with the constraints (1.2) and (1.3). Thus,

$$f_c(p,x)=(zN!)^{-1}\iint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{x}\,\delta(\bar{x})\,\delta(\bar{p})\exp(-\beta H)\,N^{-1}\sum_n \delta(p-p_n)\,\delta(x-x_n) \tag{2.2}$$

where

$$z=(N!)^{-1}\iint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{x}\,\delta(\bar{x})\,\delta(\bar{p})\exp(-\beta H). \tag{2.3}$$

Planck's constant has been omitted here, since it cancels in all relevant formulae. Because of the identity of particles the average may be taken of $\delta(x-x_N)\,\delta(p-p_N)$ instead of the symmetrized form (2.1). With the separability of the Hamiltonian $H=T+V$, this implies the factorization $f_c=\varrho_c(x)\,\theta_c(p)$, where

$$\begin{aligned}
\varrho_c(x)&=(QN!)^{-1}\int \mathrm{d}\mathbf{x}\,\delta(\bar{x})\exp(-\beta V)\,\delta(x-x_N)\\
Q&=(N!)^{-1}\int \mathrm{d}\mathbf{x}\,\delta(\bar{x})\exp(-\beta V)\\
\theta_c(p)&=(R)^{-1}\int \mathrm{d}\mathbf{p}\,\delta(\bar{p})\exp(-\beta T)\,\delta(p-p_N)\\
R&=\int \mathrm{d}\mathbf{p}\,\delta(\bar{p})\exp(-\beta T).
\end{aligned} \tag{2.4}$$

Because the kinetic energy is simply a sum of quadratic terms, $\theta_c(p)$ may be easily found. In the following development integrals are taken over all space, unless otherwise indicated. Using the Fourier representation of the δ-function,

$$\delta(\bar{p})=(2\pi)^{-1}\int \mathrm{d}k\,\exp\left(ik\sum_n p_n\right) \tag{2.5}$$

we have

$$\begin{aligned}
R\theta_c(p)&=(2\pi)^{-1}\int \mathrm{d}k\int \mathrm{d}p_1\ldots\mathrm{d}p_N\,\exp\left[ik\sum_n p_n-\beta\sum_n p_n^2/(2\sigma)\right]\delta(p-p_N)\\
&=(2\pi)^{-1}\exp[-\beta p^2/(2\sigma)]\int \mathrm{d}k\,\times\\
&\quad\times e^{ikp}\int \mathrm{d}p_1\ldots\mathrm{d}p_{N-1}\prod_{n=1}^{N-1}\exp[ikp_n-\beta p_n^2/(2\sigma)]\\
&=(2\pi)^{-1}\exp[-\beta p^2/(2\sigma)]\int \mathrm{d}k\,e^{ikp}\exp[-(N-1)\,\sigma k^2/(2\beta)]\\
&=(N-1)^{-1/2}\,(2\pi\sigma\beta^{-1})^{(N/2)-1}\exp[-\beta Np^2/(2\sigma(N-1))]
\end{aligned} \tag{2.6}$$

where the well-known formula for the Fourier transform of a Gaussian has been used twice. Since θ_c has unit normalization,

$$\int \theta_c(p)\,\mathrm{d}p = 1 \tag{2.7}$$

integration of Equation (2.6) yields

$$R = N^{-1/2}(2\pi\sigma\beta^{-1})^{(N-1)/2} \tag{2.8}$$

so that

$$\theta_c(p) = \left[\frac{\beta N}{2\pi\sigma(N-1)}\right]^{1/2} \exp\left[-\frac{\beta N p^2}{2\sigma(N-1)}\right]. \tag{2.9}$$

Thus the momentum part of the one-particle distribution is Maxwellian, as could have been predicted. The unfamiliar factors $N/(N-1)$ are explained by the fact that N particles must share the thermal kinetic energy of $N-1$ degrees of freedom, since one degree of freedom has been lost by virtue of the center of mass constraint.

In order to find the density ϱ_M, first a symmetrized form of Equation (2.4) is written:

$$Q\varrho_c(x) = (N!)^{-1} \int \mathrm{d}\mathbf{x}\, \delta(\bar{x}) \exp(-\beta V)\, N^{-1} \sum_n \delta(x - x_n). \tag{2.10}$$

Because of the symmetry the integrations may be taken over any ordering of the variables and the result multiplied by $N!$. The ordering chosen here is $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$. Defining the Fourier transform of the density ϱ_c by

$$\bar{\varrho}_c(k) = \int \mathrm{d}x \exp(ikx)\, \varrho_c(x) \tag{2.11}$$

Equation (2.10) may be written:

$$Q\bar{\varrho}_c(k) = N^{-1} \sum_n F_n(k) \tag{2.12}$$

where

$$F_n(k) = \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{x_1}^{\infty} \mathrm{d}x_2 \int_{x_2}^{\infty} \mathrm{d}x_3 \ldots \int_{x_{N-1}}^{\infty} \mathrm{d}x_N\, \delta(\bar{x}) \exp(-\beta V) \exp(ikx_n). \tag{2.13}$$

Advantage is now taken of the result (1.6) by introduction of the variables

$$\begin{aligned} u_l &= x_{l+1} - x_l, \quad 1 \leqslant l \leqslant N-1 \\ u_N &= N^{-1} \sum_m x_m = \bar{x}. \end{aligned} \tag{2.14}$$

The Jacobian of this transformation is

$$J = \frac{1}{N} \begin{vmatrix} -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{vmatrix} = \frac{1}{N} \begin{vmatrix} -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & N \end{vmatrix}. \tag{2.15}$$

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The determinant has been transformed by addition of n times the n-th row to the last row for $n=1, 2, \ldots, N-1$. This is now in tridiagonal form with the determinant equal to the product of the diagonal elements, so that $|J|=1$.

The inverse of this transformation is easily found. By writing out the summations and regrouping terms, the results

$$\begin{aligned} &\sum_{l=n}^{N-1} u_l = x_N - x_n \\ &\sum_{l=1}^{N-1} l u_l = N(x_N - u_N) \end{aligned} \tag{2.16}$$

are obtained. Therefore

$$\begin{aligned} x_n &= u_N + N^{-1} \sum_{l=1}^{N-1} l u_l - \sum_{l=n}^{N} u_l \\ &= u_N - N^{-1} \sum_{l=n}^{N=1} D_{nl} u_l \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} D_{nl} &= -l, \qquad n > l \\ &= N - l, \quad n \leqslant l. \end{aligned} \tag{2.18}$$

With the notation

$$C_l = l(N-l) \tag{2.19}$$

the potential takes the form

$$V(\mathbf{x}) = \lambda \sum_{l=1}^{N-1} C_l u_l. \tag{2.20}$$

Transforming to the new variables in Equation (2.13) yields

$$\begin{aligned} F_n(k) &= \int_{-\infty}^{\infty} \mathrm{d}u_N \int_0^{\infty} \mathrm{d}u_1 \int_0^{\infty} \mathrm{d}u_2 \ldots \int_0^{\infty} \mathrm{d}u_{N-1}\, \delta(u_N) \times \\ &\quad \times \exp\left[-\beta\lambda \sum_{l-1}^{N-1} C_l u_l - ikN^{-1} \sum_{l=1}^{N-1} D_{nl} u_l + iku_N\right] \\ &= \prod_{l=1}^{N-1} \int_0^{\infty} \mathrm{d}u_l \exp\{-(\lambda\beta C_l + ikN^{-1} D_{nl})\, u_l\} \\ &= \prod_{l=1}^{N-1} [\lambda\beta C_l + ikN^{-1} D_{nl}]^{-1}. \end{aligned} \tag{2.21}$$

The value of Q follows by setting $k=0$:

$$Q = F_n(0) = \prod_{l=1}^{N-1} [\lambda\beta l(N-l)]^{-1} = (\lambda\beta)^{-(N-1)} [(N-1)!]^{-2}. \tag{2.22}$$

The parameter $\alpha = k/(N\beta\lambda)$ is introduced, so that one has

$$\begin{aligned}\prod_{l=1}^{N-1}[C_l + i\alpha D_{nl}] &= \prod_{l=1}^{n-1}[l(N-l) - i\alpha l]\prod_{l=n}^{N-1}[l(N-l) + i\alpha(N-l)] \\ &= \prod_{l=1}^{n-1}[l(N-l-i\alpha)]\prod_{l=n}^{N-1}[(N-l)(l+i\alpha)] \\ &= (n-1)!\,\frac{\Gamma(N-i\alpha)}{\Gamma(N-n-i\alpha+1)}\,(N-n)!\,\frac{\Gamma(N+i\alpha)}{\Gamma(n+i\alpha)}. \end{aligned} \tag{2.23}$$

Therefore Equation (2.12) becomes

$$\bar{\varrho}_c(k) = \frac{[(N-1)!]^2}{\Gamma(N+i\alpha)\,\Gamma(N-i\alpha)}\,\frac{1}{N}\sum_{n=1}^{N}\frac{\Gamma(n+i\alpha)\,\Gamma(N-n+1-i\alpha)}{(n-1)!\,(N-n)!}. \tag{2.24}$$

This expression may be reduced considerably. First note the general result of the Beta-function theory

$$\int_0^1 z^{\mu-1}(1-z)^{\nu-1}\,\mathrm{d}z = \frac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)}, \quad Re(\mu) > 0, \quad Re(\nu) > 0. \tag{2.25}$$

Then with use of the binomial theorem,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\frac{\Gamma(n+i\alpha)\,\Gamma(N-n+1-i\alpha)}{(n-1)!\,(N-n)!} &= \frac{1}{N}\sum_{n=1}^{N}\frac{N!}{(n-1)!\,(N-n)!}\times \\ \times\int_0^1 z^{n+i\alpha-1}(1-z)^{N-n+i\alpha}\,\mathrm{d}z &= \int_0^1 z^{i\alpha}(1-z)^{-i\alpha}[z+(1-z)]^{N-1}\,\mathrm{d}z \\ &= \frac{\Gamma(1+i\alpha)\,\Gamma(1-i\alpha)}{\Gamma(2)}. \end{aligned} \tag{2.26}$$

Thus

$$\begin{aligned}\bar{\varrho}_c(k) &= [(N-1)!]^2\,\frac{\Gamma(1+i\alpha)}{\Gamma(N+i\alpha)}\,\frac{\Gamma(1-i\alpha)}{\Gamma(N-i\alpha)} \\ &= \prod_{l=1}^{N-1}[l^2(l+i\alpha)^{-1}(l-i\alpha)^{-1}] \end{aligned} \tag{2.27}$$

and we obtain the remarkably simple form for the Fourier transform of the density,

$$\bar{\varrho}_c(k) = \prod_{l=1}^{N-1}\frac{l^2}{l^2+\alpha^2} = \prod_{l=1}^{N-1}\frac{l^2}{l^2 + k^2/(N\beta\lambda)^2}. \tag{2.28}$$

This Fourier transform may be inverted in closed form. As a function of the complex variable k, $\bar{\varrho}_c$ is seen to have simple poles equally spaced along the imaginary axis

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from $-i\,N(N-1)\,\beta\lambda$ to $+i\,N(N-1)\,\beta\lambda$, except at $k=0$. The contour for inversion lies along the real axis:

$$\varrho_c(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \bar{\varrho}_c(k)\, e^{-ikx}\, \mathrm{d}k \tag{2.29}$$

and may be deformed upward for $x<0$, enclosing the poles on the positive portion of the imaginary axis. The residue at the pole at $+iNn\beta\lambda$ is

$$\lim_{k\to iNn\beta\lambda} (k - iNn\beta\lambda)\,\bar{\varrho}_c(k) = \frac{n}{2i} \prod_{l=1}^{N-1}{}' \frac{l^2}{l^2 - n^2} \tag{2.30}$$

where the prime on the product means to omit the term $l=n$. Now

$$\begin{aligned}\prod_{l=1}^{N-1}{}' \frac{l^2 - n^2}{l^2} &= \left[\frac{n}{(N-1)!}\right]^2 \prod_{l=1}^{n-1} (l-n) \prod_{l=n+1}^{N-1} (l-n) \prod_{l=1}^{N-1}{}' (l+n) \\ &= \left[\frac{n}{(N-1)!}\right]^2 (-1)^{n-1} (n-1)!\,(N-1-n)!\,\frac{(N+n-1)!}{(2n)\,n!}.\end{aligned} \tag{2.31}$$

With the definition

$$A_l^N = \frac{[(N-1)!]^2\,(-1)^{l-1}\,l}{(N-1-l)!\,(N-1+l)!} \tag{2.32}$$

the density may be written

$$\varrho_c(x) = N\beta\lambda \sum_{l=1}^{N-1} A_l^N e^{-N\beta\lambda l|x|}. \tag{2.33}$$

The combination of Equations (2.9) and (2.33) finally gives the canonical one-particle distribution function in closed form:

$$f_c(p, x) = \frac{(N\beta)^{3/2}\,\lambda}{[2\pi\sigma(N-1)]^{1/2}} \sum_{l=1}^{N-1} A_l^N \exp\left[-\frac{\beta N}{2\sigma(N-1)}\,p^2 - N\beta\lambda l|x|\right]. \tag{2.34}$$

There is a quite useful integral representation for ϱ_c. Note that the upper limit on the summation in Equation (2.33) may be extended to ∞, since $A_l^N=0$ for $l\geqslant N$ due to the factor $(N-1-l)!$ in the denominator. Using the result

$$\frac{[(N-1)!]^2}{(N-1-l)!\,(N-1+l)!} = \int_{-\pi/2}^{\pi/2} e^{-2ilt} \cos^{2(N-1)} t\, \mathrm{d}t \Bigg/ \int_{-(\pi/2)}^{\pi/2} \cos^{2(N-1)} t\, \mathrm{d}t \tag{2.35}$$

which follows from well-known integrals and performing the elementary summation yields

$$\varrho_c(x) = \frac{N\beta\lambda}{4} \int_{-(\pi/2)}^{\pi/2} \mathrm{sech}^2\left(\frac{N\beta\lambda|x|}{2} + it\right) \cos^{2(N-1)} t\, \mathrm{d}t \Bigg/ \int_{-(\pi/2)}^{\pi/2} \cos^{2(N-1)} t\, \mathrm{d}t. \tag{2.36}$$

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One immediate application of this formula is to compute the central density. Setting $x=0$, we have

$$\varrho_c(0) = \frac{N\beta\lambda}{2} \frac{N-1}{2N-3} \tag{2.37}$$

since $\mathrm{secht}(it) = (\cos t)^{-1}$. This also implies the sum rule for the coefficients A_l^N,

$$\sum_{l=1}^{N-1} A_l^N = \frac{1}{2} \frac{N-1}{2N-3}. \tag{2.38}$$

The partition function

$$z = QR = (\lambda\beta)^{-(N-1)} [(N-1)!]^{-2} N^{-\frac{1}{2}} (2\pi\sigma\beta^{-1})^{(N-1)/2} \tag{2.39}$$

is proportional to $\beta^{-(3/2)(N-1)}$, so the average energy is

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \log z = \tfrac{3}{2}(N-1)\,\beta^{-1}. \tag{2.40}$$

3. The Microcanonical Ensemble

The canonical results just obtained apply to a system in contact with a heat bath that keeps the system at a temperature T. The energy of such a system is not well defined, and it undergoes thermal fluctuations of the order of kT. It is more realistic, from the point of view of stellar dynamics, to assume that the total energy of the system is fixed at the value E. This requires the use of the microcanonical ensemble in the statistical mechanical treatment. Rather than the weighting function $\exp(-\beta H)$, the integrations over the phase space are then to be done with the constraint (1.4) which may be conveniently handled by the inclusion of the weighting function $\delta(E-H)$ in the phase integrals.

In this section it will be shown that it is possible to obtain the microcanonical results from the canonical results just obtained. The microcanonical one-particle distribution function is defined as the average of the quantity (2.1) over the phase space with constraints (1.2), (1.3), and (1.4). Thus

$$f_{MC}(p, x) = (\Omega N!)^{-1} \int\!\!\int \mathbf{dp}\,\mathbf{dx}\,\delta(\bar{x})\,\delta(\bar{p})\,\delta(E-H)\,N^{-1} \sum_n \delta(p-p_n)\,\delta(x-x_n) \tag{3.1}$$

where

$$\Omega = (N!)^{-1} \int\!\!\int \mathbf{dp}\,\mathbf{dx}\,\delta(\bar{x})\,\delta(\bar{p})\,\delta(E-H). \tag{3.2}$$

Note that these quantities are related to the corresponding canonical ones (2.2) and (2.3) by Laplace transformation:

$$z f_c = \int_0^\infty e^{-\beta E} (\Omega f_{MC})\,\mathrm{d}E \tag{3.3}$$

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$$z = \int_0^\infty e^{-\beta E} \Omega \, \mathrm{d}E . \tag{3.4}$$

Therefore, by inversion,

$$\Omega f_{MC} = (2\pi i)^{-1} \int_C e^{\beta E} (z f_C) \, \mathrm{d}\beta \tag{3.5}$$

$$\Omega = (2\pi i)^{-1} \int_C e^{\beta E} z \, \mathrm{d}\beta \tag{3.6}$$

where the contour C extends from $-i\infty$ to $i\infty$ to the right of all singularities. First note the general result

$$(u)_+^{\gamma-1} / \Gamma(\gamma) = (2\pi i)^{-1} \int_C e^{\beta u} \beta^{-\gamma} \, \mathrm{d}\beta . \tag{3.7}$$

The notation $(\ \ )_+$ is defined by

$$\begin{aligned} (u)_+ &= u , \quad u \geqslant 0 \\ &= 0 , \quad u < 0 . \end{aligned} \tag{3.8}$$

The above inversions may now be done in closed form, using equations (2.33), (2.40), and (3.7). The results are:

$$f_{MC}(p, x) = \frac{N\lambda}{E} \left(\frac{N}{2\pi\sigma (N-1) E} \right)^{1/2} \frac{\Gamma\left(\frac{3N}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{3N}{2} - 3\right)} \times \\ \times \sum_{l=1}^{N-1} A_l^N \left(1 - \frac{Np^2}{2\sigma (N-1) E} - \frac{N\lambda |x| \, l}{E} \right)_+^{(3N/2)-4} \tag{3.9}$$

$$\Omega = \frac{(2\pi\sigma)^{(N-1)/2} E^{(3N-5)/2}}{\lambda^{N-1} [(N-1)!]^2 N^{1/2} \Gamma\left(\frac{3}{2}(N-1)\right)} . \tag{3.10}$$

Integrating this over p and x yields the density and the momentum distributions,

$$\begin{aligned} \varrho_{MC}(x) &= \int f_{MC}(p, x) \, \mathrm{d}p \\ &= \frac{N\lambda}{E} \tfrac{1}{2}(3N-5) \sum_{l=1}^{N-1} A_l^N \left(1 - \frac{N\lambda}{E} l |x| \right)_+^{\frac{3}{2}N - \frac{7}{2}} \end{aligned} \tag{3.11}$$

$$\begin{aligned} \theta_{MC}(p) &= \int f_{MC}(p, x) \, \mathrm{d}x \\ &= \frac{\Gamma\left(\frac{3N}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{3N}{2} - 2\right)} \left(\frac{N}{2\pi\sigma (N-1) E} \right)^{1/2} \left(1 - \frac{Np^2}{2\sigma (N-1) E} \right)_+^{(3N/2)-3} . \end{aligned} \tag{3.12}$$

To obtain Equation (3.11) the formula

$$\int_{-1}^{+1} (1-t^2)^{\mu-1}\,\mathrm{d}t = \pi^{1/2}\,\frac{\Gamma(\mu)}{\Gamma(\mu+\frac{1}{2})} \tag{3.13}$$

was used, which follows from the transformation $t=2z-1$ in Equation (2.25) with $\mu=\nu$ and use of the duplication formula for Γ-functions. The normalization of ϱ_{MC} was used to evaluate

$$\sum_{l=1}^{N-1} \frac{1}{l} A_l^N = \tfrac{1}{2} \tag{3.14}$$

which was then used to obtain Equation (3.12).

The central density in the microcanonical case follows from Equations (3.11) and (2.38):

$$\varrho_{MC}(0) = \frac{N\lambda}{E}\,\frac{1}{4}\,\frac{(3N-5)(N-1)}{(2N-3)}. \tag{3.15}$$

4. The Limit of Large N

The large N limit of most interest in stellar dynamics is the one in which the total energy E and total mass $M=N\sigma$ are fixed. This will be called the *Vlasov limit*, since this is the limit that is expected to lead to the usual Vlasov results. Since there is no well-defined total energy for the canonical ensemble, in this case the limit will be taken for fixed *average* total energy $\langle E\rangle$, given by Equation (2.40). In this section the notations E and $\langle E\rangle$ will be used interchangeably.

The first step in investigating the Vlasov limit is to introduce scaled variables that depend only on the fixed quantities E and M and on the number N. It is also necessary to adopt velocity rather than momentum as the basic variable in the distribution functions. To do this the dimensionless velocity, position, and Fourier transform variables

$$\begin{aligned} \eta &= \frac{p}{\sigma V} \\ \xi &= \frac{x}{L} \\ K &= kL \end{aligned} \tag{4.1}$$

are defined, where the characteristic velocity V and length L are given by

$$L = \frac{2E}{3\pi G M^2}, \quad V^2 = \frac{4E}{3M}. \tag{4.2}$$

The particular numerical factors used here have been chosen to simplify the final

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results. Scaled distribution functions are then defined by

$$\begin{aligned} f^*(\eta,\xi) &= \sigma V L f(\sigma V\eta, L\xi) \\ \varrho^*(\xi) &= L\varrho(L\xi) \\ \theta^*(\eta) &= \sigma V\theta(\sigma V\eta) \\ \bar{\varrho}^*(K) &= \bar{\varrho}\left(\frac{K}{L}\right) = \int e^{iK\xi}\varrho^*(\xi)\,\mathrm{d}\xi\,. \end{aligned} \tag{4.3}$$

These are normalized in the following manner:

$$\iint f^*(\eta,\xi)\,\mathrm{d}\eta\,\mathrm{d}\xi = \int \varrho^*(\xi)\,\mathrm{d}\xi = \int \theta^*(\eta)\,\mathrm{d}\eta = 1\,. \tag{4.4}$$

These definitions lead to the canonical ensemble results

$$f_c^*(\eta,\xi) = 2\pi^{-1/2}\left(1-\frac{1}{N}\right)\sum_{l=1}^{N-1} A_l^N e^{-2(1-(1/N))l|\xi|-\eta^2} \tag{4.5}$$

$$\theta_c^*(\eta) = \pi^{-1/2}e^{-\eta^2} \tag{4.6}$$

$$\varrho_c^*(\xi) = 2\left(1-\frac{1}{N}\right)\sum_{l=1}^{N-1} A_l^N e^{-2(1-(1/N))l|\xi|} \tag{4.7}$$

$$= \frac{1}{2}\left(1-\frac{1}{N}\right)\cdot\frac{\displaystyle\int_{-(\pi/2)}^{\pi/2} \operatorname{sech}^2\left(\left(1-\frac{1}{N}\right)\xi + it\right)\cos^{2(N-1)} t\,dt}{\displaystyle\int_{-(\pi/2)}^{\pi/2} \cos^{2(N-1)} t\,\mathrm{d}t} \tag{4.8}$$

$$\varrho_c^*(0) = \frac{(N-1)^2}{N(2N-3)} \tag{4.9}$$

$$\bar{\varrho}_c^*(K) = \prod_{l=1}^{N-1} \frac{l^2}{l^2 + \left(\dfrac{NK}{2(N-1)}\right)^2} \tag{4.10}$$

which follow from Equations (2.34), (2.9), (2.33), (2.36), (2.37), and (2.28), respectively. For the microcanonical ensemble the results are:

$$f_{MC}^*(\eta,\xi) = \frac{4}{3N}\left(\frac{2}{3\pi(N-1)}\right)^{1/2}\frac{\Gamma\left(\dfrac{3N}{2}-\dfrac{3}{2}\right)}{\Gamma\left(\dfrac{3N}{2}-3\right)}\sum_{l=1}^{N-1} A_l^N\left(1-\frac{2\eta^2}{3(N-1)}-\frac{4l|\xi|}{3N}\right)_+^{(3N/2)-4} \tag{4.11}$$

$$\theta^*_{MC}(\eta) = \frac{\Gamma\left(\frac{3N}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{3N}{2} - 2\right)} \left(\frac{2}{3\pi(N-1)}\right)^{1/2} \left(1 - \frac{2\eta^2}{3(N-1)}\right)_+^{(3N/2)-3} \tag{4.12}$$

$$\varrho^*_{MC}(\xi) = 2\left(1 - \frac{5}{3N}\right) \sum_{l=1}^{N-1} A_l^N \left(1 - \frac{4l|\xi|}{3N}\right)_+^{\frac{3}{2}N - \frac{7}{2}} \tag{4.13}$$

$$\varrho^*_{MC}(0) = \frac{(N-1)(3N-5)}{3N(2N-3)} \tag{4.14}$$

which follow from Equations (3.9), (3.12), (3.11), and (3.15), respectively.

In Figures 1 and 2 the canonical and microcanonical ensemble densities $\varrho^*(\xi)$ are plotted for various values of N. It can be seen that as $N \to \infty$ these curves appear to approach a limit. This limit is easily found in the canonical case from Equation (4.8). Apart from the factors $(1-1/N)$, which approach unity, this equation expresses $\varrho_c^*(\xi)$ as an average of $\text{sech}^2(\xi + it)$ over the range of t, $-(\pi/2) \leqslant t \leqslant (\pi/2)$, with weighting function $\cos^{2(N-1)} t$. For large N this weighting function becomes highly peaked in the neighborhood of $t=0$, so that as $N \to \infty$ it picks out the single value at

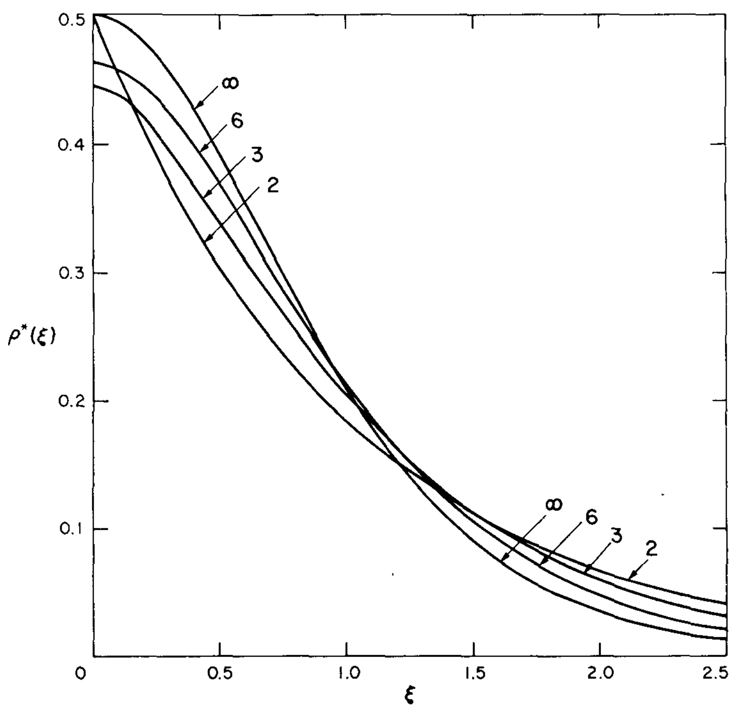

Fig. 1. Canonical density vs ξ for various values of N.

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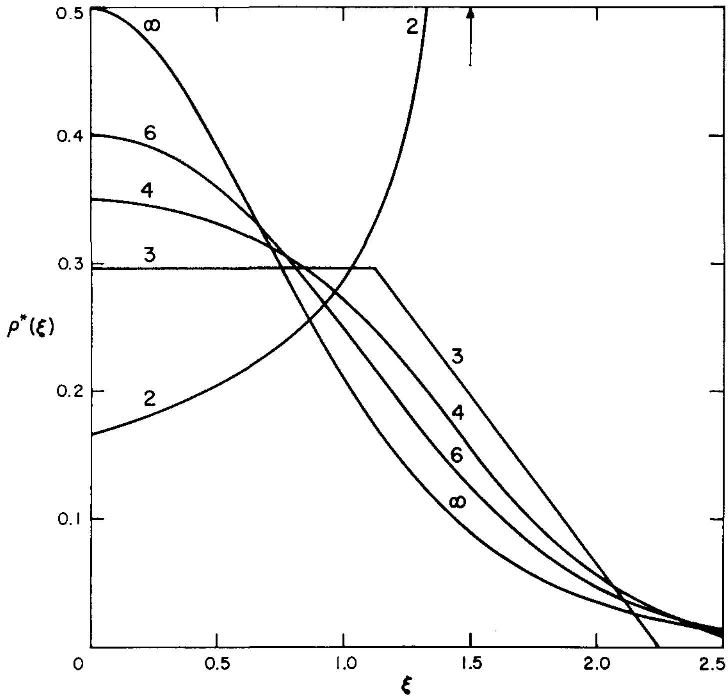

Fig. 2. Microcanonical density vs ξ for various values of N. The arrow locates the vertical asymptote for the case $N=2$.

$t=0$. Therefore,

$$\varrho_c^*(\xi) \to \tfrac{1}{2}\,\mathrm{sech}^2\xi \tag{4.15}$$

and with Equation (4.6) this implies

$$f_c^*(\eta, \xi) \to f_V^*(\eta, \xi) \tag{4.16}$$

where

$$f_V^*(\eta, \xi) = \frac{\pi^{-1/2}}{2}\, e^{-\eta^2}\,\mathrm{sech}^2\,\xi\,. \tag{4.17}$$

This may be recognized as the isothermal solution of the Vlasov equation, found by Camm (1950). The Vlasov results will be denoted by a subscript V.

This limiting form of the density appears in a quite picturesque way in terms of the Fourier transform (4.10). It is clear that as $N \to \infty$,

$$\bar{\varrho}_c^*(K) \to \prod_{l=1}^{\infty} \frac{l^2}{l^2 + \left(\frac{K}{2}\right)^2} = \frac{\pi K}{2}\,\mathrm{csch}\,\frac{\pi K}{2} \tag{4.18}$$

using the infinite product representation of $\mathrm{csch}(\pi K/2)$. The inverse transform of this again yields Equation (4.15). Apart from a slight change in the spacing of the poles, the main effect of finite N is simply to eliminate from the function (4.18), having an infinite number of poles, all but the $2(N-1)$ poles closest to the origin.

The microcanonical ensemble results also approach the isothermal solution of the Vlasov equation. This is difficult to prove rigorously, but a heuristic proof can be indicated: In the limit $N \to \infty$, one notes that

$$\begin{aligned} &A_l^N \to (-1)^{l-1}\, l \\ &\left(1 - \frac{2\eta^2}{3(N-1)} - \frac{4l|\xi|}{3N}\right)_+^{(3N/2)-4} \to e^{-\eta^2 - 2l|\xi|} \\ &\frac{4}{3N}\left(\frac{2}{3\pi(N-1)}\right)^{1/2} \frac{\Gamma\left(\frac{3N}{2} - \frac{3}{2}\right)}{\Gamma\left(\frac{3N}{2} - 3\right)} \to 2\pi^{-1/2} \end{aligned} \tag{4.19}$$

so that

$$f_{MC}^*(\eta, \xi) \to 2\pi^{-1/2} \sum_{l=1}^{\infty} (-1)^{l-1}\, l e^{-\eta^2 - 2l|\xi|} = \frac{\pi^{-1/2}}{2}\, e^{-\eta^2} \operatorname{sech}^2 \xi . \tag{4.20}$$

This proof fails to take into account that the limits (4.19) are not uniform in l, so that their replacement in the sum (4.13) is not justified. Nonetheless, the numerical results obtained seem to indicate that this is in fact the correct limit.

The approach to the Vlasov theory may be estimated from the central densities:

$$\frac{\varrho_c^*(0)}{\varrho_V^*(0)} = \frac{\left(1 - \frac{1}{N}\right)^2}{\left(1 - \frac{3}{2N}\right)} = 1 - \frac{1}{2N} + 0\left(\frac{1}{N^2}\right) \tag{4.21}$$

$$\frac{\varrho_{MC}^*(0)}{\varrho_V^*(0)} = \frac{\left(1 - \frac{1}{N}\right)\left(1 - \frac{5}{3N}\right)}{\left(1 - \frac{3}{2N}\right)} = 1 - \frac{7}{6N} + 0\left(\frac{1}{N^2}\right). \tag{4.22}$$

This approach is seen to be of order $0(1/N)$. The canonical results approach Vlasov theory faster than the microcanonical results. This is to be expected, since the velocity part of the canonical distribution is already precisely in Vlasov form.

The microcanonical ensemble for $N=2$ consists of the single system of two particles moving symmetrically about $x=0\,(\xi=0)$, along with all other systems derived from this one by a phase difference only. Each particle moves in a constant force field with a turning point occurring at a certain distance $|\xi| = \frac{3}{2}$. The density is proportional to the inverse of the velocity so that it approaches infinity at the turning point; this is indicated in Figure 2 by the arrow. The velocity distribution (see Figure 3) is uniform between certain maximum limits $|\eta| = (\frac{3}{2})^{1/2}$; this is because each particle spends the same time in each velocity range due to the constant acceleration.

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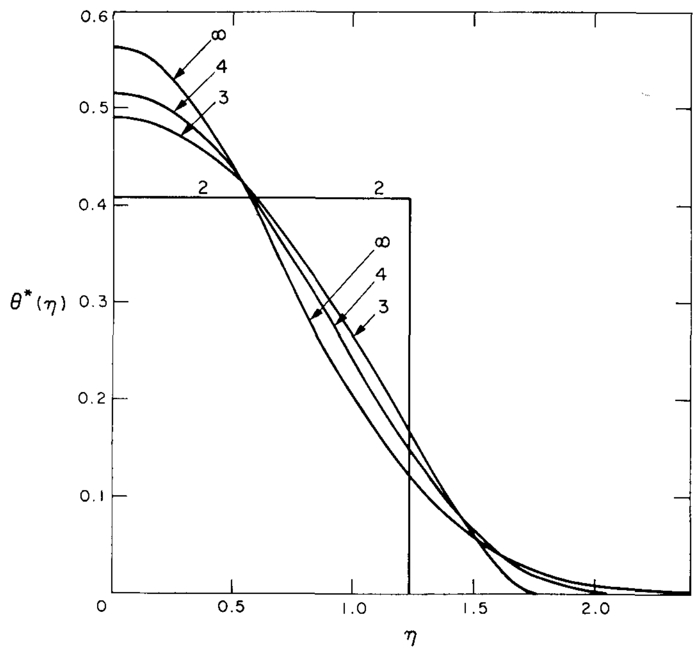

Fig. 3. Microcanonical velocity distribution vs η for various values of N. The case $N=\infty$ is also the canonical velocity distribution for all values of N.

For $N=3$ the microcanonical density is uniform over a certain range with limits $|\xi|=\frac{9}{8}$, and then falls linearly to zero at $|\xi|=\frac{9}{4}$. These points represent the respective limits of the regions where at most two or at most one particle can contribute to the density. Note that the velocity distribution in this case already approximates the Maxwellian Vlasov results quite well.

Numerical experiments with $N=3$ have been performed by Hohl (1968). The experimental velocity distribution found seems to fit the theoretical curve quite well, but the density curve shows a number of features not found in the theoretical curve. From this one may conclude that there must be an approximate integral of motion for the particular case treated by Hohl, which prevented complete ergodic behavior. It would be interesting to repeat the $N=3$ experiment with other initial conditions to see how strict these integrals may be.

For values of N larger than 3 the microcanonical density distributions appear fairly smooth, although all of these possess discontinuities in high order derivatives at certain points. The velocity distribution rapidly approaches the Vlasov result.

5. Final Remarks

Similar derivations for the two-particle distribution function have been attempted by the author. The purpose is to rigorously estimate the two-particle correlation function, which would give a definitive answer to the question of the order of magnitude of approach to Vlasov theory. This program has not as yet been successful, because of the considerable algebraic complexities involved, which make the necessary order of magnitude estimates difficult.

One possible generalization of this theory is to treat cases having two (or more) types of masses. It would be interesting to see whether such a mass spectrum affects the approach to Vlasov theory. Another possible study is suggested by the fact that the density curves found for finite N seem to differ from the Vlasov result primarily by a scale error. It is possible that the introduction of a simple N-scaling could improve the approach to Vlasov theory to, say, $0(1/N^2)$.

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